ON CHOOSABILITY OF COMPLETE MULTIPARTITE GRAPHS $K_{4,3*t,2*(k-2t-2),1*(t+1)}$*

Guo-Ping Zheng, Yu-Fa Shen†, Zuo-Li Chen, Jin-Feng Lv

School of Mathematics and Information Science and Technology
Hebei Normal University of Science and Technology
Qinhuangdao 066004, P.R. China
and
Center for Mathematics of Hebei Province
Hebei Normal University
Shijiazhuang 050016, P.R. China

e-mail: zhengguoping9199@126.com

Abstract

A graph G is said to be chromatic-choosable if $\mathrm{ch}(G) = \chi(G)$. Ohba has conjectured that every graph G with $2\chi(G) + 1$ or fewer vertices is chromatic-choosable. It is clear that Ohba's conjecture is true if and only if it is true for complete multipartite graphs. In this paper we show that Ohba's conjecture is true for complete multipartite graphs $K_{4,3*t,2*(k-2t-2),1*(t+1)}$ for all integers $t \geq 1$ and $k \geq 2t+2$, that is, $\mathrm{ch}(K_{4,3*t,2*(k-2t-2),1*(t+1)}) = k$, which extends the results $\mathrm{ch}(K_{4,3,2*(k-4),1*2}) = k$ given by Shen *et al.* (Discrete Math. 308 (2008) 136–143), and $\mathrm{ch}(K_{4,3*2,2*(k-6),1*3}) = k$ given by He *et al.* (Discrete Math. 308 (2008) 5871-5877).

Keywords: list coloring, complete multipartite graphs, chromatic-choosable graphs, Ohba's conjecture.

2010 Mathematics Subject Classifications: 05C15.

1. Introduction

The concept of *list coloring* was introduced independently by Vizing [13], and by Erdős, Rubin and Taylor [2]. For a graph $G = (V, E)$ and each vertex $u \in V(G)$, let $L(u)$ denote a set (or a list) of colors available for u; then $L = \{L(u)|u \in V(G)\}$ is said to be a *list assignment* of G. If $|L(u)| = k$ for all $u \in V(G)$, then we say that L is a *k-list assignment* of G. An *L-coloring* is a vertex-coloring c such that: $c(u) \neq c(v)$ for every $uv \in E(G)$, and $c(u) \in L(u)$ for every $u \in V(G)$. A graph G is *L-colorable* if G admits an *L*-coloring. A graph G is *k-choosable* if G is *L*-colorable for every k-list assignment L. The *choice number* $\mathrm{ch}(G)$ of a graph G is the smallest k such that G is k-choosable. A graph G is called *chromatic-choosable* [6], if $\mathrm{ch}(G) = \chi(G)$. For the chromatic-choosable graphs, there are many results and conjectures (see [14]). The following glamorous conjecture is due to Ohba.

Conjecture 1.1 (Ohba [6])**.** If $|V(G)| \leq 2\chi(G) + 1$, then $\mathrm{ch}(G) = \chi(G)$.

It seems that verifying Conjecture 1.1 is not easy for all graphs. As a general situation, Reed and Sudakov [8] proved the following weaker version of this conjecture.

Theorem 1.1 (Reed and Sudakov [8])**.** *If* $|V(G)| \leq \frac{5}{3}\chi(G) - \frac{4}{3}$, *then* $\mathrm{ch}(G) = \chi(G)$.

Because every χ-chromatic graph is a subgraph of a complete χ-partite graph, Ohba's conjecture is true if and only if it is true for complete multipartite graphs. Moreover, if a complete k-partite graph is chromatic-choosable, then all k-chromatic subgraphs of G are chromatic-choosable. Thus Conjecture 1.1 is equivalent to the following conjecture.

Conjecture 1.2. If G is a complete k-partite graph with $|V(G)| = 2k + 1$, then $\mathrm{ch}(G) = \chi(G) = k$.

