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PACKING OF THREE COPIES OF A DIGRAPH INTO THE TRANSITIVE TOURNAMENT

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Abstract

In this paper, we show that if the number of arcs in an oriented graph $\overrightarrow{G}$ (of order n) without directed cycles is sufficiently small (not greater than $\frac{2}{3}n - 1$), then there exist arc disjoint embeddings of three copies of $\overrightarrow{G}$ into the transitive tournament TT_n. It is the best possible bound.

Keywords: packing of digraphs, transitive tournament.

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1. Introduction. Results

Let $\overrightarrow{G}$ be a digraph of order n with the vertex set $V(\overrightarrow{G})$ and the arc set $E(\overrightarrow{G})$. A digraph $\overrightarrow{G}$ is called *transitive* when it satisfies the condition of transitivity: if (u, v) and (v, w) are two arcs of $\overrightarrow{G}$ then (u, w) is the arc, too. For any vertex $v \in V(\overrightarrow{G})$ let us denote by $d^+(v)$ the *outdegree* of v, i.e., the number of vertices of $\overrightarrow{G}$ that are adjacent from v. By $d^-(v)$ we denote the *indegree* of v, i.e., the number of vertices adjacent to v. The *degree* of a vertex v, denoted by $d(v)$, is the sum $d(v) = d^-(v) + d^+(v)$. A digraph without directed cycles of length two is called an *oriented graph.* Replacing every arc (u, v) in an oriented graph $\overrightarrow{G}$ by an edge uv yields its *underlying graph.*

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A *tournament* is an oriented graph such that its underlying graph is complete. A transitive tournament of order n will be denoted by TT_n. As it is unique up to isomorphism, throughout the paper, we will view TT_n as shown in Figure 1. And we can denote the vertices in TT_n by consecutive integers in such way that if $i < j$, then (i, j) is an arc of TT_n. The vertices 1, 2 and n will be called the *first*, the *second* and the *last vertex* of TT_n, respectively. We define the *length of an arc* (i, j) as the difference $j - i$.

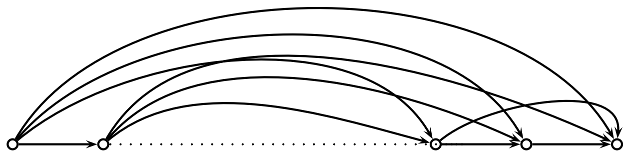

Figure 1: Transitive tournament TT_n

An (*oriented*) *path* between two distinct vertices u and v in an oriented graph $\overrightarrow{G}$ is a finite sequence

$$u = v_0, v_1, \ldots, v_{k-1}, v_k = v$$

of vertices, beginning with u and ending with v and edges $v_{i-1}v_i \in E(\overrightarrow{G})$ for $i \in \{1, ..., k\}$. A *semipath* between two distinct vertices u and v is a path between u and v in the underlying graph G.

A vertex $x \in V(\overrightarrow{G})$ is an *end-vertex* if its degree $d(x) = 1$. An arc beginning or ending in x we call an *end-arc*.

Let u and v be end-vertices. The arcs $u'u$, $v'v$ (or uu', vv') are called *independent* when $u' \neq v'$.

Let $\overrightarrow{G}(V, E)$ be an oriented graph of order n. An *embedding of* $\overrightarrow{G}$ *into* TT_n is a couple (σ, σ') in which σ is a bijection $V \to \{1, \ldots, n\} = V(TT_n)$ and σ' is an injection $E \to E(TT_n)$ induced by σ (i.e., for any edge $ij \in E$, $\sigma'(ij) = \sigma(i)\sigma(j)$). We will speak more simply of the embedding σ of $\overrightarrow{G}$. If $V(\overrightarrow{G}) = k < n$ we can also speak about an embedding of $\overrightarrow{G}$ by adding $(n - k)$ isolated points to $\overrightarrow{G}$ and we say that $\overrightarrow{G}$ is embeddable into TT_n if $\overrightarrow{G'} := \overrightarrow{G} \cup \{\text{isolated vertices}\}$ is embeddable.

A *k-packing* of k oriented graphs $\overrightarrow{G_1}$, $\overrightarrow{G_2}$, …, $\overrightarrow{G_k}$ of order n into TT_n is a k-tuple $(\sigma_1, \ldots, \sigma_k)$ in which σ_i is an embedding of $\overrightarrow{G_i}$ for $1 \leq i \leq k$ such that the k sets $\sigma_i'(E_i)$ are disjoint.

We say that $\overrightarrow{G}$ is *k-packable* into TT_n if a packing of k copies of $\overrightarrow{G}$ into TT_n exists.

There are many results concerning packing of graphs. The basic result was proved, independently, in [2], [3] and [6].

