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The lattice of connected subgraphs of a connected graph

1. Introduction. In [2], Leclerc states that the lattice L_T of all connected subgraphs of a tree T with p points has the following properties:

(i) L_T has p irreducible elements;

(ii) all maximal chains joining the least and the greatest elements of L_T are of length p;

(iii) the atoms of L_T are the $\vee$-irreducible elements of L_T;

(iv) for each element $a \in L_T$, $a \neq 0$, the filter $[a)$ is a distributive sublattice of L_T.

Leclerc constructs further a connected graph G, which is not a tree but whose lattice of connected and induced subgraphs has properties (i) and (iv). The purpose of this paper is to characterize the lattice of connected, induced subgraphs of a connected graph. After the characterization an immediate generalization is given in terms of ideals of graphs [3], [4].

By a graph $G = \big(P(G), X(G)\big)$ we shall mean a connected, undirected and finite graph without loops and multiple lines, where $P(G)$ is its set of points and $X(G)$ its set of lines. We shall follow the terminology and definitions given by Harary in [1].

The concepts of lattice theory used here can be found in monograph [5] of Szász. Let L be a lattice; $[b) = \{x \mid b \leqslant x, b, x \in L\}$ and it is called the *principal filter* of L generated by b, and as well known, it is a convex sublattice of L. Dually, $(b] = \{x \mid x \leqslant b, x, b \in L\}$.

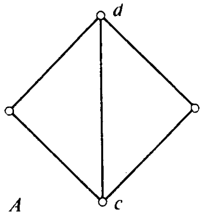
Let G be a given graph. The lattice L_G of its connected, induced subgraphs, if it exists, is determined by the set-theoretical inclusion order: $G_1 \subseteq G_2 \Leftrightarrow P(G_1) \subseteq P(G_2)$. As we shall consider induced subgraphs only, the property $P(G_1) \subseteq P(G_2)$ implies that $X(G_1) \subseteq X(G_2)$. The 0-element of L_G is the empty graph.

2. The lattice of connected subgraphs. Let G be a given graph and $K_1, K_2, \ldots, K_k$ its decomposition into k induced subgraphs. We shall call G a tree structure if G can be decomposed into subgraphs $K_1, \ldots, K_k$ such that each subgraph is a complete subgraph of G (or a point), $k \geqslant 1$, each two subgraphs K_i and K_j ($i \neq j$) are connected by at most one line of G, and there are in G no cycles connecting two points x_i and x_j of G: $x_i \in K_i$ and $x_j \in K_j$, $i \neq j$. Thus every complete graph is a tree structure as well as every tree, where each K_i is a point.

We begin with a lemma determining the cases where the connected and induced subgraphs do not constitute a lattice with respect to the set-theoretical inclusion order.

Lemma 1. *Let G be a connected graph $|P(G)| \geqslant 4$. If G is not a tree structure, its connected and induced subgraphs do not constitute a lattice with respect to the set-theoretical inclusion order.*

Proof. If G is a graph satisfying the conditions of the lemma, then G contains an induced subgraph isomorphic to A or B, where A is the graph of Fig. 1 and B is a cycle of points $\{x_1, \ldots, x_n, x_{n+1}\} \subseteq P(G)$ with $x_{n+1} = x_1$, $n \geqslant 4$ and there are no lines in B other than the lines of the cycle. We shall consider the subgraph A only; the proof for B is similar.



A

Fig. 1

Let $G\{a\}$ and $G\{b\}$ be connected subgraphs of A (and of G) induced by points a and b, respectively. Moreover, $G\{a\}$, $G\{b\} \subseteq G\{a, b, c\}$ and $G\{a\}$, $G\{b\} \subseteq G\{a, b, d\}$, whence $G\{a\}$, $G\{b\} \subseteq \{a, b, c\} \wedge G\{a, b, d\}$. As $\{a, b\}$ does not induce a connected subgraph of A and G, the meet $G\{a, b, c\} \wedge G\{a, b, d\}$ is not uniquely defined, and so the connected and induced subgraphs of G do not constitute a lattice L_G.

As one can easily see, the induced subgraphs of a complete graph constitute a Boolean lattice. The connected subgraphs of a tree T constitute a lattice L_T as noted in [2].

We shall give our characterization in two lemmas.