At present, for some special classes of complete multipartite graphs, Conjecture 1.2 have been verified (see [1, 3, 4, 7, 9, 10, 11]). We denote by K_{l*r} the complete r-partite graph with l vertices in each part, and denote by $K_{l*r,m*s,n*t,...}$ the complete $(r + s + t + \cdots)$-partite graph $K_{l*r} \vee K_{m*s} \vee K_{n*t} \vee \cdots$, where $\vee$ denotes 'join'. We need the following results from [4, 10].

Theorem 1.2 (Shen *et al.* [10]). *For every integer $k \geq 4$,* $\mathrm{ch}(K_{4,3,2*(k-4),1*2}) = k$.

Theorem 1.3 (He *et al.* [4]). *For every integer $k \geq 6$,* $\mathrm{ch}(K_{4,3*2,2*(k-6),1*3}) = k$.

In this paper, we extend the results of Theorem 1.2 and Theorem 1.3 to the more general graphs $K_{4,3*t,2*(k-2t-2),1*(t+1)}$ for all integers $t \geq 0$ and $k \geq 2t + 2$. Namely, we show that $\mathrm{ch}(K_{4,3*t,2*(k-2t-2),1*(t+1)}) = k$ for all integers $t \geq 0$ and $k \geq 2t + 2$. We will prove our main result in Section 3. In Section 2, we state some lemmas as a preparation for proving our main result.

2. SOME LEMMAS

For a graph $G = (V, E)$ and a subset $W \subset V$, let $G[W]$ denote the subgraph of G induced by W. For a list assignment L of G, let $L|_W$ denote L restricted to W, and $L(W)$ denote the union $\bigcup_{u \in W} L(u)$. If A is a set of colors, let $L \backslash A$ denote the list assignment obtained from L by removing the colors in A from each $L(u)$ with $u \in V(G)$. When A consists of a single color a, we write $L - a$ instead of $L \backslash \{a\}$. We say that G with L satisfies *Hall's condition* in G, if $|L(W)| \geq |W|$ for every subset $W \subset V(G)$. It is clear that if G with L satisfies Hall's condition, then by Hall's marriage theorem, there exists an L-coloring for G in which all vertices receive distinct colors.

In [5], Kierstead proved the following lemma (our statement is stronger than Kierstead's, but the proof is identical).

Lemma 2.1 (Kierstead [5]). *Let L be a list assignment for a graph $G = (V, E)$. Then G is L-colorable if $G[W]$ is $L|_W$-colorable for some maximal non-empty subset $W \subseteq V(G)$ such that $|L(W)| < |W|$.*

From Lemma 2.1, Kierstead obtained a corollary as follows.

Corollary 2.1 (Kierstead [5]). *A graph $G = (V, E)$ is k-choosable if G is L-colorable for every k-list assignment L such that $|\bigcup_{u \in V} L(u)| < |V|$.*

Corollary 2.1 is only stated for k-choosability, where every vertex has a list of the same size k. By a similar method, in [9] we extended k-choosability to f-*choosability* (see [2,12]), and obtained a more general version of Corollary 2.1, which can be applied even when different vertices may have lists of

different sizes. Furthermore, by the more general version of Corollary 2.1, we obtained a lemma in [9] as follows. For brevity, we denote by $[t]$ the set $\{1, 2, \ldots, t\}$ for an integer $t \geq 1$.

Lemma 2.2 (Shen *et al.* [9])**.** *Let* $G = K_{3*t, 1*(1+t)}$ $(t \geq 0)$ *with* $2t + 1$ *parts:* $V_i = \{x_i, y_i, z_i\}$ *for* $i \in [t]$, *and* $V_i = \{z_i\}$ *for* $i \in [2t + 1]\backslash[t]$. *If* L *is a list assignment of* G *such that in* $\{L(x_i), L(y_i), L(z_i)\}$ *there are two lists both with size* $2t$ *and the third one with size* $2t + 1$ *for each* $i \in [t]$, $|L(z_{t+1})| = 2t + 1$, *and* $|L(z_i)| = 2t$ *for each* $i \in [2t + 1]\backslash[t + 1]$, *then* G *is* L-colorable.