Theorem 1. *Let G, H be graphs of order n. If $|E(G)| \leq n-2$ and $|E(H)| \leq n-2$ then G and H are packable into K_n.*

B. Bollobás and S.E. Eldridge made the following conjecture.

Conjecture 2. *Let $G_1, G_2, \ldots, G_k$ be k graphs of order n. If $|E(G_i)| \leq n-k$, $i = 1, \ldots, k$, then $G_1, G_2, \ldots, G_k$ are packable into K_n.*

The case $k = 3$ of Conjecture 2 was proved by H. Kheddouci, S. Marshall, J.F. Saclé and M. Woźniak in [5].

If one restrains the study to the packing of three copies of the same graph, the hypothesis on size can slightly improved. The following theorem was proved in [7].

Theorem 3. *Let G be a graph of order n, $G \neq K_3 \cup 2K_1$, $G \neq K_4 \cup 4K_1$. If $|E(G)| \leq n-2$, then a 3-packing of G into K_n exists.*

The main result of this paper is similar to the basic result of Conjecture 2 for case $k = 3$ but for an acyclic digraph and its 3-packing into TT_n.

The motivation for us is the paper by A. Görlich, M. Pilśniak, M. Woźniak [4] where the existence of a 2-packing of $\overrightarrow{G}$ into TT_n was shown. More precisely, the following result was proved therein.

Theorem 4. *Let $\overrightarrow{G}$ be an acyclic digraph of order n such that $|E(\overrightarrow{G})| \leq \frac{3(n-1)}{4}$. Then $\overrightarrow{G}$ is 2-packable into TT_n.*

The basic references of studies addressing packing problems can be found in [1, 8, 9, 10].

2. Some Lemmas

Before starting the proof of the main theorem we need some preliminary lemmas.

Lemma 5. *Let $\overrightarrow{G}$ be a digraph isomorphic to a path of length k. If $k = \lfloor \frac{2}{3}n - 1 \rfloor$, then $\overrightarrow{G}$ is 3-packable into TT_n.*

Proof. Notice that for $n \leq 3$ the length of a path is zero or one and it is clear that it is 3-packable into TT_n.

We use induction on the order of the transitive tournament. For $n = 4$ the length of a path $\overrightarrow{P}$ is one, let $\overrightarrow{P} = v_0, v_1$. We can define its embedding $\sigma_1(v_0) = 1$ and $\sigma_1(v_1) = 4$ in TT_4 and the embeddings σ_2 and σ_3 as follows: $\sigma_2(v_0) = 2$, $\sigma_3(v_0) = 3$ and $\sigma_2(v_1) = 3$, $\sigma_3(v_0) = 4$.

For $n = 5$ the length of a path $\overrightarrow{P}$ is two, let $\overrightarrow{P} = v_0, v_1, v_2$. We can define its embedding $\sigma_1(v_0) = 3$, $\sigma_1(v_1) = 4$ and $\sigma_1(v_2) = 5$ in TT_5 and the embeddings σ_2 and σ_3 as follows: $\sigma_2(v_0) = \sigma_3(v_0) = 1$, and $\sigma_2(v_1) = 2$, $\sigma_3(v_1) = 3$, and $\sigma_2(v_2) = 4$, $\sigma_3(v_2) = 5$.

For $n = 6$ the length of a path $\overrightarrow{P}$ is three, let $\overrightarrow{P} = v_0, v_1, v_2, v_3$. We can define its embedding $\sigma_1(v_0) = 1$, $\sigma_1(v_1) = 4$, $\sigma_1(v_2) = 5$ and $\sigma_1(v_3) = 6$ in TT_6 and the embeddings σ_2 and σ_3 as follows: $\sigma_2(v_0) = \sigma_3(v_0) = 1$ $\sigma_2(v_1) = 2$, $\sigma_3(v_1) = 3$, and $\sigma_2(v_2) = 3$, $\sigma_3(v_2) = 4$, and $\sigma_2(v_3) = 5$, $\sigma_3(v_3) = 6$.

