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Some remarks on families of Orlicz classes

1. In [1] there were investigated connections between sets $\bigcap_{i=1}^{\infty} L^{\varphi_i}\langle a, b\rangle$, $\bigcup_{i=1}^{\infty} L^{\varphi_i}\langle a, b\rangle$ and $L^{\varphi}\langle a, b\rangle$, where $L^{\varphi}\langle a, b\rangle$ and $L^{\varphi_i}\langle a, b\rangle$, $i = 1, 2, \ldots$, are Orlicz classes. There were given necessary and sufficient conditions for inclusions of these sets. This was generalized further in [2] for a class of modular spaces (see also [3]).

These problems were considered in connection with investigation of families of modulars depending on a parameter, as well as with investigation of some families of functions integrable with a parameter.

1.1. Let Ξ, E and H be three abstract non-empty sets, and let $\mathscr{E}$, $\mathscr{X}$ and $\mathscr{Y}$ be σ-algebras of subsets of sets E, Ξ and H, respectively. Let μ be a finite measure on $\mathscr{E}$, and let m, n be measures on $\mathscr{X}$, $\mathscr{Y}$, respectively. We denote the respective measure spaces by $(E, \mathscr{E}, \mu)$, $(\Xi, \mathscr{X}, m)$ and $(H, \mathscr{Y}, n)$.

It will be assumed that for any sequence (ε_i) of positive numbers satisfying the inequality $\varepsilon_1+\varepsilon_2+\ldots < \mu(E)$ there is a sequence of pairwise disjoint sets $E_i \in \mathscr{E}$ such that $\mu(E_i) = \varepsilon_i$ for $i = 1, 2, \ldots$ Moreover, we consider two fixed non-empty families $\mathscr{Z}$ and $\mathscr{Z}^*$ of subsets of $\mathscr{X}$ respectively $\mathscr{Y}$, such that $0 < m(Z) < \infty$ and $0 < n(Z^*) < \infty$ for all $Z \in \mathscr{Z}$ and $Z^* \in \mathscr{Z}^*$.

1.2. We shall say that the family $\mathscr{Z}$ is *σ-absorbed* if there exists an increasing sequence of sets (Z_i), $Z_i \in \mathscr{Z}$, such that for every $Z \in \mathscr{Z}$ there is an index k for which $Z \subset Z_k$.

The family $\mathscr{Z}^*$ will be called *σ-absorbing*, if there exists a sequence of sets (Z_i^*), $Z_i^* \in \mathscr{Z}^*$, such that for every $Z^* \in \mathscr{Z}^*$ there is an index k_0 for which $Z_k^* \subset Z^*$ for all $k \geqslant k_0$.

1.3. Let us take two functions $\varphi\colon \Xi \times R_+ \to R_+$ and $\Psi\colon H \times R_+ \to R_+$, where $R_+ = \langle 0, \infty)$, satisfying the following conditions:

1° $\varphi(\cdot, u)$ is $\mathscr{X}$-measurable in the variable $\xi \in \Xi$ and $\psi(\cdot, u)$ is $\mathscr{Y}$-measurable in the variable $\eta \in H$ for every $u \in R_+$.

2° $\varphi(\xi, \cdot)$ is a φ-function (see e.g. [2]) for m-almost every $\xi \in \Xi$ and $\psi(\eta, \cdot)$ is a φ-function for n-almost every $\eta \in H$.

3° $\varphi(\cdot, u)$ is m-integrable over Z for all $Z \in \mathscr{Z}$, $u \in R_+$, and $\psi(\cdot, u)$ is n-integrable over Z^* for all $Z^* \in \mathscr{Z}^*$, $u \in R_+$.