3.   Ohba's Conjecture is True for Graphs $K_{4,3*t,2*(k-2t-2),1*(t+1)}$

In order to prove that $\mathrm{ch}(K_{4,3*t,2*(k-2t-2),1*(t+1)}) = \chi(K_{4,3*t,2*(k-2t-2),1*(t+1)}) = k$ $(t \geq 0, k \geq 2t + 2)$ by induction, we show that $\mathrm{ch}(K_{4,3*t,1*(t+1)}) = \chi(K_{4,3*t,1*(t+1)}) = 2t + 2$ first.

Theorem 3.1. *For each integer* $t \geq 0$, $\mathrm{ch}(K_{4,3*t,1*(t+1)}) = 2t + 2$.

Proof. For $G = K_{4,3*t,1*(t+1)}$, denote its k parts as $V_1 = \{x_1, y_1, z_1, w_1\}$, $V_i = \{x_i, y_i, z_i\}$ for $i \in [t+1] \backslash [1]$, $V_i = \{z_i\}$ for $i \in [2t+2] \backslash [t+1]$. Let L be a $(2t + 2)$-list assignment of G. We will prove by induction on t that G is L-colorable.

The case where $t = 0$ is trivial. If $t = 1$ then Theorem 3.1 holds by Theorem 1.2. So we may assume $t \geq 2$ and suppose that Theorem 3.1 is true for smaller values of t. If there exists $i \in [t+1]$ such that $\bigcap_{u \in V_i} L(u) \neq \emptyset$, then we choose a color $c_1 \in \bigcap_{u \in V_i} L(u)$ to color all the vertices in V_i, and a different color $c_2 \in L(z_{2t+2})$ to color the vertex z_{2t+2}. Let $G' = G - V_i - z_{2t+2}$ and $L' = L - c_1 - c_2$. Clearly, G' is a subgraph of $K_{4,3*(t-1),1*t}$ and $|L'(u)| \geq 2t$ for each $u \in V(G')$. Thus, we can finish the proof applying the induction hypothesis. So we suppose that

$$(1) \qquad\qquad \bigcap_{u \in V_i} L(u) = \emptyset \ \text{ for all } \ i \in [t+1].$$

Case 1. There exist three vertices in V_1, say x_1, y_1, z_1, such that $L(x_1) \cap L(y_1) \cap L(z_1) \neq \emptyset$.

We choose a color $c_1 \in L(x_1) \cap L(y_1) \cap L(z_1)$ to color all the vertices x_1, y_1, z_1, and a different color $c_2 \in L(z_{2t+2})$ to color the vertex z_{2t+2}. Let

$G' = G - x_1 - y_1 - z_1 - z_{2t+2}$ and $L' = L - c_1 - c_2$. Clearly, $|L'(u)| \geq 2t$ for every $u \in V(G')$. By (1), $|L'(w_1)| \geq 2t + 1$, and for each $i \in [t + 1] \setminus [1]$ at least one of the sets $L'(x_i)$, $L'(y_i)$, $L'(z_i)$ contain at least $2t + 1$ colors, without loss of generality, say $|L'(z_i)| \geq 2t + 1$ for all such i. Therefore $G' = K_{3*t,1*(t+1)}$ and L' satisfies requirements of Lemma 2.2. Thus G' is L'-colorable by Lemma 2.2, and hence G is L-colorable.

