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On the Banach principle and its application to the theory of differential equations

K. Goebel [3] has proved the following theorem:

Let A be an arbitrary set and let M be a metric space with the metric ϱ. Suppose that S, T are two transformations defined on the set A with the values in M. If $S[A] \subset T[A]$ and $T[A]$ is a complete subspace of M and if for all $x, y \in A$

$$\varrho(Sx, Sy) \leqslant k\varrho(Tx, Ty),$$

where $0 \leqslant k < 1$ holds, then

1° *there exists $x_0 \in A$ such that*

$$(*) \qquad Sx_0 = Tx_0;$$

2° *if x_0 satisfies (*) and $Tx = Tx_0$, then $Sx = Tx = Sx_0 = Tx_0$;*

3° *if each of elements x_0, y_0 satisfies (*), then $Tx_0 = Ty_0$.*

This theorem generalizes the well-known Banach fixed-point principle and is connected with Bielecki's method [1] of changing the norm in the theory of differential equations.

In this paper we give a version of the Goebel's result which enables us to get the global theorems on continuous dependence of a differential or differential-like equation solution on initial conditions, its right-hand side and parameter.

1. THEOREM. *Suppose that A is an arbitrary set and let B and M be metric spaces. Assume, moreover, that S, T are two transformations defined on the set $A \times B$ with the values in M such that for all $y \in B$*

1° *$\{S(x, y): x \in A\} \subset \{T(x, y): x \in A\}$ and $\{T(x, y): x \in A\}$ is a complete subspace of M,*

2° *there exists $\alpha(y) \in [0, 1)$ such that*

$$\varrho\big(S(x_1, y),\ S(x_2, y)\big) \leqslant \alpha(y)\varrho\big(T(x_1, y),\ T(x_2, y)\big)$$

for every $x_1, x_2 \in A$, where ϱ denotes the metric on M, 3° *the equation* $S(x, y) = T(x, y)$ *has at most one solution* $x \in A$.

If the functions $S(x, \cdot)$ *and* $T(x, \cdot)$ *are continuous on* B *and if* $y \mapsto a(y)$ *is a continuous function on* B, *then there exists a unique function* $\psi\colon B \to A$ *such that* $S(\psi(y), y) = T(\psi(y), y)$ *for every* $y \in B$ *and functions* $S(\psi(\cdot), \cdot)$ *and* $T(\psi(\cdot), \cdot)$ *are continuous on* B.

Proof. Let us fix $y \in B$. In view of Goebel's theorem, there exists $x_0 \in A$ such that $S(x_0, y) = T(x_0, y)$ and therefore x_0 is determined uniquely. Consequently, there exists a unique function $\psi\colon B \to A$ such that $S(\psi(y), y) = T(\psi(y), y)$ for $y \in B$.

Let $y_1, y_2 \in B$. We have

$$\begin{aligned}\varrho\big(T(\psi(y_1), y_1),\ T(\psi(y_2), y_2)\big) &= \varrho\big(S(\psi(y_1), y_1),\ S(\psi(y_2), y_2)\big)\\ &\leqslant a(y_1)\cdot\varrho\big(T(\psi(y_1), y_1),\ T(\psi(y_2), y_2)\big)+\\ &\quad + a(y_1)\cdot\varrho\big(T(\psi(y_2), y_2),\ T(\psi(y_2), y_1)\big)+\\ &\quad + \varrho\big(S(\psi(y_2), y_1),\ S(\psi(y_2), y_2)\big)\end{aligned}$$

hence

$$\begin{aligned}\varrho\big(S(\psi(y_1), y_1),\ S(\psi(y_2), y_2)\big) &= \varrho\big(T(\psi(y_1), y_1),\ T(\psi(y_2), y_2)\big)\\ &\leqslant \frac{a(y_1)}{1-a(y_1)}\cdot\varrho\big(T(\psi(y_2), y_2),\ T(\psi(y_2), y_1)\big)+\\ &\quad + \frac{1}{1-a(y_1)}\cdot\varrho\big(S(\psi(y_2), y_1),\ S(\psi(y_2), y_2)\big)\\ &< \frac{1}{1-a(y_1)}\big[\varrho\big(T(\psi(y_2), y_2),\ T(\psi(y_2), y_1)\big)+\\ &\quad + \varrho\big(S(\psi(y_2), y_1),\ S(\psi(y_2), y_2)\big)\big].\end{aligned}$$

Finally, $T(\psi(\cdot), \cdot)$ and $S(\psi(\cdot), \cdot)$ are continuous on B.

