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Determinant version of Taylor theorems and some generalizations

Abstract. It is shown that Taylor's theorem in either of its derivative or integral remainder forms has a simple determinant formulation and proof. Also, some generalizations involving several functions of a single variable are derived.

In what follows all functions are real-valued and of a single real variable.

Without loss of generality and for the sake of clarity of the exposition we give below the determinant formulation and proof for the special case of $n = 2$ of Taylor's theorem in its derivative remainder form.

THEOREM 1. *Let f be a function whose 2-nd derivative is continuous on the closed interval $[a, b]$ with $a < b$ and let the third derivative f''' exist in the open interval (a, b). Then there exists c with $a < c < b$ such that:*

$$(1)\qquad \begin{vmatrix} f(b) & b^3 & b^2 & b & 1 \\ f(a) & a^3 & a^2 & a & 1 \\ f'(a) & 3a^2 & 2a & 1 & 0 \\ f''(a) & 6a & 2 & 0 & 0 \\ f'''(c) & 6 & 0 & 0 & 0 \end{vmatrix} = 0.$$

Proof. Let us consider function F given by:

$$F(x) = \begin{vmatrix} f(b) & b^3 & b^2 & b & 1 \\ f(a) & a^3 & a^2 & a & 1 \\ f'(a) & 3a^2 & 2a & 1 & 0 \\ f''(a) & 6a & 2 & 0 & 0 \\ f(x) & x^3 & x^2 & x & 1 \end{vmatrix}.$$

If in the last row of the above determinant we replace x by a we obtain the second row and therefore $F(a) = 0$. Similarly, if in the last row of the above determinant we replace x by b we obtain the first row and therefore $F(b) = 0$. Thus, $F(a) = F(b) = 0$. Moreover, from the hypothesis of Theorem 1 it follows that F is continuous on $[a, b]$ and F' exists in (a, b). Hence, by Rolle's theorem there exists e such that

$$(2)\qquad a < e < b$$

and

(3) $$F'(e) = \begin{vmatrix} f(b) & b^3 & b^2 & b & 1 \\ f(a) & a^3 & a^2 & a & 1 \\ f'(a) & 3a^2 & 2a & 1 & 0 \\ f''(a) & 6a & 2 & 0 & 0 \\ f'(e) & 3e^2 & 2e & 1 & 0 \end{vmatrix} = 0.$$

Next, let us consider function F' given by:

$$F'(x) = \begin{vmatrix} f(b) & b^3 & b^2 & b & 1 \\ f(a) & a^3 & a^2 & a & 1 \\ f'(a) & 3a^2 & 2a & 1 & 0 \\ f''(a) & 6a & 2 & 0 & 0 \\ f'(x) & 3x^2 & 2x & 1 & 0 \end{vmatrix}.$$

If in the last row of the above determinant we replace x by a we obtain the third row and therefore $F'(a) = 0$. On the other hand, (3) shows that $F'(e) = 0$. Thus, $F'(a) = F'(e) = 0$. Moreover, from the hypothesis of Theorem 1 it follows that F' is continuous on $[a, b]$ and F'' exists in (a, b). Hence, by Rolle's theorem there exists h such that

(4) $$a < h < e$$

and

(5) $$F''(h) = \begin{vmatrix} f(b) & b^3 & b^2 & b & 1 \\ f(a) & a^3 & a^2 & a & 1 \\ f'(a) & 3a^2 & 2a & 1 & 0 \\ f''(a) & 6a & 2 & 0 & 0 \\ f''(h) & 6h & 2 & 0 & 0 \end{vmatrix} = 0.$$

Finally, let us consider function F'' given by:

$$F''(x) = \begin{vmatrix} f(b) & b^3 & b^2 & b & 1 \\ f(a) & a^3 & a^2 & a & 1 \\ f'(a) & 3a^2 & 2a & 1 & 0 \\ f''(a) & 6a & 2 & 0 & 0 \\ f''(x) & 6x & 2 & 0 & 0 \end{vmatrix}.$$