Case 2. No color appears on more than two vertices in the part V_1. We suppose that $|\bigcup_{u \in V(G)} L(u)| < |V(G)|$ by Corollary 2.1. Thus there must exist two vertices in V_{t+1}, say x_{t+1} and y_{t+1}, such that $L(x_{t+1}) \cap L(y_{t+1}) \neq \emptyset$. Choose a color $c_1 \in L(x_{t+1}) \cap L(y_{t+1})$ to color both x_{t+1} and y_{t+1}. Let $G' = G - x_{t+1} - y_{t+1}$ and $L' = L - c_1$. We only need to show that G' is L'-colorable. Let W be a maximal subset of $V(G')$ such that $|L'(W)| < |W|$. By Lemma 2.1, it suffices to show that $G'[W]$ is $L'|_W$-colorable. We claim that $|V_1 \cap W| \leq 3$. Otherwise, $|L'(W)| \geq (|L(x_1)| + |L(y_1)| + |L(z_1)| + |L(w_1)| - 2)/2 = (8t+6)/2 = 4t+3 = |V'(G)| \geq |W|$. This is a contradiction. Without loss of generality, let $w_1 \notin W$. As $G'[W]$ is a subgraph of $G' - w_1$, it suffices to show that $G' - w_1$ is L'-colorable. Choose a color $c_2 \in L'(z_{2t+2})$ to color the vertex z_{2t+2}. Let $G'' = G' - w_1 - z_{2t+2}$, $L'' = L' - c_2$. Clearly, $|L''(u)| \geq 2t$ for every $u \in V(G'')$. By (1) and the condition of Case 2, it is easy to see that $|L''(z_{t+1})| = 2t + 1$, and for each $i \in [t]$ at least one of the sets $L''(x_i)$, $L''(y_i)$, $L''(z_i)$ contain at leat $2t + 1$ colors, without loss of generality, say $|L''(z_i)| \geq 2t + 1$ for all such i. Therefore $G'' = K_{3*t,1*(t+1)}$ and L'' satisfies requirements of Lemma 2.2. Thus G'' is L''-colorable by Lemma 2.2, and hence $G' - w_1$ is L'-colorable. ∎

Theorem 3.2. *For each integer $t \geq 0$ and $k \geq 2t + 2$,* $\mathrm{ch}(K_{4,3*t,2*(k-2t-2),1*(t+1)}) = k$.

Proof. For $G = K_{4,3*t,1*(t+1)}$, denote its k parts as $V_1 = \{x_1, y_1, z_1, w_1\}$, $V_i = \{x_i, y_i, z_i\}$ for $i \in [t + 1] \setminus [1]$, $V_i = \{z_i\}$ for $i \in [2t + 2] \setminus [t + 1]$, and $U_j = \{u_j, v_j\}$ for $j \in [k - 2t - 2]$. Let L be a k-list assignment of G. We will prove by induction on t and k that G is L-colorable.

At first, we use induction on t. If $t = 0$, by the result $\mathrm{ch}(K_{s+3,2*(k-s-1),1*s}) = k$ for $s \geq 0$ [1], then Theorem 3.2 holds (let $s = 1$). If $t = 1$, then Theorem 3.2 is just Theorem 1.2. So we may suppose that $t \geq 2$ and suppose that Theorem 3.2 is true for smaller values of t. If there exists $i \in [t + 1]$ such that $\bigcap_{u \in V_i} L(u) \neq \emptyset$, then we can choose a color $c_1 \in \bigcap_{u \in V_i} L(u)$ to color all the vertices in V_i, and a different

color $c_2 \in L(z_{2t+2})$ to color the vertex z_{2t+2}. Let $G' = G - V_i - z_{2t+2}$ and $L' = L - c_1 - c_2$. Clearly, G' is a subgraph of $K_{4,3*(t-1),2*(k-2t),1*t}$, and we can finish the proof applying the induction hypothesis. So we can suppose that

$$(2) \qquad \bigcap_{u \in V_i} L(u) = \emptyset \ \text{ for all } \ i \in [t+1].$$

Then under the above supposition we use induction on k to prove that G is L-colorable for the given t. If $k = 2t+2$ then Theorem 3.2 is just Theorem 3.1. So we suppose that $k \geq 2t+3$ and Theorem 3.2 is true for smaller value of k. If there exists $j \in [k-2t-2]$ such that $L(u_j) \cap L(v_j) \neq \emptyset$ then we can choose a color $c_1 \in L(u_j) \cap L(v_j)$ to color both u_j and v_j, and apply induction to $G - U_j$ and $L - c_1$, we can obtain that G is L-colorable. So we can suppose that

$$(3) \qquad L(u_j) \cap L(v_j) = \emptyset \ \text{ for all } \ j \in [k-2t-2].$$

Case 1. There exist three vertices in V_1, say x_1, y_1, z_1, such that $L(x_1) \cap L(y_1) \cap L(z_1) \neq \emptyset$.