Let $a = (Z, Z^*, c, u_0)$, where $Z \in \mathscr{Z}$, $Z^* \in \mathscr{Z}^*$, $c > 0$, $u_0 > 0$. The function ψ will be called *a-weaker than* φ if

$$\int_{Z^*} \psi(\eta, u)\, dn \leqslant c \int_{Z} \varphi(\xi, u)\, dm \quad \text{for all } u \geqslant u_0. \tag{1}$$

1.4. We shall denote by X the set of all $\mathscr{E}$-measurable functions $x\colon E \to R = (-\infty, \infty)$ such that the functions $\varphi(\xi, |x(t)|)$ and $\psi(\eta, |x(t)|)$ are $\mathscr{X} \times \mathscr{E}$-measurable in $\Xi \times E$ or $\mathscr{Y} \times \mathscr{E}$-measurable in $H \times E$, respectively.

The following lemma is easy to verify:

LEMMA. *Let (u_i) be a sequence of positive numbers and let (E_i) be a sequence of pairwise disjoint sets belonging to $\mathscr{E}$. Then the function*

$$x(t) = \begin{cases} u_i & \text{for } t \in E_i, \\ 0 & \text{for } t \notin E_i,\ i = 1, 2, \ldots, \end{cases} \tag{2}$$

belongs to X.

The following notation will be used:

$$g_\varphi(\xi) = \int_E \varphi(\xi, |x(t)|)\, d\mu, \qquad g_\psi(\eta) = \int_E \psi(\eta, |x(t)|)\, d\mu$$

for $\xi \in \Xi$, $\eta \in H$, where $x \in X$. Moreover, we denote by $L^1(Z, \mathscr{X}, m)$ and $L^1(Z^*, \mathscr{Y}, n)$ the spaces of functions m-integrable over the set $Z \in \mathscr{Z}$ or n-integrable over the set $Z^* \in \mathscr{Z}^*$, respectively.

2. The subject of this paper is to find the connections between m-integrability of g_φ over a set $Z \in \mathscr{Z}$ and n-integrability of g_ψ over a set $Z^* \in \mathscr{Z}^*$. There will be presented following theorems giving necessary and sufficient condition concerning this problem.

2.1. THEOREM A. *Let the family $\mathscr{Z}$ be σ-absorbed, and let the family $\mathscr{Z}^*$ be σ-absorbing. The following two conditions are equivalent:*

A.1. *if $g_\varphi \in L^1(Z, \mathscr{X}, m)$ for all $Z \in \mathscr{Z}$, then there exists $Z^* \in \mathscr{Z}^*$ such that $g_\psi \in L^1(Z^*, \mathscr{Y}, n)$,*

A.2. *the exist sets $Z \in \mathscr{Z}$, $Z^* \in \mathscr{Z}^*$ and numbers $c, u_0 > 0$ such that ψ is a-weaker than φ with $a = (Z, Z^*, c, u_0)$.*

Proof. Let us denote

$$T = \{t\colon |x(t)| \geqslant u_0,\ t \in E\}; \tag{3}$$

then

$$\begin{aligned}\int_{Z^*} g_\psi(\eta)\, dn &\leqslant \int_T \Big(\int_{Z^*} \psi(\eta, |x(t)|)\, dn\Big) d\mu + \mu(E) \cdot \int_{Z^*} \psi(\eta, u_0)\, dn \\ &\leqslant c \int_T \Big(\int_Z \varphi(\xi, |x(t)|)\, dm\Big) d\mu + \mu(E) \cdot \int_{Z^*} \psi(\eta, u_0)\, dn \\ &\leqslant c \int_Z g_\varphi(\xi)\, dm + \mu(E) \cdot \int_{Z^*} \psi(\eta, u_0)\, dn.\end{aligned}$$

Thus we obtain the inequality

$$\int_{Z^*} g_\psi(\eta)\,dn \leqslant c\cdot \int_{Z} g_\varphi(\xi)\,dm + \mu(E)\int_{Z^*} \psi(\eta, u_0)\,dn. \tag{4}$$

Now, let us suppose A.2, then, by the above inequality, we obtain A.1.