Lemma 2. *The connected induced subgraphs of a tree structure G constitute a lattice L_G satisfying the following conditions:*

1° *for each $a \in L_G$, $a \neq 0$, $[a)$ is a distributive sublattice of L_G;*

2° *the only $\vee$-irreducible elements of L_G are 0 and the atoms of L_G;*

3° *the length of each chain between a ($\neq 0$) and 0 is equal to the number of atoms in the sublattice $(a]$ of L_G.*

Proof. Let G_1 and G_2 be two connected and induced subgraphs of G.

If $P(G_1) \cap P(G_2) \neq \emptyset$, the points in $P(G_1) \cap P(G_2)$ evidently induce a unique connected subgraph of G, the same holds if the intersection is empty. Let us consider the join $G_1 \vee G_2$. If $P(G_1) \cap P(G_2) \neq \emptyset$, then, as there exist no cycles in G joining two points $x_i \in K_i$ and $x_j \in K_j$, $i \neq j$, the set $P(G_1) \cup P(G_2)$ of points induces a unique connected subgraph G_3 of G such that $P(G_3) = P(G_1) \cup P(G_2)$. Let $P(x, y)$ be a shortest path joining $x \in P(G_1)$ and $y \in P(G_2)$. If $P(G_1) \cap P(G_2) = \emptyset$, the union $\bigcup \{P(x, y) | x \in P(G_1)$ and $y \in \in P(G_2)\}$ of points induces a connected subgraph G_3 of G such that $P(G_3) = \bigcup \{P(x, y) | x \in P(G_1)$ and $y \in P(G_2)\}$ and G_3 is the least connected and induced subgraph containing G_1 and G_2 in G. Indeed, as $P(G_1) \cap P(G_2) = \emptyset$, it may happen that $P(G_1) \cap P(K_i) \neq \emptyset$ or $P(G_2) \cap P(K_i) \neq \emptyset$ but not both except for at most one value of i according to the definition of the tree structure. Assume that $K_{i0} \cap P(G_1) = R_1 \neq \emptyset \neq K_{i0} \cap P(G_2) = R_2$ for some value i_0 of i. Then all the shortest paths $P(x, y | x \in P(G_1)$ and $y \in P(G_2))$ go through the lines of the complete graph induced by the point set $R_1 \cup R_2$, and our assertion holds. Assume now that there is no such subgraph K_{i0} in G. As G is a tree structure, there is a unique sequence $K_{i1}, K_{i2}, \ldots, K_{ir}$ of subgraphs of G joining all the points of G_1 and G_2, where only, for i_1, $P(G_1) \cap K_{i1} \neq \emptyset$, for i_r, $P(G_2) \cap K_{ir} \neq \emptyset$. Let $z \in K_{i1}$ be the endpoint of the unique line joining K_{i1} and K_{i2} and $w \in K_{ir}$ the endpoint of the unique line joining K_{ir} and K_{ir-1}. The shortest path between z and w is unique in G and all the shortest paths $P(x, y | x \in P(G_1)$ and $y \in P(G_2))$ go through the lines joining z to the points of $K_{i1} \cap P(G_1)$. An analogous fact holds for w and the points of $K_{ir} \cap P(G_2)$. So the graph thus obtained is the least possible connected subgraph of G containing G_1 and G_2. Obviously it is also induced by $\bigcup \{P(x, y) | x \in P(G_1)$ and $y \in P(G_2)\}$. Hence there exists a lattice L_G when G is a tree structure.

Trivially, the atoms of L_G are $\vee$-irreducible in L_G. Let $x \in L_G$ be neither an atom nor the 0-element of L_G but a $\vee$-irreducible element, and let G_x be the connected induced subgraph of G corresponding to x. We denote by G_g the greatest connected induced subgraph of G contained in G_x properly, i.e. x covers g in L_G. As the subgraphs under consideration are induced, G_x is obtained from G_g by adding to $P(G_g)$ a point v of G. Thus $P(G_x) = P(G_g) \cup \{v\}$, and as G_x is an induced subgraph of G, $G_x = G_g \vee G\{v\}$. So G_x is $\vee$-reducible, which is a contradiction.