If in the last row of the above determinant we replace x by a we obtain the fourth row and therefore $F''(a) = 0$. On the other hand, (5) shows that $F''(h) = 0$. Thus, $F''(a) = F''(h) = 0$. Moreover, from the hypothesis of Theorem 1 it follows that F'' is continuous on $[a, b]$ and F''' exists in (a, b). Hence, by Rolle's theorem there exists c such that

$$a < c < h \tag{6}$$

and

$$F'''(c) = \begin{vmatrix} f(b) & b^3 & b^2 & b & 1 \\ f(a) & a^3 & a^2 & a & 1 \\ f'(a) & 3a^2 & 2a & 1 & 0 \\ f''(a) & 6a & 2 & 0 & 0 \\ f'''(c) & 6 & 0 & 0 & 0 \end{vmatrix} = 0. \tag{7}$$

Clearly, (7) establishes (1) and from (2), (4), (6) it follows that $a < c < b$. Thus, Theorem 1 is proved.

It can be readily verified (by expanding the determinant appearing in (1)) that (1) implies

$$\frac{1}{2!3!}\begin{vmatrix} f(b) & \dots & 1 \\ \dots & \dots & \dots \\ f'''(c) & \dots & 0 \end{vmatrix} = 0$$

$$= f(b) - f(a) - \frac{f'(a)}{1}(b-a) - \frac{f''(a)}{2!}(b-a)^2 - \frac{f'''(c)}{3!}(b-a)^3$$

and therefore,

$$f(b) = f(a) + \frac{f'(a)}{1}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \frac{f'''(c)}{3!}(b-a)^3$$

which is the classical formulation of Taylor's theorem in its derivative remainder form for the case of $n = 2$.

Based on the above, it is obvious how to give the determinant formulation and proof of Taylor's theorem in its derivative remainder form for the general case. Without giving the proof, we state below the determinant version of that theorem.

THEOREM 2. *Let f be a function whose n-th derivative is continuous on the closed interval $[a, b]$ with $a < b$ and let the $(n+1)$-st derivative $f^{(n+1)}$ exist in the* *open interval* (a, b). *Then there exists* c *with* $a < c < b$ *such that*:

$$\begin{vmatrix} f(b) & b^{n+1} & b^n & \dots & b & 1 \\ f(a) & a^{n+1} & a^n & \dots & a & 1 \\ f'(a) & (n+1)a^n & na^{n-1} & \dots & 1 & 0 \\ f''(a) & n(n+1)a^{n-1} & (n-1)na^{n-2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f^{(n+1)}(c) & (n+1)! & 0 & \dots & 0 & 0 \end{vmatrix} = 0.$$

Remark 1. We observe that the first row of the determinant appearing in (1) consists of the values for $x = b$ of five functions:

(8) $$f(x),\ x^3,\ x^2,\ x,\ 1.$$

Clearly, of the five functions above, the last four are very special cases of functions. With a proof verbatim of that of Theorem 1, we can prove the following generalization of Theorem 1, where instead of special function listed in (8) we consider five arbitrary functions $f(x)$, $g(x)$, $u(x)$, $v(x)$, $w(x)$.

THEOREM 3. *Let* f, g, u, v, w *be functions whose* 2-*nd derivatives are continuous on the closed interval* $[a, b]$ *with* $a < b$ *and let the third derivatives* f''', g''', u''', v''', w''' *exist in the open interval* (a, b). *Then there exists* c *with* $a < c < b$ *such that*:

(9) $$\begin{vmatrix} f(b) & g(b) & u(b) & v(b) & w(b) \\ f(a) & g(a) & u(a) & v(a) & w(a) \\ f'(a) & g'(a) & u'(a) & v'(a) & w'(a) \\ f''(a) & g''(a) & u''(a) & v''(a) & w''(a) \\ f'''(c) & g'''(c) & u'''(c) & v'''(c) & w'''(c) \end{vmatrix} = 0.$$

Naturally, Theorem 2 can be similarly generalized.