Now, let us suppose that A.2 does not hold. Then taking in (1), $Z^* = Z_i^*$, $Z = Z_i$, $c = 2^i$, u_0 sufficiently large, we see there exists a sequence $u_i \uparrow \infty$ such that

$$\int_{Z_i^*} \psi(\eta, u_i)\,dn > 2^i \int_{Z_i} \varphi(\xi, u_i)\,dm \qquad \text{for } i = 1, 2, \ldots,$$

where Z_i^* and Z_i are defined as in 1.2. Since $\varphi(\xi, u_i)\uparrow\infty$ for m-almost all $\xi\in\Xi$, we obtain $\int_{Z_i} \varphi(\xi, u_i)\,dm\uparrow\infty$. Hence, we may choose the sequence (u_i) in such a manner that $\int_{Z_i} \varphi(\xi, u_i)\,dm > 1$ for $i = 1, 2, \ldots$ Let us choose

$$\varepsilon_i = \frac{\mu(E)}{2^i \int_{Z_i} \varphi(\xi, u_i)\,dm}, \qquad i = 1, 2, \ldots; \tag{5}$$

then $\sum_{i=1}^{\infty} \varepsilon_i < \mu(E)$. Hence there exist pairwise disjoint sets $E_i\in\mathscr{E}$ such that $\mu(E_i) = \varepsilon_i$ for $i = 1, 2, \ldots$ Defining x as in (2) and taking arbitrary $Z\in\mathscr{Z}$ we find an index k such that $Z \subset Z_k$. Since $Z_k \subset Z_i$ for $i \geqslant k$, we have

$$\begin{aligned}\int_{Z} g_\varphi(\xi)\,dm &\leqslant \sum_{i=1}^{k-1} \mu(E_i)\int_{Z_k} \varphi(\xi, u_i)\,dm + \sum_{i=k}^{\infty} \mu(E_i)\int_{Z_k} \varphi(\xi, u_i)\,dm \\ &\leqslant \sum_{i=1}^{k-1} \varepsilon_i \int_{Z_k} \varphi(\xi, u_i)\,dm + \frac{\mu(E)}{2^{k-1}} < \infty.\end{aligned}$$

Hence $g_\varphi\in L^1(Z, \mathscr{X}, m)$ for every $Z\in\mathscr{Z}$. On the other hand, we have

$$\begin{aligned}\int_{Z^*} g_\psi(\eta)\,dn &\geqslant \sum_{i=k_0}^{\infty} \varepsilon_i \int_{Z^*} \psi(\eta, u_i)\,dn \geqslant \sum_{i=k_0}^{\infty} \varepsilon_i \int_{Z_i^*} \psi(\eta, u_i)\,dn \\ &\geqslant \sum_{i=k_0}^{\infty} \varepsilon_i\cdot 2^i \int_{Z} \varphi(\xi, u_i)\,dm = \infty,\end{aligned}$$

and so $g_\psi\notin L^1(Z^*, \mathscr{Y}, n)$. Consequently: A.1 is not satisfied.

THEOREM B. *The following two conditions are equivalent:*

B.1. *if there exists a set $Z\in\mathscr{Z}$ such that $g_\varphi\in L^1(Z, \mathscr{X}, m)$, then $g_\psi\in L^1(Z^*, \mathscr{Y}, n)$ for all $Z^*\in\mathscr{Z}^*$,* B.2. *for any* $Z \in \mathscr{Z}$ *and* $Z^* \in \mathscr{Z}^*$ *there exist constants* $c, u_0 > 0$ *such that* ψ *is* a*-weaker than* φ *with* $a = (Z, Z^*, c, u_0)$.

Proof. Supposing B.2, we apply inequality (4). By B.2, Z and Z^* may be chosen arbitrarily, and $g_\varphi \in L^1(Z, \mathscr{X}, m)$ implies the right-hand side of the above inequality to be finite for a set $Z \in \mathscr{Z}$. Hence $\int\limits_{Z^*} g_\psi(\eta)\,dn < \infty$ for any $Z^* \in \mathscr{Z}^*$, and this proves B.1.