Remark. Let $y_0 \in B$ and let for $y = y_0$ condition 2° be satisfied. If T is one-to-one, then there exists at most one element $x \in A$ such that $S(x, y_0) = T(x, y_0)$.

From the corollary we obtain [2]:

Let E *be an arbitrary metric space and let* M *be a complete metric space with the metric* ϱ. *Suppose that* $S\colon M \times E \to M$ *is a transformation such that*

(i) *for all* $y \in E$ *there exists* $a(y) \in [0, 1)$ *such that* $\varrho\big(S(x_1, y),\ S(x_2, y)\big) \leqslant a(y)\cdot\varrho(x_1, x_2)$ *for* $x_1, x_2 \in M$;

(ii) *for all* $x \in M$ *the function* $S(x, \cdot)$ *is continuous on* E.

If $y \mapsto \alpha(y)$ is a continuous function on E, then there exists a unique continuous function $\psi\colon E \to M$ such that $S(\psi(y), y) = \psi(y)$ for an arbitrary $y \in E$.

2. Now, we are going to give some examples of applications of Theorem 1.

I. Consider the differential equation

$$x' = f(t, x) \tag{1}$$

(cf. [1], [3]). We introduce

ASSUMPTION (I). *Suppose that*

1° *the function $f\colon [0, \infty) \times (-\infty, \infty) \to (-\infty, \infty)$ satisfies the Carathéodory's conditions and Lipschitz conditions: there exists a locally integrable function $L\colon [0, \infty) \to [0, \infty)$ such that*

$$|f(t, x_1) - f(t, x_2)| \leqslant L(t)|x_1 - x_2|$$

for every $t \geqslant 0$ and $-\infty < x_1, x_2 < \infty$;

2° $\int_0^t f(s, 0)\,ds = O\left(\exp \int_0^t L(s)\,ds\right)$ *for* $t \geqslant 0$.

Denote by $C[0, \infty)$ the Banach space of bounded continuous functions on $[0, \infty)$ with the usual norm $\|x\| = \sup\{|x(t)|\colon t \geqslant 0\}$. In [3] it has been proved that assuming (I) equation (1) has for every η exactly one solution $x \in C[0, \infty)$ with the initial condition $x(0) = \eta$. Let us prove:

2.1. *Let assumption* (I) *be satisfied. Then*

1° *equation* (1) *has for every $P = \eta \in (-\infty, \infty)$ exactly one solution $x_P \in C[0, \infty)$ with the initial condition $x_P(0) = \eta$ and $x_P(t) = O\left(\exp\left(p \cdot \int_0^t L(s)\,ds\right)\right)$, where $p > 1$;*

2° *if $(P_n) \to P_0$, where $P_n, P_0 \in (-\infty, \infty)$, then $\lim_{n\to\infty} x_{P_n}(t) = x_{P_0}(t)$ uniformly in every finite subinterval from $[0, \infty)$.*

Proof. Let $p > 1$,

$$A = \left\{x \in C[0, \infty)\colon x(t) = O\left(\exp\left(p \cdot \int_0^t L(s)\,ds\right)\right) \quad \text{for } t \geqslant 0\right\}$$

and let $B = (-\infty, \infty)$ with the Euclidean metric. The transformations T and S are defined by

$$T(x, P)(t) = x(t)\exp\left(-p \cdot \int_0^t L(s)\,ds\right),$$

$$S(x, P)(t) = \left(\eta + \int_0^t f(s, x(s))\,ds\right) \cdot \exp\left(-p \cdot \int_0^t L(s)\,ds\right)$$

for $x \in A$ and $P = \eta \in B$. Then $S, T\colon A \times B \to C[0, \infty)$ and $\{T(x, P)\colon x \in A\} = C[0, \infty)$ for every $P \in B$.