Now, let $n \geq 7$ and we assume that our result is true for all $n' < n$. Let $v_0, \ldots, v_k$ be the path $\overrightarrow{P}$ of length $k = \lfloor \frac{2}{3}n - 1 \rfloor$ in TT_n. By induction, there exist the embeddings σ'_1, σ'_2 and σ'_3 of path $v_0, \ldots, v_{k-2}$ into TT_{n-3}. Moreover, we can assume that vertices $\sigma'_1(v_{k-2}) = \sigma'_3(v_{k-2}) = n - 3$ and the number of $\sigma'_2(v_{k-2})$ in TT_{n-3} is less than $n - 3$. Now we add three vertices to TT_{n-3} at the end. Two vertices v_{k-1}, v_k of the path obtain the numbers: $n - 1$ and n, so $\sigma_1(v_{k-1}) = n - 1$, $\sigma_1(v_k) = n$. We define the embeddings σ_2 and σ_3 in TT_n as follows: $\sigma_2(v_{k-1}) = \sigma_3(v_{k-1}) = n - 2$ and $\sigma_2(v_k) = n - 1$, $\sigma_3(v_k) = n$, and $\sigma_1(v_i) = \sigma'_1(v_i)$, $\sigma_2(v_i) = \sigma'_2(v_i)$, $\sigma_3(v_i) = \sigma'_3(v_i)$ for $i \in \{0, ..., k - 2\}$.

Thus, by induction, the proof is complete. ∎

The following result may be proved in a similar way as Lemma 4.15 in [8].

Lemma 6. *Let $\overrightarrow{G}$ be an acyclic digraph of order n. Suppose that*

(a) $x'x$, $y'y$, $z'z$, *or*

(b) xx', yy', zz'

are three independent end-arcs in $E(\overrightarrow{G})$. If $\overrightarrow{H} := \overrightarrow{G} - \{x, y, z\}$ is 3-packable into TT_{n-3}, then $\overrightarrow{G}$ is 3-packable into TT_n.

Lemma 7. *Let $\overrightarrow{G}$ be an acyclic digraph of order n. Suppose that z is an isolated vertex and*

(c) $\quad x'x,\ y'y,$ *or*

(d) $\quad xx',\ yy'$

are two independent end-arcs in $E(\overrightarrow{G})$. *If* $\overrightarrow{H} := \overrightarrow{G} - \{x, y, z\}$ *is* 3*-packable into* TT_{n-3}, *then* $\overrightarrow{G}$ *is* 3*-packable into* TT_n.

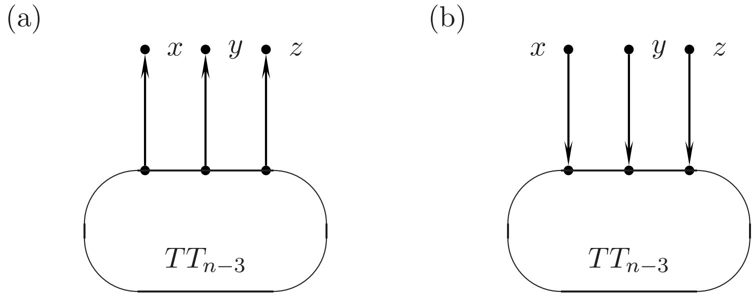

Figure 2. Two cases from Lemma 6

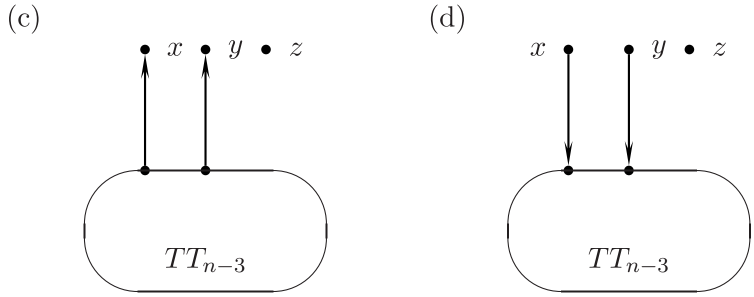

Figure 3. Two cases from Lemma 7

Proof. This lemma follows immediately from Lemma 6, (see Figure 2 and Figure 3). ∎

Lemma 8. *Let* $\overrightarrow{G}$ *be an acyclic digraph of order* n. *Suppose that* y, z *are two isolated vertices and* x *is a vertex such that*

(e) $\quad d^-(x) \geq 2,\ d^+(x) = 0,$ *or*

(f) $\quad d^+(x) \geq 2,\ d^-(x) = 0.$

If $\overrightarrow{H} := \overrightarrow{G} - \{x, y, z\}$ *is* 3*-packable into* TT_{n-3}, *then* $\overrightarrow{G}$ *is* 3*-packable into* TT_n.

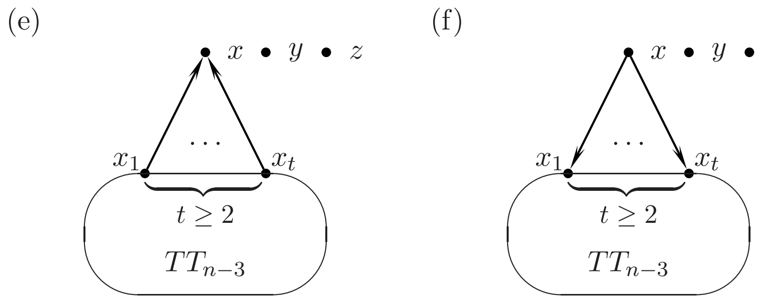

Figure 4. Two cases from Lemma 8

Proof. Without loss of generality we can consider only the case (e).
By assumption there exist arc disjoint embeddings σ_1', σ_2' and σ_3' of $\overrightarrow{H}$ into TT_{n-3}. Add three vertices to TT_{n-3} at the end and we obtain the transitive tournament TT_n.