Now, supposing B.2 is not true, there exist sets $Z \in \mathscr{Z}$, $Z^* \in \mathscr{Z}^*$ such that inequality (1) does not hold, where $c = 2^i$, $u = u_i \uparrow \infty$, $i = 1, 2, \ldots$, i.e.,

$$\int\limits_{Z^*} \psi(\eta, u_i)\,dn > 2^i \int\limits_{Z} \varphi(\xi, u_i)\,dm \quad \text{for } i = 1, 2, \ldots,$$

and we may suppose $\int\limits_{Z} \varphi(\xi, u_i)\,dm > 1$ for $i = 1, 2, \ldots$ Choosing

$$\varepsilon_i = \frac{\mu(E)}{2^i \int\limits_{Z} \varphi(\xi, u_i)\,dm}, \tag{6}$$

we have $\sum\limits_{i=1}^{\infty} \varepsilon_i < \mu(E)$. Let $E_i \in \mathscr{E}$ be pairwise disjoint and such that $\mu(E_i) = \varepsilon_i$ for $i = 1, 2, \ldots$, and let x be defined as in (2). Then

$$\int\limits_{Z} g_\varphi(\xi)\,dm = \sum_{i=1}^{\infty} \mu(E_i) \cdot \int\limits_{Z} \varphi(\xi, u_i)\,dm = \mu(E) < \infty,$$

but

$$\int\limits_{Z^*} g_\psi(\eta)\,dn = \sum_{i=1}^{\infty} \varepsilon_i \int\limits_{Z^*} \psi(\eta, u_i)\,dn \geqslant \sum_{i=1}^{\infty} \varepsilon_i 2^i \int\limits_{Z} \varphi(\xi, u_i)\,dm = \infty,$$

contrary to B.1.

THEOREM C. *Let the family* $\mathscr{Z}$ *be* σ*-absorbed. The following two conditions are equivalent*:

C.1. *if* $g_\varphi \in L^1(Z, \mathscr{X}, m)$ *for all* $Z \in \mathscr{Z}$, *then* $g_\psi \in L^1(Z^*, \mathscr{Y}, n)$ *for all* $Z^* \in \mathscr{Z}^*$,

C.2. *for every* $Z^* \in \mathscr{Z}^*$ *there exist* $Z \in \mathscr{Z}$ *and numbers* $c, u_0 > 0$ *such that* ψ *is* a*-weaker than* φ *with* $a = (Z, Z^*, c, u_0)$.

Proof. Let us suppose C.2. Then, by inequality (4), we obtain C.1, immediately. Now, let us suppose C.2 does not hold, then there exists $Z^* \in \mathscr{Z}^*$ such that for any $Z \in \mathscr{Z}$, $c, u_0 > 0$, it is not true that ψ is a-weaker than φ, for $a = (Z, Z^*, c, u_0)$. We take $Z = Z_i$ as defined in 1.3, $c = 2^i$, u_0 sufficiently large. Then there exists a sequence $u_i \uparrow \infty$ such that

$$\int\limits_{Z^*} \psi(\eta, u_i)\,dn > 2^i \int\limits_{Z_i} \varphi(\xi, u_i)\,dm \quad \text{for } i = 1, 2, \ldots,$$

where we may suppose that $\int_{Z_i} \varphi(\xi, u_i)dm > 1$. Defining numbers $\varepsilon_i > 0$ by (5) and choosing sets E_i and function x as in the proof of Theorem A, we have for any $Z \in \mathscr{Z}$ with $Z \subset Z_k$:

$$\int_Z g_\varphi(\xi)dm \leqslant \sum_{i=1}^{k=1} \varepsilon_i \int_{Z_k} \varphi(\xi, u_i)dm + \frac{\mu(E)}{2^{k-1}} < \infty,$$

and

$$\int_{Z^*} g_\psi(\eta)dn \geqslant \sum_{i=1}^{\infty} \varepsilon_i 2^i \int_{Z^*} \varphi(\xi, u_i)dm = \infty,$$

contrary to C.1.