Fix $P_0 = \eta_0 \in B$. Then $S(x, P_0) = T(\bar{x}, P_0)$ for $x \in A$ and $\bar{x}(t) = \eta_0 + \int_0^t f(s, x(s))\,ds$ for $t \geqslant 0$. Since

$$|\bar{x}(t)| \leqslant |\eta_0| + \left|\int_0^t f(s, 0)\,ds\right| + \int_0^t L(s)\,|x(s)|\,ds$$

and

$$\int_0^t L(s)\exp\Big(p\cdot\int_0^s L(u)\,du\Big)\,ds = p^{-1}\Big(\exp\Big(p\cdot\int_0^t L(u)\,du\Big) - 1\Big),$$

we obtain $\bar{x} \in A$ and therefore $\{S(x, P_0)\colon x \in A\} \subset \{T(x, P_0)\colon x \in A\}$. It can be easily seen (cf. [4]), that

$$\|S(x_1, P_0) - S(x_2, P_0)\| \leqslant p^{-1}\cdot\|T(x_1, P_0) - T(x_2, P_0)\|$$

for $x_1, x_2 \in A$. Now, we shall prove that the equation $S(x, P_0) = T(x, P_0)$ has at most one solution in A.

Let $x_i \in A$ and let $S(x_i, P_0) = T(x_i, P_0)$ for $i = 1, 2$. Then

$$|x_1(t) - x_2(t)| \leqslant \int_0^t L(s)\,|x_1(s) - x_2(s)|\,ds$$
$$\leqslant p^{-1}\cdot\|x_1 - x_2\|_p\Big(\exp\Big(p\cdot\int_0^t L(s)\,ds\Big) - 1\Big),$$

where

$$\|x_1 - x_2\|_p = \sup\Big\{|x_1(t) - x_2(t)|\exp\Big(-p\cdot\int_0^t L(s)\,ds\Big)\colon\ t \geqslant 0\Big\}.$$

Hence

$$p\cdot\|x_1 - x_2\|_p \leqslant \|x_1 - x_2\|_p \sup\Big\{1 - \exp\Big(-p\cdot\int_0^t L(s)\,ds\Big)\colon\ t \geqslant 0\Big\}$$

and therefore $\|x_1 - x_2\|_p = 0$.

By Theorem 1 there exists a unique function $h\colon B \to A$ such that $T(h(P), P) = S(h(P), P)$ for $P \in B$ and if $P_n = \eta_n$, $P_0 = \eta_0 \in B$, $(\eta_n) \to \eta_0$, then

$$\|h(P_n) - h(P_0)\|_p = \sup_{t \geqslant 0}\Big|\Big(\eta_n + \int_0^t f\big(s, h(P_n)(s)\big)\,ds\Big) -$$
$$-\Big(\eta_0 + \int_0^t f\big(s, h(P_0)(s)\big)\,ds\Big)\Big|\cdot\exp\Big(-p\int_0^t L(s)\,ds\Big) \to 0 \quad \text{as } n \to \infty.$$

II. Let $a > 0$. Let us denote by C_t $(t \in [0, a])$ the set of all bounded continuous functions on $(-\infty, t]$. We introduce

ASSUMPTION (II). *Let $G(t, x)$ be a functional defined for $t \in [0, a]$, and $x \in C_t$. Suppose that*