Now, we define the embeddings of $\overrightarrow{G}$: $\sigma_1(v) = \sigma_1'(v)$, $\sigma_2(v) = \sigma_2'(v)$, $\sigma_3(v) = \sigma_3'(v)$ for all vertices of $\overrightarrow{H}$, and $\sigma_1(x) = n-2$, $\sigma_2(x) = n-1$, $\sigma_3(x) = n$. This is the correct 3-packing of $\overrightarrow{G}$ into TT_n, which completes the proof. ■

Lemma 9. *Let $\overrightarrow{G}$ be an acyclic digraph of order n. Suppose that y, z are two isolated vertices in $\overrightarrow{G}$, the end-vertices $x_1, \ldots, x_k$ are adjacent to a vertex x, which is such that $d^+(x) = t \geq 1$, $d^-(x) = k \geq 2$ and $k + t \geq 4$.*

If $\overrightarrow{H} := \overrightarrow{G} - \{x, y, z, x_1, \ldots, x_k\}$ is 3-packable into TT_{n-3-k}, then $\overrightarrow{G}$ is 3-packable into TT_n.

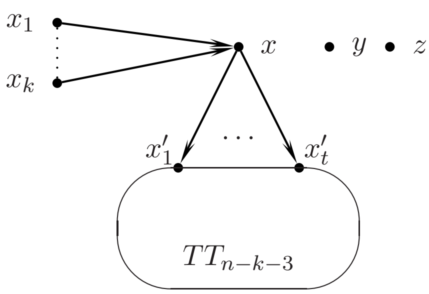

Figure 5. The case from Lemma 9

Proof. Let us imagine a transitive tournament TT_{n-3-k} with the vertices numbered from $k+4$ to n. Let us assume that embeddings σ_1', σ_2' and σ_3' of

$\overrightarrow{H}$ exist in TT_{n-3-k}. Let us add $k+3$ vertices to TT_{n-3-k} at the beginning and we obtain the transitive tournament TT_n.

Now, we define the embeddings σ_1, σ_2 and σ_3 of $\overrightarrow{G}$ into TT_n as follows: $\sigma_1(x_i) = \sigma_2(x_i) = \sigma_3(x_i) = i$ for $i \in \{1, \ldots, k\}$, $\sigma_1(x) = k+1$, $\sigma_2(x) = k+2$, $\sigma_3(x) = k+3$, and $\sigma_1(v) = \sigma_1'(v)$, $\sigma_2(v) = \sigma_2'(v)$, $\sigma_3(v) = \sigma_3'(v)$ for all the remaining vertices. We obtain a 3-packing of $\overrightarrow{G}$. ■

Lemma 10. *Let $\overrightarrow{G}$ be an acyclic digraph of order n. Suppose that x, y are two isolated vertices in $\overrightarrow{G}$, a, b are two end-vertices adjacent to a vertex c. Let d be a vertex adjacent from c such that $d^-(c) = 2$, $d^+(c) = 1$, $d^-(d) = 1$, $d^+(d) \geq 1$. If $\overrightarrow{H} := \overrightarrow{G} - \{x, y, a, b, c, d\}$ is 3-packable into TT_{n-6}, then $\overrightarrow{G}$ is 3-packable into TT_n.*

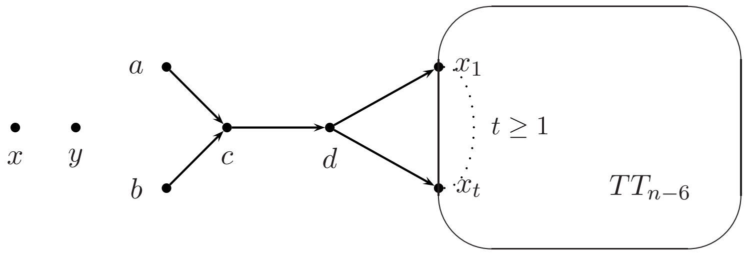

Figure 6. The case from Lemma 10

Proof. Let us imagine a transitive tournament TT_{n-6} with the vertices numbered from 7 to n. Let us assume that embeddings σ_1', σ_2' and σ_3' of $\overrightarrow{H}$ exist in TT_{n-6}. Let us add the vertices a, b, c, d, x, y to TT_{n-6} at the beginning and we obtain a transitive tournament TT_n.

