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The Schur and Steinhaus Theorems for 4-Dimensional Matrices in Ultrametric Fields

Abstract. Throughout this paper, K denotes a ds-complete, non-trivially valued, ultrametric field. Entries of double sequences, double series and 4-dimensional matrices are in K. We prove the Schur and Steinhaus theorems for 4-dimensional matrices in such fields.

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1. Introduction. Throughout, K denotes a ds-complete, non-trivially valued, ultrametric field with valuation $|\cdot|$ (ds-completeness will be defined in the sequel). If $A = (a_{m,n,k,\ell})$ is a 4-dimensional infinite matrix, $a_{m,n,k,\ell} \in K$, $m, n, k, \ell = 0, 1, 2, \ldots$, by the A-transform of a double sequence $x = \{x_{k,\ell}\}$, $x_{k,\ell} \in K$, $k, \ell = 0, 1, 2, \ldots$, we mean the double sequence $Ax = \{(Ax)_{m,n}\}$,

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty} a_{m,n,k,\ell} x_{k,\ell}, \quad m, n = 0, 1, 2, ...,$$

where we suppose that the double series on the right converge. The double sequence $x = \{x_{k,\ell}\}$ is said to be summable A or A-summable to ℓ if

$$\lim_{m+n\to\infty} (Ax)_{m,n} = \ell.$$

Let c_{ds}, ℓ_{ds}^{∞} respectively denote the spaces of convergent double sequences and bounded double sequences. If $A = (a_{m,n,k,\ell})$ is such that $\{(Ax)_{m,n}\} \in c_{ds}$ whenever $x =$

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$\{x_{k,\ell}\} \in c_{ds}$, A is said to be convergence-preserving. If, further, $\lim_{m+n\to\infty}(Ax)_{m,n} = \lim_{k+\ell\to\infty} x_{k,\ell}$, we say that A is regular.

Natarajan and Srinivasan proved the following theorem.

THEOREM 1.1 ([2]) *$A = (a_{m,n,k,\ell})$ is regular if and only if*

$$\lim_{m+n\to\infty} a_{m,n,k,\ell} = 0, \quad k,\ell = 0,1,2,\ldots; \tag{1}$$

$$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{\infty} a_{m,n,k,\ell} = 1; \tag{2}$$

$$\lim_{m+n\to\infty} \sup_{k\geqslant 0} |a_{m,n,k,\ell}| = 0, \ell = 0,1,2,\ldots; \tag{3}$$

$$\lim_{m+n\to\infty} \sup_{\ell\geqslant 0} |a_{m,n,k,\ell}| = 0, k = 0,1,2,\ldots; \tag{4}$$

and

$$\sup_{m,n,k,\ell} |a_{m,n,k,\ell}| < \infty. \tag{5}$$

A is called a Schur matrix if $\{(Ax)_{m,n}\} \in c_{ds}$ whenever $x = \{x_{k,\ell}\} \in \ell^{\infty}_{ds}$. The main object of this paper is to get necessary and sufficient conditions for A to be a Schur matrix and then deduce Steinhaus theorem.

DEFINITION 1.2 The double sequence $\{x_{m,n}\}$ in K is called a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ (the set of all non-negative integers) such that the set

$$\{(m,n),(k,\ell) \in \mathbb{N}^2 : |x_{m,n} - x_{k,\ell}| \geqslant \epsilon, m,n,k,\ell \geqslant N\}$$

is finite.

It is easy to prove the following result.

THEOREM 1.3 *The double squence $\{x_{m,n}\}$ in K is Cauchy if and only if*

$$\lim_{m+n\to\infty} |x_{m+1,n} - x_{m,n}| = 0; \tag{6}$$

and

$$\lim_{m+n\to\infty} |x_{m,n+1} - x_{m,n}| = 0. \tag{7}$$

Definition 1.4 If every Cauchy double sequence of an ultrametric normed linear space X converges to an element of X, X is said to be double sequence complete or ds-complete.

For $x = \{x_{m,n}\} \in \ell_{ds}^{\infty}$, define $||x|| = \sup_{m,n} |x_{m,n}|$. One can easily prove that ℓ_{ds}^{∞} is an ultrametric normed linear space which is ds-complete. With the same definition of norm for elements of c_{ds}, c_{ds} is a closed subspace of ℓ_{ds}^{∞}. Natarajan [1] proved Schur's theorem and Steinhaus' theorem for 2-dimensional matrices over complete, non-trivially valued, ultrametric fields. We now prove the main results of the paper.