Let us consider a filter $[a)$ of L_G, $a \neq 0$, and let $d, f, h \in [a)$. As is well known, the distributivity of $[a)$ follows already from $d \wedge (h \vee f) \leqslant (d \wedge h) \vee (d \wedge f)$. In $[a)$, the intersection of the point sets of two subgraphs is always non-empty, and, as shown above, if $P(G_1) \cap P(G_2) \neq \emptyset$, then $G_1 \vee G_2$ and $G_1 \wedge G_2$ are induced by $P(G_1) \cup P(G_2)$ and $P(G_1) \cap P(G_2)$ respectively. The validity of the assertion is now obvious, as $P(G_d) \cap (P(G_h) \cup P(G_f)) \supseteq (P(G_d) \cap P(G_h)) \cup \cup (P(G_d) \cap P(G_f))$ and the corresponding graphs are induced by the point sets thus obtained.

We prove the last assertion by induction over the number $|P(G)|$ of points in the tree structure G. Obviously each connected and induced subgraph of a tree structure is a tree structure again, and hence the general assertion follows from the result proved by induction. Obviously the length of all maximal chains from 0 to the 1-element of a tree structure G is 1 and 2 when $|P(G)| = 1, 2$, respectively. We assume that the assertion is true for all tree structures with $|P(G)| \leqslant n-1$.

Lrt G be a tree structure and $|P(G)| = n$, $n \geqslant 3$. As the only $\vee$-irreducible elements of L_G are the atoms and the 0-element, the element 1 of L_G corresponding to G is the join of at least two elements s and t covered by 1 in L_G. As 1 covers s, G_s is obtained from G by removing a suitable point from G; the same fact holds also for G_t. Furthermore, as $n \geqslant 3$, $P(G_s) \cap P(G_t) \neq \emptyset$, and hence G is generated by the point set $P(G_s) \cup P(G_t)$. As G_t and G_s are tree structures and $|P(G_s)| = |P(G_t)| = n-1$, the maximal chains from 0 to s and to t are of length $n-1$. As 1 covers s and t, the chains from 0 to 1 through s and t are of length n. The proof above can be repeated for each two elements p and q covered by 1, whence all chains from 0 to 1 are of length n. This completes the proof.

LEMMA 3. *A finite lattice L is the lattice of connected and induced subgraphs of a tree structure G if L satisfies conditions* 1°–3° *of Lemma* 2.

Proof. We shall prove the lemma by induction over the number of atoms of L. If the number of atoms is 1 or 2, then L is generated by a complete graph of 1 or 2 points, respectively, and the lemma holds. We shall assume that the lemma is valid for all lattices L satisfying conditions 1°–3° and having $n-1$ or less atoms.

Let us consider a lattice L satisfying conditions 1°–3° and having n atoms, $n \geqslant 3$. According to 2°, 1 covers at least two elements s and t, the join of which is 1. As the length of chains between 0 and 1 is n, the chain between 0 and s (and 0 and t) is $n-1$. Clearly, conditions 1°–3° are valid in the sublattices $(s]$ and $(t]$ of L, and hence the sublattices are the lattices of connected induced subgraphs of some tree structures G_s and G_t. The tree structure G_s determined by $(s]$ can be found as follows: Let $\{a_1, a_2, \ldots, a_{n-1}\}$ be the atoms of $(s]$. In G_s two points a_i and a_j which are atoms of $(s]$ are joined by a line if and only if $a_i \vee a_j$ covers a_i and a_j in $(s]$. The tree structure G_t can be found similarly. Let a_s be the atom of $(s]$ not being in $(t]$ and a_t that of $(t]$ not being in $(s]$. If a_s were equal to a_t, then $G_s = G_t$, and therefore $(t] = (s]$, which is a contradiction. As $n \geqslant 3$, $P(G_t) \cap P(G_s) \neq \emptyset$, and as G_s and G_t are tree structures, a_s and a_t are joined by a path of points in $P(G_s) \cap P(G_t)$. The atoms of L determine a graph G in the same way as the atoms of $(s]$ the graph G_s above. If in the graph G, a_s and a_t belong to a complete induced subgraph of G, or they are not joined by a line, then the graph G of L is a tree structure.

As all the dual atoms of L, like s, determine the lattice of connected and induced subgraphs of the corresponding graph (in the case of s, of the graph G_s), L is the lattice of connected induced subgraphs of G.