THEOREM D. *Let the family $\mathscr{Z}^*$ be σ-absorbing. The following two conditions are equivalent:*

D.1. *if there exists $Z \in \mathscr{Z}$ for which $g_\varphi \in L^1(Z, \mathscr{X}, m)$, then there exists $Z^* \in \mathscr{Z}^*$ such that $g_\psi \in L^1(Z^*, \mathscr{Y}, n)$,*

D.2. *for every $Z \in \mathscr{Z}$ there exist $Z^* \in \mathscr{Z}^*$ and constants $c, u_0 > 0$ such that ψ is a-weaker than φ with $a = (Z, Z^*, c, u_0)$.*

Proof. The sufficiency of D.2 follows from inequality (4). In order to prove the necessity, let us suppose D.2 is not true. Then there exists $Z \in \mathscr{Z}$ such that for any $Z^* \in \mathscr{Z}^*$, $c, u_0 > 0$, the condition ψ is a-weaker than φ is not satisfied for $a = (Z, Z^*, c, u_0)$. Let $Z^* = Z_i^*$ be defined as in 1.3, $c = 2^i$, u_0 sufficiently large; then there is a sequence $u_i \uparrow \infty$ such that $\int_Z \varphi(\xi, u_i)dm > 1$ and

$$\int_{Z_i^*} \psi(\eta, u_i)dn > 2^i \int_Z \varphi(\xi, u_i)dm \quad \text{for } i = 1, 2, \ldots$$

Now, let ε_i be defined by (6) and let E_i and the function x be chosen as in the proof of Theorem B. Then we have

$$\int_Z g_\varphi(\xi)dm = \sum_{i=1}^{\infty} \mu(E_i) \int_Z \varphi(\xi, u_i)dm = \mu(E) < \infty.$$

Let $Z^* \in \mathscr{Z}^*$ be arbitrary and let k_0 be chosen in such a manner that $Z_k^* \subset Z^*$ for all $k \geqslant k_0$. Then

$$\int_{Z^*} g_\psi(\eta)dn \geqslant \sum_{i=k_0}^{\infty} \varepsilon_i \int_{Z_i^*} \psi(\eta, u_i)dn \geqslant \sum_{i=k_0}^{\infty} \varepsilon_i 2^i \int_Z \varphi(\xi, u_i)dm = \infty,$$

a contradiction to D.1.

2.2. Let us remark that in case of $\mu(E) = \infty$, the condition about sets $E_i \in \mathscr{E}$ in 1.1 is to be replaced by the following one: for any sequence of numbers $\varepsilon_i > 0$ there exists a sequence of pairwise disjoint sets $E_i \in \mathscr{E}$

such that $\mu(E_i) = \varepsilon_i$ for $i = 1, 2, \ldots$ Then it is easily observed that Theorems A–D remain true taking $u_0 = 0$.

3. Now, let $\Xi = H =$ the set of positive integers, $\mathscr{X} = \mathscr{Y} =$ the set of all subsets of $\Xi = H$, $m = n =$ the measure assigning to each one-point set the number 1. Now, in place of $\varphi(\xi, u)$ and $\psi(\eta, u)$ we may write $\varphi_i(u)$, $\psi_i(u)$, $i = 1, 2, \ldots$ The conditions $g_\varphi \in L^1(Z, \mathscr{X}, m)$ and $g_\psi \in L^1(Z^*, \mathscr{Y}, n)$ mean that

$$\sum_{i\in Z} \int_E \varphi_i\big(|x(t)|\big) d\mu < \infty \quad \text{or} \quad \sum_{i\in Z^*} \int_E \psi_i\big(|x(t)|\big) d\mu < \infty .$$