2. Main Results. In the sequel, let us suppose that K is a non-trivially valued, ultrametric field which is ds-complete.

Theorem 2.1 (Schur) *The necessary and sufficient conditions for a 4-dimensional infinite matrix $A = (a_{m,n,k,\ell})$ to transform double sequences in ℓ_{ds}^{∞} into double sequences in c_{ds}, i.e., $\{(Ax)_{m,n}\} \in c_{ds}$ whenever $x = \{x_{k,\ell}\} \in \ell_{ds}^{\infty}$ are:*

$$\lim_{k+\ell\to\infty} a_{m,n,k,\ell} = 0, \quad m,n = 0,1,2,\ldots; \tag{8}$$

$$\lim_{m+n\to\infty} \sup_{k,\ell\geqslant 0} |a_{m+1,n,k,\ell} - a_{m,n,k,\ell}| = 0; \tag{9}$$

and

$$\lim_{m+n\to\infty} \sup_{k,\ell\geqslant 0} |a_{m,n+1,k,\ell} - a_{m,n,k,\ell}| = 0. \tag{10}$$

Proof Sufficiency. Let (8), (9), (10) hold and $x = \{x_{k,\ell}\} \in \ell_{ds}^{\infty}$. We first note that (8), (9), (10) together imply that (5) holds. In view of (5) and (8),

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty} a_{m,n,k,\ell} x_{k,\ell}, \quad m,n = 0,1,2,\ldots$$

is defined, the double series on the right being convergent.
Now,

$$\begin{aligned} |(Ax)_{m+1,n} - (Ax)_{m,n}| &= \left| \sum_{k,\ell=0}^{\infty} (a_{m+1,n,k,\ell} - a_{m,n,k,\ell}) x_{k,\ell} \right| \\ &\leqslant M \sup_{k,\ell\geqslant 0} |a_{m+1,n,k,\ell} - a_{m,n,k,\ell}| \\ &\to 0, \quad m+n \to \infty, \quad \text{using}(9), \end{aligned}$$

where $|x_{k,\ell}| \leqslant M$, $k,\ell = 0,1,2,\ldots$, $M > 0$. Similarly it follows that

$$|(Ax)_{m,n+1} - (Ax)_{m,n}| \to 0, \quad m+n \to \infty, \ \text{using}(10).$$

Thus $\{(Ax)_{m,n}\}$ is a Cauchy double sequence in K. Since K is ds-complete, $\{(Ax)_{m,n}\}$ converges and so $\{(Ax)_{m,n}\} \in c_{ds}$, completing the sufficiency part of the proof.
Necessity. Let $\{(Ax)_{m,n}\} \in c_{ds}$ whenever $x = \{x_{k,\ell}\} \in \ell_{ds}^{\infty}$. Consider the double sequence $\{x_{k,\ell}\}$ where $x_{k,\ell} = 1$, $k, \ell = 0, 1, 2, \dots$. $\{x_{k,\ell}\} \in \ell_{ds}^{\infty}$ so that, by hypothesis,

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty} a_{m,n,k,\ell}, \quad m, n = 0, 1, 2, \dots$$

is defined. Since the series on the right converges, (8) holds. Suppose (9) does not hold. Then there exists $\ell_0 \in \mathbb{N}$ such that

$$\lim_{m+n\to\infty} \sup_{k\geqslant 0} |a_{m+1,n,k,\ell_0} - a_{m,n,k,\ell_0}| = 0$$

does not hold. So there exists $\epsilon > 0$ such that the set

$$\{(m,n) \in \mathbb{N}^2 : \sup_{k\geqslant 0} |a_{m+1,n,k,\ell_0} - a_{m,n,k,\ell_0}| > \epsilon\} \text{is infinite.}$$

Thus we can choose pairs of integers $m_p, n_p \in \mathbb{N}$ such that $m_1 + n_1 < m_2 + n_2 < \dots < m_p + n_p < \dots$ and

$$\sup_{k\geqslant 0} |a_{m_p+1,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| > \epsilon, \quad p = 1, 2, \dots . \tag{11}$$

Using (8),

$$\lim_{k\to\infty} |a_{m_1+1,n_1,k,\ell_0} - a_{m_1,n_1,k,\ell_0}| = 0.$$

Consequently there exists $r_1 \in \mathbb{N}$ such that

$$\sup_{k\geqslant r_1} |a_{m_1+1,n_1,k,\ell_0} - a_{m_1,n_1,k,\ell_0}| < \frac{\epsilon}{2}. \tag{12}$$

Because of (11) and (12), we have,

$$\sup_{0\leqslant k<r_1} |a_{m_1+1,n_1,k,\ell_0} - a_{m_1,n_1,k,\ell_0}| > \epsilon, \tag{13}$$

so that there exists k_1, $0 \leqslant k_1 < r_1$ with

$$|a_{m_1+1,n_1,k_1,\ell_0} - a_{m_1,n_1,k_1,\ell_0}| > \epsilon. \tag{14}$$

By hypothesis, (1) holds. So we can suppose that

$$\sup_{0\leqslant k<r_1} |a_{m_2+1,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| < \frac{\epsilon}{2}. \tag{15}$$

By (11) we have,

$$\sup_{k\geqslant 0} |a_{m_2+1,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| > \epsilon. \tag{16}$$