Assume that this is not the case in G, i.e. a_s and a_t belong to a connected and induced subgraph of G which is isomorphic to one of the graphs A and B in the proof of Lemma 1. If a_s and a_t are not adjacent in G, then in the case of B we can choose two points h and f of B, $h, f \in P(G_s) \cap P(G_t)$, which belong to the opposite sides of the cycle B joining a_s and a_t. In the lattice $(t]$ of induced and connected subgraphs of G_t the element $h \vee f \geqslant a_t$, and in the lattice $(s]$ of G_s, $h \vee f \geqslant a_s$. This is a contradiction, as $(t]$ and $(s]$ are sublattices of L and $a_t \notin (s]$ and $a_s \notin (t]$. In the case of sublattice A, if $a_s \vee a_t \vee h \vee f = m < 1$, we obtain a contradiction, as $(m]$ is the lattice of connected induced subgraphs of a tree structure according to the assumption. Assume that $a_s \vee a_t \vee h \vee f = 1$. Then $a_s \vee a_t \vee h \vee f \in [h \vee f)$, and $[h \vee f)$ is a distributive sublattice of the lattice $[h)$. The chain from 0 to $a_t \vee h \vee f$ is of length three in the lattice $(t]$, as $\{h, f, a_t\}$ induces a complete subgraph of G_t; the same holds for $a_s \vee h \vee f$ in $(s]$. Moreover, the chains from $h \vee f$ to $a_t \vee h \vee f$ and to $a_s \vee h \vee f$ are of length one. As $[h \vee f)$ is distributive, these lengths imply that the chain from $h \vee f$ to $a_s \vee a_t \vee h \vee f$ is of length two, and hence the chain between 0 and 1 in L is of length 4. So L contains 4 atoms, and as $(t]$ and $(s]$ are Boolean lattices as the lattices of connected and induced subgraphs of complete graphs, L contains the structure S_1 of

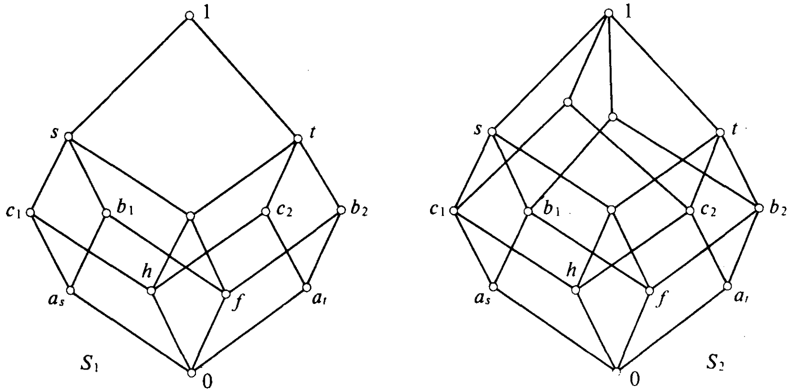


Fig. 2

Fig. 2, where $[f)$ is not a distributive sublattice of S_1. As a_s and a_t are not joined by a line in G, the structure S_1 can be modified further as follows: In S_2 $b_1 \vee b_2$ is added ($[f)$ is not distributive) as well as $c_1 \vee c_2$ ($[a_s)$ is not distributive). The same holds if $c_1 \vee b_2$ (or $c_2 \vee b_1$ or both) is added into S_1: $[f)$ is not distributive. Thus we obtain a contradiction also in the case of the graph A.

It remains to consider the case where a_s and a_t are joined by a line in G. If G is not a tree structure, it contains a connected induced subgraph isomorphic to the subgraph A or B of Lemma 1. In the case of subgraph A we obtain a contradiction as in the case of B above, when a_s and a_t were not adjacent in G. If the join of all the atoms of B is m_1 and less than 1 in L, we obtain a contradiction as with $(m]$ above. Further, we can prove similarly as above that L has $|P(B)| = n$ atoms, and hence $B = G$, i.e. B is the graph of L. As $a_t \notin (s]$ and $a_s \notin (t]$, $a_s \vee a_t \notin (s], (t]$. But $1 \in [a_s \vee a_t)$, and so L contains a third dual atom $u \neq s, t$. According to the induction assumption, $(u]$ is the lattice of connected induced subgraphs of a graph ($\subseteq B$), and the corresponding graph contains $n-1$ atoms. B is a cycle and a_s and a_t are joined by a line in B. We denote by b_t the other point joined by a line in B to a_t, $b_t \neq a_s$, and by b_s the corresponding other point joined to a_s. In $(s]$, $a_s \vee b_t$ is greater than b_s and not greater than a_t, and in $(u]$, if $b_t \in (u]$, $a_s \vee b_t$ is greater than a_t. Similarly, in $(t]$, $a_t \vee b_s$ is greater than a_s, and in $(u]$, if $b_s \in (u]$, $a_t \vee b_s$ is greater than a_s. Both of these observations imply a contradiction as $(s]$, $(t]$ and $(u]$ are sublattice of L. As $(u]$ contains $n-1$ atoms, b_s or b_t belongs to $(u]$. This completes the proof.