1° *there exists a continuous function q: $[0, a] \to [0, \infty)$ such that*

$$|G(t, v_1) - G(t, v_2)| \leqslant q(t) \cdot \sup\{|v_1(s) - v_2(s)|: s \in (-\infty, t]\}$$

for every $(t, v_1), (t, v_2) \in [0, a] \times C_t$,

2° *for fixed $h \in C_t$ the function $G(\cdot, h)$ is continuous on $[0, t]$.*

Consider the differential-functional equation

$$(2) \qquad \begin{aligned} x(t) &= \varphi(t) && \text{for } t \in (-\infty, 0], \\ x'(t) &= f\big(t, G(t, x)\big) && \text{for } t \in [0, a], \end{aligned}$$

where φ: $(-\infty, 0] \to (-\infty, \infty)$ and f: $[0, a] \times (-\infty, \infty) \to (-\infty, \infty)$.

Let us denote:

by $C(-\infty, a]$ — the Banach space of bounded continuous functions on $(-\infty, a]$ with the usual norm $\|\cdot\|$;

by $\mathscr{F}$ — the space of bounded continuous functions f on $[0, a] \times (-\infty, \infty)$ satisfing the Lipschitz condition: there exists a constant $L_f > 0$ such that $|f(t, x_1) - f(t, x_2)| \leqslant L_f |x_1 - x_2|$ for $t \in [0, a]$ and $-\infty < x_1, x_2 < \infty$, with the norm

$$|||f||| = \sup\{|f(t, x)|: (t, x) \in [0, a] \times (-\infty, \infty)\}.$$

2.2. *Let assumption* (II) *be satisfied and let φ be a defined and continuous bounded function on $(-\infty, 0]$.*

Then, for an arbitrary $f \in \mathscr{F}$ there exists a unique function $x_f \in C(-\infty, a]$, equal identically to the function φ on the set $(-\infty, 0]$ and such that

$$x_f'(t) = f\big(t, G(t, x_f)\big) \quad \text{for } t \in [0, a].$$

Assume, moreover, that $\sup\{L_f: f \in \mathscr{F}\} < \infty$. *Then the function $f \mapsto x_f$ maps continuously $\mathscr{F}$ into $C(-\infty, a]$.*

Proof. Let $p > 1$ and let A denote the set of continuous functions x: $(-\infty, a] \to (-\infty, \infty)$, which are equal identically to the function φ on the set $(-\infty, 0]$ and such that

$$\sup\{|x(t)| \cdot \exp(-pt): t \in (-\infty, a]\} < \infty.$$

For each $(x, f) \in A \times \mathscr{F}$, define

$$T(x, f)(t) = x(t) \cdot \exp(-pt)$$

and

$$S(x, f)(t) = \begin{cases} \varphi(t) \cdot \exp(-pt) & \text{for } t \in (-\infty, 0], \\ \Big(\varphi(0) + \int\limits_0^t f\big(s, G(s, x)\big)\, ds\Big) \cdot \exp(-pt) & \text{for } t \in [0, a]. \end{cases}$$

Evidently, S and T map the set $A\times\mathscr{F}$ into the space $C(-\infty, a]$ and $\{S(x,f)\colon x\in A\}\subset\{T(x,f)\colon x\in A\}$ for $f\in\mathscr{F}$.

Let $x_n\in A$ and let $\|T(x_n,f)-y_0\|\to 0$ as $n\to\infty$. Put

$$x_0(t)=\begin{cases}\varphi(t) & \text{for } t\in(-\infty,0],\\ y_0(t)\cdot\exp(pt) & \text{for } t\in[0,a].\end{cases}$$

Since

$$y_0(t)=\varphi(t)\cdot\exp(-pt) \quad \text{for } t\in(-\infty,0],$$

we have $T(x_0,f)=y_0$ and $x_0\in A$. Consequently, $\{T(x,f)\colon x\in A\}$ is a complete subspace of $C(-\infty,a]$.