Using (8),

$$\lim_{k\to\infty} |a_{m_2+1,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| = 0$$

so that there exists $r_2 \in \mathbb{N}$, $r_2 > r_1$ such that

$$\sup_{k\geqslant r_2} |a_{m_2+1,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| < \frac{\epsilon}{2}. \tag{17}$$

From (15), (16), (17), we have,

$$\sup_{r_1\leqslant k<r_2} |a_{m_2+1,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| > \epsilon.$$

Thus there exists $k_2 \in \mathbb{N}$ such that $r_1 \leqslant k_2 < r_2$ and

$$|a_{m_2+1,n_2,k_2,\ell_0} - a_{m_2,n_2,k_2,\ell_0}| > \epsilon. \tag{18}$$

Inductively, we can choose strictly increasing sequences $\{r_p\}$, $\{k_p\}$ such that $r_{p-1} \leqslant k_p < r_p$,

$$\sup_{0\leqslant k<r_{p-1}} |a_{m_p+1,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| < \frac{\epsilon}{2}; \tag{19}$$

$$\sup_{r_p\leqslant k<\infty} |a_{m_p+1,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| < \frac{\epsilon}{2}; \tag{20}$$

and

$$|a_{m_p+1,n_p,k_p,\ell_0} - a_{m_p,n_p,k_p,\ell_0}| > \epsilon. \tag{21}$$

Now, define $\{x_{k,\ell}\} \in \ell^\infty_{ds}$, where

$$x_{k,\ell} = \begin{cases} 1, & \text{if } k = k_p, \ell = \ell_0, p = 1, 2, \ldots; \\ 0, \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned}
(Ax)_{m_p+1,n_p} - (Ax)_{m_p,n_p} &= \sum_{k,\ell=0}^{\infty} (a_{m_p+1,n_p,k,\ell} - a_{m_p,n_p,k,\ell})x_{k,\ell} \\
&= \sum_{k=0}^{\infty} (a_{m_p+1,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0} \\
&= \sum_{0\leqslant k<r_{p-1}} (a_{m_p+1,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0} \\
&\quad + \sum_{r_{p-1}\leqslant k<r_p} (a_{m_p+1,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0} \\
&\quad + \sum_{k\geqslant r_p} (a_{m_p+1,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0})x_{k,\ell_0}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{0 \leqslant k < r_{p-1}} (a_{m_p+1,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}) x_{k,\ell_0} \\
&\quad + (a_{m_p+1,n_p,k_p,\ell_0} - a_{m_p,n_p,k_p,\ell_0}) \\
&\quad + \sum_{k \geqslant r_p} (a_{m_p+1,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}) x_{k,\ell_0}
\end{aligned}$$

so that

$$\begin{aligned}
&(a_{m_p+1,n_p,k_p,\ell_0} - a_{m_p,n_p,k_p,\ell_0}) \\
&\qquad = \{(Ax)_{m_p+1,n_p} - (Ax)_{m_p,n_p}\} \\
&\qquad\quad - \sum_{0 \leqslant k < r_{p-1}} (a_{m_p+1,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}) x_{k,\ell_0} \\
&\qquad\quad - \sum_{k \geqslant r_p} (a_{m_p+1,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}) x_{k,\ell_0}.
\end{aligned}$$

In view of (19), (20), (21), we have,

$$\begin{aligned}
\epsilon &< |a_{m_p+1,n_p,k_p,\ell_0} - a_{m_p,n_p,k_p,\ell_0}| \\
&\leqslant Max \left[|(Ax)_{m_p+1,n_p} - (Ax)_{m_p,n_p}|, \frac{\epsilon}{2}, \frac{\epsilon}{2} \right],
\end{aligned}$$

from which it follows that

$$|(Ax)_{m_p+1,n_p} - (Ax)_{m_p,n_p}| > \epsilon, \quad p = 1, 2, \ldots .$$

Consequently $\{(Ax)_{m,n}\} \notin c_{ds}$, a contradiction. Thus (9) holds. It can similarly be proved that (10) also holds. This completes the proof of the theorem. ∎

We now deduce the following theorem.

THEOREM 2.2 (STEINHAUS) *A 4-dimensional infinite matrix $A = (a_{m,n,k,\ell})$ cannot be both a regular and a Schur matrix, i.e., there exists a bounded divergent double sequence which is not A-summable.*

PROOF If $A = (a_{m,n,k,\ell})$ is regular, then (1) and (2) hold. If A were a Schur matrix too, then, $\{a_{m,n,k,\ell}\}_{m,n=0}^{\infty}$ is uniformly Cauchy with respect to $k, \ell = 0, 1, 2, \ldots$. Since K is ds-complete, $\{a_{m,n,k,\ell}\}_{m,n=0}^{\infty}$ converges uniformly to 0. In other words, we have,

$$\lim_{m+n \to \infty} \sum_{k,\ell=0}^{\infty} a_{m,n,k,\ell} = 0,$$

a contradiction, completing the proof. ∎

References

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