Now, let $\mathscr{Z} = \mathscr{Z}^*$ be the family of sets $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}, \ldots$ Then it is easily seen that $\mathscr{Z}$ is σ-absorbed with $Z_i = \{1, 2, \ldots, i\}$, and $\mathscr{Z}^*$ is σ-absorbing with $z_i^* = \{1\}$ for $i = 1, 2, \ldots$ Moreover, since the sets $Z \in \mathscr{Z}$ are finite, so $g_\varphi \in L^1(Z, \mathscr{X}, m)$ means that $x \in L^{\varphi_i}(E, \mathscr{E}, \mu)$ for all $i \in Z$, and $g_\psi \in L^1(Z^*, \mathscr{Y}, n)$ means that $x \in L^{\psi_i}(E, \mathscr{E}, \mu)$ for all $i \in Z^*$, where $L^{\varphi_i}(E, \mathscr{E}, \mu)$ and $L^{\psi_i}(E, \mathscr{E}, \mu)$ are Orlicz classes generated by φ-functions φ_i and ψ_i, respectively.

Moreover, if $Z = \{1, 2, \ldots, k\}$, $Z^* = \{1, 2, \ldots, l\}$, $c, u_0 > 0$, then (ψ_i) is α-weaker than (φ_i) if and only if

$$\sum_{j=1}^{l} \psi_j(u) \leqslant c \cdot \sum_{i=1}^{k} \varphi_i(u) \quad \text{for } u \geqslant u_0. \tag{7}$$

Consequently, the following corollaries follow from Theorems A–D:

3.1. Corollary A. *The following two conditions are equivalent*:

A′ $\bigcap_{i=1}^{\infty} L^{\varphi_i}(E, \mathscr{E}, \mu) \subset \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{i} L^{\psi_j}(E, \mathscr{E}, \mu)$,

A″ *there exist indices k, l and numbers $c, u_0 > 0$ such that* (7) *holds.*

Corollary B. *The following two conditions are equivalent*:

B′ $\bigcup_{i=1}^{\infty} \bigcap_{j=1}^{i} L^{\varphi_j}(E, \mathscr{E}, \mu) \subset \bigcap_{i=1}^{\infty} L^{\psi_i}(E, \mathscr{E}, \mu)$,

B″ *for any two positive integers k, l there exist constants $c, u_0 > 0$ such that* (7) *holds.*

Corollary C. *The following two conditions are equivalent*:

C′ $\bigcap_{i=1}^{\infty} L^{\varphi_i}(E, \mathscr{E}, \mu) \subset \bigcap_{i=1}^{\infty} L^{\psi_i}(E, \mathscr{E}, \mu)$,

C″ *for every index l there exists an index k and numbers $c, u_0 > 0$ such that* (7) *holds.*

Corollary D. *The following two conditions are equivalent*:

D′ $\bigcup_{i=1}^{\infty} \bigcap_{j=1}^{i} L^{\varphi_j}(E, \mathscr{E}, \mu) \subset \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{i} L^{\psi_j}(E, \mathscr{E}, \mu)$, D″ *for every index k there exists an index l and numbers $c, u_0 > 0$ such that* (7) *holds.*

3. Let us remark that taking as $\mathscr{Z} = \mathscr{Z}^*$ the family of all finite non-empty sets of positive integers, we see that $\mathscr{Z}$ remains σ-absorbed, but is not σ-absorbing. Hence Theorem B may be applied. Thus, in Corollary B one may replace the conditions B′ and B″ by the following ones:

$\overline{\mathrm{B}}'$ $\bigcup_{i=1}^{\infty} L^{\varphi_i}(E, \mathscr{E}, \mu) \subset \bigcap_{i=1}^{\infty} L^{\psi_i}(E, \mathscr{E}, \mu)$,

$\overline{\mathrm{B}}''$ *for any two finite sets of positive integers Z, Z^* there exist constants $c, u_0 > 0$ such that*

$$\sum_{j\in Z} \psi_j(u) \leqslant c \sum_{i\in Z^*} \varphi_i(u) \quad \textit{for } u \geqslant u_0.$$

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