We can define the embedding σ_1 of $\overrightarrow{G}$ into TT_n as follows: $\sigma_1(a) = 1$, $\sigma_1(b) = 2$, $\sigma_1(c) = 3$, $\sigma_1(d) = 4$, $\sigma_1(x) = 5$, $\sigma_1(y) = 6$ and $\sigma_1(v) = \sigma_1'(v)$ for all the remaining vertices. Now, we define the embeddings σ_2 and σ_3 of $\overrightarrow{G}$ into TT_n as follows: $\sigma_2(a) = \sigma_3(a) = 1$, $\sigma_2(b) = \sigma_3(b) = 2$, $\sigma_2(c) = 4$ and $\sigma_3(c) = 5$, $\sigma_2(d) = 5$ and $\sigma_3(d) = 6$ and $\sigma_2(v) = \sigma_2'(v)$, $\sigma_3(v) = \sigma_3'(v)$ for all the remaining vertices. So a 3-packing of $\overrightarrow{G}$ into TT_n exists. ■

Lemma 11. *Let $\overrightarrow{G}$ be an acyclic digraph of order n. Suppose that a_k $(k > 1)$ is a vertex in $\overrightarrow{G}$ such that a path of length $k-1$ from a_1 to a_k exists and $d^+(a_k) \geq 2$. Moreover, suppose that $y_1, \ldots, y_{k'}$ $(k' = \lfloor \frac{k+3}{2} \rfloor)$ are isolated vertices in $\overrightarrow{G}$.*

If $\overrightarrow{H} := \overrightarrow{G} - \{y_1, \ldots, y_{k'}, a_1, \ldots, a_k\}$ *is* 3*-packable into* $TT_{n-k-k'}$, *then* $\overrightarrow{G}$ *is* 3*-packable into* TT_n.

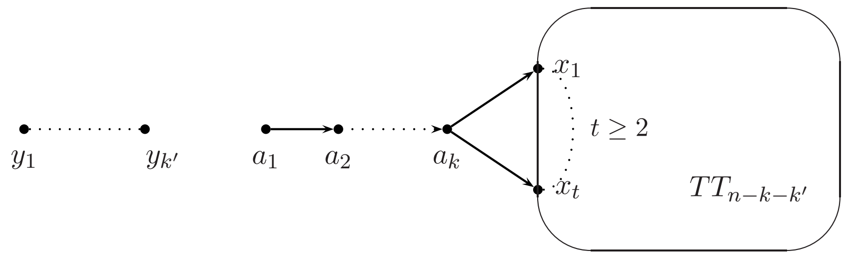

Figure 7. The case from Lemma 11

Proof. Let us imagine a transitive tournament $TT_{n-k-k'}$ with the vertices numbered from $k+k'+1$ to n. Let us assume that there are embeddings σ_1', σ_2' and σ_3' of $\overrightarrow{H}$ into $TT_{n-k-k'}$. Let us add $k+k'$ vertices to $TT_{n-k-k'}$ at the beginning and we obtain the transitive tournament TT_n.

In Lemma 5 we show that the path of length $k-1$ is 3-packable into $TT_{\lfloor \frac{3}{2}k+\frac{1}{2} \rfloor}$. So there are embeddings σ_1'', σ_2'' and σ_3'' of this path into $TT_{k+k'-1}$. Now we extend the embeddings σ_1'', σ_2'' and σ_3'' to embeddings σ_1^*, σ_2^* and σ_3^* into $TT_{k+k'}$ with the last isolated vertex added. We will modify these embeddings if necessary so that $\sigma_1^*(a_k) \neq \sigma_2^*(a_k) \neq \sigma_3^*(a_k)$.

We consider three cases:

1. In the case of two embedings of a path, the vertex a_k is embedded in the same vertex of $TT_{k+k'-1}$, for example $\sigma_1''(a_k) \neq \sigma_2''(a_k) = \sigma_3''(a_k)$,
2. In the case of three embeddings of a path, the vertex a_k is embedded in the same vertex of $TT_{k+k'-1}$ but not in the last, say $\sigma_1''(a_k) = \sigma_2''(a_k) = \sigma_3''(a_k) = i < k+k'-1$,
3. In the case of three embeddings of a path, the vertex a_k is embedded in the last vertex of $TT_{k+k'-1}$.

In the first case we may choose for $\sigma_2^*(a_k)$ the last vertex of $TT_{k+k'}$.

In the second case we may choose $\sigma_2^*(a_k) = k+k'-1$ and $\sigma_3^*(a_k) = k+k'$.