The lemmas above imply the following characterization:

THEOREM 1. *A finite lattice L is the lattice of connected induced subgraphs of a graph if and only if conditions 1°–3° of Lemma 2 hold.*

3. On ideals in graphs. The purpose of this section is to give a generalization of the considerations above. The ideal concept of graphs [4] is a natural generalization of the corresponding concept defined for trees by Nebeský in [3].

We define a binary operation, denoted by SP, on the point set $P(G)$ of a given graph G as follows:

$$SP(x, y) = \{z \mid z \in P(G) \text{ and } z \text{ is on a shortest path joining } x \text{ and } y \text{ in } G\}.$$

In particular, $\{x, y\} \subseteq SP(x, y)$, and $SP(x, x) = \{x\}$. In general, let U, W be two sets of $P(G)$. Then $SP(U, W)$ will denote the union of the sets $SP(u, w)$, where $u \in U$ and $w \in W$. A set $U \subseteq P(G)$ is called an *ideal of* G, if $SP(U, U) = U$ and $U \neq \emptyset$.

Let U be an ideal of G. The subgraph G_U of G generated by U in G, the ideal graph of U, is defined as follows: $P(G_U) = U$ and a line $(x, y) \in X(G_U)$ if and only if it belongs to a shortest path joining two points of U in G. The following lemma shows that the ideal graphs have the fundamental properties of the subgraphs considered before.

LEMMA 4. *Let U be an ideal of a connected graph G. The ideal graph G_U is connected and induced subgraph of G.*

Proof. As G is a connected graph, the definition of the SP-operation implies that for each two points x, $y \in U$ the points on the shortest paths joining x and y in G belong to U. The connectivity of G_U follows now from the definition of $X(G_U)$. If x, $y \in U$ and $(x, y) \in X(G)$, then $(x, y) \in X(G_U)$ according to the definition of $X(G_U)$, and hence G_U is induced by $U = P(G_U)$ in G.

Clearly $P(G)$ is an ideal in G as well as $\{x\}$ for each $x \in P(G)$. Further, if G' is a complete subgraph of G, then $P(G')$ is an ideal in G, and if G is a tree, then for every connected subgraph G'' of G, $P(G'')$ is an ideal of G.

The following two lemmas show that there exists a lattice of ideal graphs for each connected graph G. As above, we assume that the empty graph is an ideal graph contained in each ideal graph of G. Further, $G_U \leqslant G_W \Leftrightarrow U \subseteq W$.

LEMMA 5. *Let U and W be two ideals of a graph G. Then either $U \cap W$ is an ideal of G or $U \cap W = \emptyset$.*

Proof. Assume that $U \cap W \neq \emptyset$. If it contains a point only, $U \cap W$ is an ideal of G. Let $x \neq y$ and $x, y \in U \cap W$. As U and W are ideals of G, $SP(x, y) \subseteq U, W$, and so $SP(U \cap W, U \cap W) \subseteq U \cap W$. The converse relation is obvious, and hence $SP(U \cap W, U \cap W) = U \cap W$, which proves the assertion.

Lemma 5 shows that the meet $G_U \wedge G_W = G_U \cap G_W$ of two ideal graphs G_U, G_W of a graph G is always an ideal graph or the empty graph.

LEMMA 6. *Let U and W be two ideals of a graph G. There exists a least ideal V in G containing the ideals U and W.*

Proof. As $P(G)$ is an ideal of G, there always exists an ideal of G containing U and W. Let $V_1, V_2, \ldots, V_h$ be the sequence of all distinct ideals of G containing U and W. As G is finite, h is finite as well, and according to Lemma 5, $\bigcap \{V_i | i = 1, \ldots, h\}$ is an ideal of G. Clearly, it is the least ideal containing U and W. This completes the proof.

For example, the lattice S_1 in Fig. 2 is the lattice of all ideal graphs of a graph isomorphic to the graph A of Fig. 1.

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