Let the function $f\in\mathscr{F}$ satisfy the Lipschitz condition with a constant L_f. For $t\in[0,a]$ and $x_1,x_2\in A$, we have

$$\begin{aligned}
&|S(x_1,f)(t)-S(x_2,f)(t)|\\
&\quad\leqslant \exp(-pt)\cdot\int_0^t\big|f\big(s,G(s,x_1)\big)-f\big(s,G(s,x_2)\big)\big|\,ds\\
&\quad\leqslant L_f\cdot\exp(-pt)\cdot\int_0^t q(s)\sup_{0\leqslant u\leqslant s}|x_1(u)-x_2(u)|\,ds\\
&\quad\leqslant L_f\cdot\sup_{0\leqslant t\leqslant a}q(t)\cdot\exp(-pt)\int_0^t\exp(ps)\cdot\sup\{\exp(-ps)\,|x_1(s)-\\
&\qquad\qquad -x_2(s)|\colon s\in[0,a]\}\,ds\\
&\quad\leqslant p^{-1}L_f\sup_{0\leqslant t\leqslant a}q(t)\,\|T(x_1,f)-T(x_2,f)\|
\end{aligned}$$

and it follows

$$\|S(x_1,f)-S(x_2,f)\|\leqslant p^{-1}L_f\sup_{0\leqslant t\leqslant a}q(t)\cdot\|T(x_1,f)-T(x_2,f)\|.$$

Fix $x\in A$. Let $f_n\in\mathscr{F}$ and $|||f_n-f_0|||\to 0$. We have

$$\begin{aligned}
&|S(x,f_n)(t)-S(x,f_0)(t)|\\
&\qquad\leqslant\exp(-pt)\cdot\int_0^t\big|f_n\big(s,G(s,x)\big)-f_0\big(s,G(s,x)\big)\big|\,ds\leqslant a\,|||f_n-f_0|||
\end{aligned}$$

for $t\in[0,a]$ and it follows

$$\|S(x,f_n)-S(x,f_0)\|\to 0 \quad \text{as } n\to\infty.$$

The function

$$f\mapsto p^{-1}\sup_{0\leqslant t\leqslant a}q(t)\cdot\sup_{f\in\mathscr{F}}L_f$$

is constant on $\mathscr{F}$. The application of Theorem 1 completes the proof.

Remark 1. Let $h\colon [0, a] \to (-\infty, a]$ be a continuous function such that $h(t) \leqslant t$ for $t \in [0, a]$. Then the functional

$$G(t, x) = x\big(h(t)\big)$$

satisfies assumption (II) and equation (2) takes the form

$$\begin{aligned} x(t) &= \varphi(t) && \text{for } t \in (-\infty, 0], \\ x'(t) &= f\Big(t, x\big(h(t)\big)\Big) && \text{for } t \in [0, a]. \end{aligned}$$

Remark 2. If

$$G(t, x) = \int_0^\infty x(t-s)\, d_s r(t, s),$$

we get

$$\begin{aligned} x(t) &= \varphi(t) && \text{for } t \in (-\infty, 0], \\ x'(t) &= f\Big(t, \int_0^\infty x(t-s)\, d_s r(t, s)\Big) && \text{for } t \in [0, a], \end{aligned}$$

where the integral is in the sense of Stieltjes. In order that the above functional G satisfy assumption (II), it is necessary to choose a suitable function which appears in the functional's definition.

Suppose that the function $r\colon [0, a] \times [0, \infty) \to (-\infty, \infty)$ satisfies the following conditions

1° $r(t, 0) = 0$ for $t \in [0, a]$;

2° there exists a continuous function $V\colon [0, a] \to [0, \infty)$ such that the total variation of the function r with respect to the second variable verifies the inequality

$$\bigvee_{s=0}^{\infty} r(t, s) \leqslant V(t) \quad \text{for } t \in [0, a];$$

3° for every $\varepsilon > 0$ there exists a number $K > 0$ such that

$$\bigvee_{s=K}^{\infty} r(t, s) < \varepsilon \quad \text{for } t \in [0, a];$$

4° for every $k > 0$ and $\tau \in [0, a]$

$$\lim_{t \to \tau} \int_0^k |r(t, s) - r(\tau, s)|\, ds = 0, \quad \text{where } t \in [0, a].$$