In the third case we must have $\sigma_1''(a_{k-1}) > \sigma_2''(a_{k-1}) > \sigma_3''(a_{k-1})$. If $\sigma_1''(a_k) - \sigma_1''(a_{k-1}) > 1$, then we may assume $\sigma_1^*(a_k)$ is in the $k+k'-2$ vertex, and $\sigma_2^*(a_k) = k+k'$. If $\sigma_1''(a_k) - \sigma_1''(a_{k-1}) = 1$ (in $TT_{k+k'-1}$), then either we may assume $\sigma_2^*(a_k)$ is in the $k+k'-2$ vertex or we may assume $\sigma_3^*(a_k)$ is in the $k+k'-2$ vertex and the other one in $k+k'$ vertex.

Now $\sigma_1^*(a_k) \neq \sigma_2^*(a_k) \neq \sigma_3^*(a_k)$ and we can define the embeddings σ_1, σ_2 and σ_3 of $\overrightarrow{G}$ into TT_n as follows: $\sigma_1(a_i) = \sigma_1^*(a_i)$, $\sigma_2(a_i) = \sigma_2^*(a_i)$, $\sigma_3(a_i) = \sigma_3^*(a_i)$ for all $i \in \{1, \ldots, k\}$, $\sigma_1(y_j) = \sigma_1^*(y_j)$, $\sigma_2(y_j) = \sigma_2^*(y_j)$, $\sigma_3(y_j) = \sigma_3^*(y_j)$ for all $j \in \{1, \ldots, k'\}$ and $\sigma_1(v) = \sigma_1'(v)$, $\sigma_2(v) = \sigma_2'(v)$, $\sigma_3(v) = \sigma_3'(v)$ for all the remaining vertices. ■

3. The Main Result

In this section, we consider the existence of a 3-packing of $\overrightarrow{G}$ into TT_n and we prove the following theorem.

Theorem 12. *Let $\overrightarrow{G}$ be an acyclic digraph of order n such that $|E(\overrightarrow{G})| \leq \frac{2}{3}n - 1$. Then $\overrightarrow{G}$ is 3-packable into TT_n.*

3..1 The bound in Theorem 12 is the best possible

First, we show that the size condition in Theorem 12 cannot be weakened.

Let us consider a path of length k and suppose that a 3-packing of such a path into TT_n exists, where $n > k$. It means that $\overrightarrow{G}$, $\overrightarrow{G}'$ and $\overrightarrow{G}''$ are three arc disjoint subgraphs of the transitive tournament TT_n isomorphic to such a path. Let k_1, k_1' and k_1'' denote the number of arcs of length one in $\overrightarrow{G}$, $\overrightarrow{G'}$ and $\overrightarrow{G''}$, k_2, k_2' and k_2'' denote the number of arcs of length two and k_3, k_3' and k_3'' denote the number of arcs of length greater than two, respectively. Thus

$$(*) \qquad \left.\begin{aligned} k_1 + k_2 + k_3 &= k, \\ k_1' + k_2' + k_3' &= k, \\ k_1'' + k_2'' + k_3'' &= k. \end{aligned}\right\}$$

Since $\overrightarrow{G}$, $\overrightarrow{G'}$ and $\overrightarrow{G''}$ are subgraphs of TT_n, we have

$$k_1 + 2k_2 + 3k_3 \leq n - 1,$$

$$k_1' + 2k_2' + 3k_3' \leq n - 1,$$

$$k_1'' + 2k_2'' + 3k_3'' \leq n - 1.$$

By adding the last three inequalities we get

$$k_1 + k_1' + k_1'' + 2k_2 + 2k_2' + 2k_2'' + 3k_3 + 3k_3' + 3k_3'' \leq 3n - 3.$$

But on the other hand, since $\overrightarrow{G}$, $\overrightarrow{G'}$ and $\overrightarrow{G''}$ are arc disjoint and the total number of arcs of length 1 in TT_n is equal to $(n-1)$, we have:

$$2(k_1 + k_1' + k_1'') \leq 2(n-1)$$

and since the total number of arcs of length 2 in TT_n is equal to $(n-2)$, we have:

$$k_2 + k_2' + k_2'' \leq n - 2.$$

By adding these three inequalities and using $(*)$ we get

$$9k \leq 6n - 7.$$

Finally, we obtain

$$k \leq \frac{2}{3}n - 1.$$

3..2 Proof of Theorem 12

At the beginning, we can notice that for $n \leq 4$ an oriented graph satisfying the assumption of Theorem 12 has zero or one arc and, obviously, is 3-packable into TT_n. For $n = 5$ an oriented graph satisfying the assumption of Theorem 12 has at most two arcs and it is also easily seen that it is 3-packable.