We choose a color $c_1 \in L(x_1) \cap L(y_1) \cap L(z_1)$ to color all the vertices x_1, y_1, z_1. Let $G' = G - x_1 - y_1 - z_1$ and $L' = L - c_1$. Clearly, $|L'(u)| \geq k-1$ for every $u \in V(G')$. By (2), $|L'(w_1)| = k$, and for each $i \in [t+1] \setminus [1]$ at least one of the sets $L'(x_i)$, $L'(y_i)$ and $L'(z_i)$ contain k colors, without loss of generality, say $|L'(z_i)| = k$ for all such i. Similarly, by (3), for each $j \in [k-2t-2]$ at least one of the sets $L'(u_j)$, $L'(v_j)$ contains k colors, so that $|L'(u_j) \cup L'(v_j)| \geq 2k-1$. We wish to show that G' is L'-colorable.

Let W be a maximal subset of $V(G')$ such that $|L'(W)| < |W|$. By Lemma 2.1, it suffices to show that $G'[W]$ is $L'|_W$-colorable. Note that

$$(4) \qquad |W \cap U_j| \leq 1 \ \text{ for all } \ j \in [k-2t-2],$$

since otherwise, we have that $2k-1 \leq |L'(u_j) \cup L'(v_j)| \leq |L'(W)| < |W| \leq |V(G')| = 2k-2$, a contradiction.

Let $U = \bigcup \{U_j | j \in [k-2t-2]\}$ and $m = |\{j | W \cap U_j \neq \emptyset, j \in [k-2t-2]\}| \leq k-2t-2$. It follows from (4) that $m = |W \cap U|$. Color the vertices of $W \cap U$ with m distinct colors. Let the set of these m colors be C, and $G'' = G' - U$, $W' = W \backslash U$, $L'' = L' \backslash C$. It suffices to prove that G'' is L''-colorable, since this will imply that $G''[W']$ is $L''|_{W'}$-colorable (as $G''[W']$ is a subgraph of G''), so that $G'[W]$ is $L'|_W$-colorable. We choose

a color $c_2 \in L''(z_{2t+2})$ to color the vertex z_{2t+2}. Let $G''' = G'' - z_{2t+2}$ and $L''' = L'' - c_2$. Since $|L'(u)| \geq k - 1$ for every $u \in V(G')$, it follows that $|L'''(u)| \geq k - 1 - m - 1 \geq 2t$ for every $u \in V(G''')$. And since $|L'(w_1)| = k$, $|L'(z_i)| = k$ for every $i \in [t + 1] \setminus [1]$, it follows that $|L'''(w_1)| \geq k - m - 1 \geq 2t + 1$, $|L'''(z_i)| \geq k - m - 1 \geq 2t + 1$ for all such i. Therefore $G''' = K_{3*t,1*(t+1)}$ and L''' satisfies requirements of Lemma 2.2. Thus G''' is L'''-colorable by Lemma 2.2, and hence G'' is L''-colorable.