Under the above assumptions if $x \in C(-\infty, a]$, then the Stieltjes integral has its meaning $\int_0^\infty x(t-s)\, d_s r(t, s)$ and it is a continuous function of the variable t (cf. [1], [4]). Moreover,

$$|G(t, x_1) - G(t, x_2)| \leqslant \sup_{s\geqslant 0} |x_1(t-s) - x_2(t-s)| \bigvee_{s=0}^{\infty} r(t, s)$$
$$\leqslant V(t)\cdot \sup_{s\geqslant 0} \{ \sup_{u\leqslant t-s} |x_1(u) - x_2(u)|\}$$
$$\leqslant V(t)\cdot \sup_{u\leqslant t} |x_1(u) - x_2(u)|$$

for $t\in[0, a]$ and $x_1, x_2 \in C(-\infty, a]$. Consequently, the functional G satisfies (II).

III. Let $a, b > 0$ and $P = [0, a]\times[0, b]$. Consider the following partial differential equation

$$\frac{\partial^2 z(x, y)}{\partial x \partial y} = f\big(x, y, z(x, y)\big), \tag{3}$$

where f is defined and continuous over $P\times(-\infty, \infty)$.

Let that the functions σ and τ be respectively of the class $C^1[0, a]$ and $C^1[0, b]$ satisfying the condition $\sigma(0) = \tau(0)$. Then the Darboux problem for equation (3) is equivalent to solution of the following integral equation

$$z(x, y) = z_0(x, y) + \int_0^x \int_0^y f\big(u, v, z(u, v)\big) du dv,$$

where $z_0(x, y) = \sigma(x) + \tau(y) - \sigma(0)$.

Let us denote:

by $C(P)$ — the Banach space of continuous functions on P with the usual norm $\|\cdot\|$;

by $\mathscr{F}$ — the space of bounded continuous functions f on $P\times(-\infty, \infty)$ satisfing the Lipschitz condition: there exists a constant $L_f > 0$ such that $|f(x, y, z_1) - f(x, y, z_2)| \leqslant L_f|z_1 - z_2|$ for $(x, y)\in P$ and $z_1, z_2 \in (-\infty, \infty)$, with the usual norm $|||\cdot|||$;

by $\mathscr{X}$ — the space of points $(\sigma, \tau)\in C^1[0, a]\times C^1[0, b]$ such that $\sigma(0) = \tau(0)$, with the usual metric.

2.3. *For an arbitrary* $f\in\mathscr{F}$ *and* $\sigma\in C^1[0, a]$, $\tau\in C^1[0, b]$ *such that* $\sigma(0) = \tau(0)$ *there exists a unique function* $z_{(f,\sigma,\tau)}\in C(P)$ *satisfy equation* (3) *on* P *and such that* $z_{(f,\sigma,\tau)}(x, 0) = \sigma(x)$ *for* $x\in[0, a]$ *and* $z_{(f,\sigma,\tau)}(0, y) = \tau(y)$ *for* $y\in[0, b]$.

If $\sup\{L_f\colon f\in\mathscr{F}\} < \infty$, *then the function*

$$(f, \sigma, \tau) \mapsto z_{(f,\sigma,\tau)}$$

maps continuously $\mathscr{F}\times\mathscr{X}$ *into* $C(P)$.

The proof of this result follows exactly the same pattern as that of 2.1 and 2.2. In this case, we define the transformations T and S on

$C(P)\times\mathscr{F}\times\mathscr{X}$ by:

$$T\big(z,f,(\sigma,\tau)\big)(x,y) = z(x,y)\exp\big(-p(x+y)\big)$$

and

$$S\big(z,f,(\sigma,\tau)\big)(x,y)$$
$$= \Big[\sigma(x)+\tau(y)-\sigma(0)+\int_0^x\int_0^y f\big(u,v,z(u,v)\big)\,du\,dv\Big]\exp\big(-p(x+y)\big),$$

where $p>1$.

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