Now, let us assume that $\overrightarrow{G}$ is a counterexample of Theorem 12 for minimum possible $n \geq 6$.

Let us notice that for $6 \leq n \leq 9$, if $\overrightarrow{G}$ does not have any isolated vertex and has, of course, at most $\frac{2}{3}n - 1$ edges, then $\overrightarrow{G}$ has only tree-components and at least three of them are isolated arcs. So by Lemma 6, we get a contradiction with the minimality of $\overrightarrow{G}$.

As above, if $\overrightarrow{G}$ (for $6 \leq n \leq 9$) has only one isolated vertex, then $\overrightarrow{G}$ has at least two isolated arcs (for $7 \leq n \leq 9$) or one isolated arc and one end-arc ($n = 6$). So by Lemma 7, we get a contradiction with the minimality of $\overrightarrow{G}$. Hence in the next part of the proof we can assume that for $n \leq 9$ $\overrightarrow{G}$ has at least two isolated vertices.

It is obvious that every oriented graph $\overrightarrow{G}$, for $n \geq 10$ which satisfies the conditions of Theorem 12 is not connected and at least $\lceil \frac{n}{3} + \frac{7}{9} \rceil$ of its components are oriented trees (including, the isolated points as trivial oriented trees). If in $\overrightarrow{G}$ there are more than four non-trivial oriented trees as its components, then $\overrightarrow{G}$ has at least five independent end-vertices. So three of them have to be such as in case (a) or (b) in Lemma 6. We get a contradiction with the minimality of $\overrightarrow{G}$. Hence $\overrightarrow{G}$ has at most four components being non-trivial oriented trees and at least $\lceil \frac{n}{3} + \frac{7}{9} \rceil$ of its components are oriented trees. For order $n \geq 10$ we obtain an isolated point in $\overrightarrow{G}$.

Now, if in $\overrightarrow{G}$ there are more than two non-trivial oriented trees as its components, then $\overrightarrow{G}$ has at least three independent end-vertices. So two of them have to be such as in case (c) or (d) in Lemma 7 and since in $\overrightarrow{G}$ there is an isolated vertex, we get a contradiction with the minimality of $\overrightarrow{G}$.

Hence from this moment in the proof (for order $n \geq 6$) $\overrightarrow{G}$ has at most two components being non-trivial oriented trees and at least $\max\{2, \lceil \frac{n}{3} - \frac{11}{9} \rceil\}$ of its components are isolated vertices.

Let $\overrightarrow{H}$ be a non-trivial connected component of $\overrightarrow{G}$ of the greatest order. Let a vertex $x \in V(\overrightarrow{H})$ be such that $d^-(x) = 0$. It is easily seen that there is not more than one vertex adjacent from x, since if there is more than one, then $\overrightarrow{G}$ satisfies the assumptions of Lemma 8 and it leads to a contradiction with the minimality of $\overrightarrow{G}$.

It means that $d^+(x) = 1$. If y is a neighbour of x, $\overrightarrow{G}$ satisfies one of the following properties:

1. $d^-(y) \geq 3$;
2. $d^-(y) = 2$ and $d^+(y) \geq 2$;
3. $d^-(y) = 2$ and $d^+(y) \leq 1$;
4. there is a path $(a_1 = x, a_2 = y, \ldots, a_k)$, $k \geq 2$ and $d^+(a_k) \geq 2$;
5. $\overrightarrow{G}$ is an oriented path.

It is easily seen that in the first, the second and the third case we may assume that all vertices adjacent to y are end-vertices. If not, in the graph $\overrightarrow{G}$ either there are two end-vertices like in Lemma 7 or there is a vertex with indegree zero and outdegree greater than or equal to 2, hence it satisfies the assumptions of Lemma 8. In both the cases we obtain a contradiction with the minimality of $\overrightarrow{G}$.

Case 1. It is obvious that in this case such a graph is 3-packable since either $d^+(y) = 0$ and it satisfies the assumptions of Lemma 8 or $d^+(y) > 0$ and the assumptions of Lemma 9.

Case 2. Such a graph is 3-packable since it satisfies the assumptions of Lemma 9.

Case 3. As in the first case, if $d^+(y) = 0$, it satisfies the assumptions of Lemma 8.