Case 2. No color appears on more than two vertices in the part V_1.
We suppose that $|\bigcup_{u \in V(G)} L(u)| < |V(G)|$ by Corollary 2.1. Thus there must exist two vertices in V_{t+1}, say x_{t+1} and y_{t+1}, such that $L(x_{t+1}) \cap L(y_{t+1}) \neq \emptyset$. Choose a color $c_1 \in L(x_{t+1}) \cap L(y_{t+1})$ to color x_{t+1} and y_{t+1}. Let $G' = G - x_{t+1} - y_{t+1}$ and $L' = L - c_1$. We only need to show that G' is L'-colorable. Let W be a maximal subset of $V(G')$ such that $|L'(W)| < |W|$. By Lemma 2.1, it suffices to show that $G'[W]$ is $L'|_W$-colorable. By a similar argument in Case 1, we also have the inequality (4). Moreover, we claim that $|V_1 \cap W| \leq 3$. Otherwise, $|L'(W)| \geq (|L(x_1)| + |L(y_1)| + |L(z_1)| + |L(w_1)| - 2)/2 = (4k - 2)/2 = 2k - 1 = |V'(G)| \geq |W|$. This is a contradiction. Without loss of generality, let $w_1 \notin W$, $G'' = G' - U - w_1$ and $L'' = L' \setminus C$, where the meanings of U and C are the same as in Case 1. It suffices to show that G'' is L''-colorable. Choose a color $c_2 \in L'(z_{2t+2})$ to color the vertex z_{2t+2}. Let $G''' = G'' - z_{2t+2}$, $L''' = L'' - c_2$. Clearly, $|L'''(u)| \geq 2t$ for every $u \in V(G''')$. By (2) and the condition of Case 2, it is easy to see that $|L'''(z_{t+1})| \geq 2t + 1$, and for each $i \in [t]$ at least one of the sets $L'''(x_i)$, $L'''(y_i)$, $L'''(z_i)$ contain at least $2t + 1$ colors, without loss of generality, say $|L'''(z_i)| \geq 2t + 1$ for all such i. Therefore $G''' = K_{3*t,1*(t+1)}$ and L''' satisfies requirements of Lemma 2.2. Thus G''' is L'''-colorable by Lemma 2.2, and hence G'' is L''-colorable. ∎

References

[1] H. Enotomo, K. Ohba, K. Ota and J. Sakamoto, *Choice number of some complete multipartite graphs*, Discrete Math. **244** (2002) 55–66.

[2] P. Erdős, A.L. Rubin and H. Taylor, *Choosability in graphs*, Congr. Numer. **26** (1979) 125–157.

[3] S. Gravier and F. Maffray, *Graphs whose choice number is equal to their chromatic number*, J. Graph Theory **27** (1998) 87–97.

[4] W. He, L. Zhang, Daniel W. Cranston, Y. Shen and G. Zheng, *Choice number of complete multipartite graphs $K_{3*3,2*(k-5),1*2}$ and $K_{4,3*2,2*(k-6),1*3}$*, Discrete Math. **308** (2008) 5871–5877.

[5] H.A. Kierstead, *On the choosability of complete multipartite graphs with part size three*, Discrete Math. **211** (2000) 255–259.

[6] K. Ohba, *On chromatic choosable graphs*, J. Graph Theory **40** (2002) 130–135.

[7] K. Ohba, *Choice number of complete multipartite graphs with part size at most three*, Ars Combinatoria **72** (2004) 133–139.

[8] B. Reed and B. Sudakov, *List colouring when the chromatic number is close to the order of the graph*, Combinatorica **25** (2005) 117–123.

[9] Y. Shen, W. He, G. Zheng and Y. Li, *Ohba's conjecture is ture for graph with independence number at most three*, Applied Mathematics Letters **22** (2009) 938–942.

[10] Y. Shen, W. He, G. Zheng, Y. Wang and L. Zhang, *On choosability of some complete multipartite graphs and Ohba's conjecture*, Discrete Math. **308** (2008) 136–143.

[11] Y. Shen, G. Zheng and W. He, *Chromatic choosability of a class of complete multipartite graphs*, J. Mathematical Research and Exposition **27** (2007) 264–272.

[12] Zs. Tuza, *Graph colorings with local constrains—A survey*, Discuss. Math. Graph Theory **17** (1997) 161–228.

[13] V.G. Vizing, *Coloring the vertices of a graph in prescribed colors* (in Russian), Diskret. Anal. **29** (1976) 3–10.

[14] D.R. Woodall, List colourings of graphs, in: J.W.P. Hirschfeld (ed.), Surveys in Combinatorics, 2001, London Math. Soc. Lecture Note Series, vol. 288 (Cambridge University Press, Cambridge, UK, 2001) 269–301.