Let $d^+(y) = 1$ and z be a vertex adjacent from y. If $d(z) = 1$, assume first that $\overrightarrow{H}$ is a not unique non-trivial component of $\overrightarrow{G}$. In the second non-trivial component $\overrightarrow{K}$ of $\overrightarrow{G}$ there is a vertex $v \in V(\overrightarrow{K})$ such that $d^-(v) = 0$. For the same reason as before the outdegree of v must be equal to 1. And then there are two end-arcs: one ending in x and the other ending in v, so by Lemma 7 $\overrightarrow{G}$ is 3-packable, which contradicts the minimality of $\overrightarrow{G}$. Hence in this case $\overrightarrow{G}$ has a unique non-trivial component $\overrightarrow{H}$. So $\overrightarrow{H}$ has three arcs and in $\overrightarrow{G}$, which satisfies the assumption of Theorem 12, there are two isolated vertices. Three copies of such a graph can be packed in the same way as in the proof of Lemma 10, but $\overrightarrow{G}$ is not 3-packable, so $d^-(z) > 1$.

If $d^-(z) > 1$, then two end-vertices, like in Lemma 7, exist in the graph $\overrightarrow{G}$ and $\overrightarrow{G}$ is 3-packable. If $d^-(z) = 1$ and $d^+(z) \geq 1$ such a graph is 3-packable since it satisfies the assumptions of Lemma 10.

Case 4. We may observe that if $d^-(a_i) > 1$, for any $i > 2$, in the graph $\overrightarrow{G}$ either there are two end-vertices like in Lemma 7 or there is a vertex with indegree zero and outdegree greater than or equal to 2, hence it satisfies the assumptions of Lemma 8. In both the cases we obtain a contradiction with the minimality of $\overrightarrow{G}$.

It is obvious that in the fourth case such a graph is 3-packable since it satisfies the assumptions of Lemma 11.

Case 5. Such a graph is 3-packable since it satisfies the assumptions of Lemma 5.

Therefore the set of counterexamples is empty and the proof of Theorem 12 is complete. ∎

4. A Conjecture — m-Packable into TT_n

Finally we can make a general conjecture.

Conjecture 13. *Let $\overrightarrow{G}$ be an acyclic digraph of order n such that $|E(\overrightarrow{G})| \leq \frac{m+1}{2m}n - \frac{m^2+5}{6m}$. Then $\overrightarrow{G}$ is m-packable into TT_n.*

We show only that the size condition in Theorem 13 cannot be weakened. Let us consider a path of length k. Then we suppose that there is an

m-embedding of such a path into TT_n, where $n > k$. It means that $\overrightarrow{G}_1$, $\overrightarrow{G}_2, \ldots, \overrightarrow{G}_m$ are m arc disjoint subgraphs of the transitive tournament TT_n isomorphic to such a path. Let for $1 \le i \le m-1$, $k_1^i, k_2^i, \ldots, k_m^i$ denote the numbers of arcs in $\overrightarrow{G}_1, \overrightarrow{G}_2, \ldots, \overrightarrow{G}_m$ of length i in TT_n and $k_1^m, k_2^m, \ldots, k_m^m$ denote the number of arcs in $\overrightarrow{G}_1, \overrightarrow{G}_2, \ldots, \overrightarrow{G}_m$ of length greater than $m-1$, respectively. Thus

$$(*) \qquad \left.\begin{aligned} k_1^1 + k_1^2 + \ldots + k_1^m &= k, \\ k_2^1 + k_2^2 + \ldots + k_2^m &= k, \\ &\ldots \\ k_m^1 + k_m^2 + \ldots + k_m^m &= k. \end{aligned}\right\}$$

Since $\overrightarrow{G}_1, \overrightarrow{G}_2, \ldots, \overrightarrow{G}_m$ are subgraphs of TT_n we have for each $\overrightarrow{G}_m$

$$k_i^1 + 2k_i^2 + \ldots + mk_i^m \le n-1.$$

By adding those inequalities we get

$$\sum_{i=1}^{m} k_i^1 + 2\sum_{i=1}^{m} k_i^2 + \ldots + m\sum_{i=1}^{m} k_i^m \le mn - m.$$

But on the other hand, since $\overrightarrow{G}_1, \overrightarrow{G}_2, \ldots, \overrightarrow{G}_m$ are arc disjoint and the total number of arcs of length 1 is equal to $n-1$ we have:

$$(m-1)\sum_{i=1}^{m} k_i^1 \le (m-1)(n-1),$$

Since the total number of arcs of length 2 is equal to $n-2$ we have:

$$(m-2)\sum_{i=1}^{m} k_i^2 \le (m-2)(n-2)$$

and similar inequalities, up to

$$\ldots$$

$$\sum_{i=1}^{m} k_i^{m-1} \le (n-m+1).$$

By adding these inequalities and using $(*)$ we obtain

$$m^2k \leq (m{+}m{-}1{+}m{-}2{+}\ldots{+}1)n-(m{+}1(m{-}1){+}2(m{-}2){+}\ldots{+}(m{-}1)1)$$

hence finally

$$k \leq \frac{m+1}{2m}n - \frac{m^2+5}{6m}.$$

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