Remark 2. A special case of (9) deserves a particular attention. Let us replace in (9) functions $u(x)$, $v(x)$, $w(x)$ respectively with the particular functions x^2, x and 1. As a result we obtain:

(10) $$\begin{vmatrix} f(b) & g(b) & b^2 & b & 1 \\ f(a) & g(a) & a^2 & a & 1 \\ f'(a) & g'(a) & 2a & 1 & 0 \\ f''(a) & g''(a) & 2 & 0 & 0 \\ f'''(c) & g'''(c) & 0 & 0 & 0 \end{vmatrix} = 0.$$

It can be readily verified (by expanding the determinant appearing in (10)) that (10) implies:

$$(11)\qquad \frac{f(b)-f(a)-\frac{f'(a)}{1}(b-a)-\frac{f''(a)}{2!}(b-a)^2}{g(b)-g(a)-\frac{g'(a)}{1}(b-a)-\frac{g''(a)}{2!}(b-a)^2}=\frac{f'''(c)}{g'''(c)}$$

provided the denominator in the left-hand side of equality (11) does not vanish and provided f''' and g''' do not vanish simultaneously.

It is remarkable that in the right-hand side of equality (11) the same c occurs in $f'''(c)$ and $g'''(c)$. Clearly, (11) is a first step generalization of the classical generalized mean-value theorem. By considering larger (dimension-wise) determinants than (10), however, with the same pattern as indicated in (10), we can easily derive further and further generalizations of the classical generalized mean-value theorem. Obviously, the classical mean-value theorem of the differential calculus is the special case of Theorem 1 given by:

$$\begin{vmatrix} f(b) & b & 1 \\ f(a) & a & 1 \\ f'(c) & 1 & 0 \end{vmatrix}=0$$

and the classical generalized mean-value theorem of the differential calculus is the special case of Theorem 3 given by:

$$\begin{vmatrix} f(b) & g(b) & 1 \\ f(a) & g(a) & 1 \\ f'(c) & g'(c) & 0 \end{vmatrix}=0.$$

Next, again, without loss of generality, we give below the determinant formulation and proof for the special case of $n=2$ of Taylor's theorem in its integral remainder form.

First, however, let us observe that if f''' exists then it can be readily verified that the derivative of the determinant appearing in (12) satisfies the equalities given in (12):

$$(12)\qquad \frac{d}{dx}\begin{vmatrix} f(b) & b^2 & b & 1 \\ f(x) & x^2 & x & 1 \\ f'(x) & 2x & 1 & 0 \\ f''(x) & 2 & 0 & 0 \end{vmatrix}=\begin{vmatrix} f(b) & b^2 & b & 1 \\ f(b) & x^2 & x & 1 \\ f'(x) & 2x & 1 & 0 \\ f'''(x) & 0 & 0 & 0 \end{vmatrix}=\begin{vmatrix} 0 & b^2 & b & 1 \\ 0 & x^2 & x & 1 \\ 0 & 2x & 1 & 0 \\ f'''(x) & 0 & 0 & 0 \end{vmatrix}$$

$$=f'''(x)(b-x)^2.$$

Now, we prove:

THEOREM 4. *Let f be a function whose third derivative f''' is continuous on the closed interval $[a, b]$ with $a < b$. Then for every $x \in [a, b]$ it is the case that:*

$$\text{(13)} \quad \begin{vmatrix} f(b) & b^2 & b & 1 \\ f(x) & x^2 & x & 1 \\ f'(x) & 2x & 1 & 0 \\ f''(x) & 2 & 0 & 0 \end{vmatrix} = \int_x^b \frac{d}{dt} \begin{vmatrix} f(b) & b^2 & b & 1 \\ f(t) & t^2 & t & 1 \\ f'(t) & 2t & 1 & 0 \\ f''(t) & 2 & 0 & 0 \end{vmatrix} dt = \int_x^b f'''(t)(b-t)^2\, dt.$$

Proof. Clearly, the left most determinant appearing in (13) is a function of x. Let us denote it by $F(x)$. Obviously, $F(b) = 0$. Moreover, from the hypothesis of the theorem it follows that $F'(x)$ is continuous on $[a, b]$. Therefore $F(x) = \int_x^b F'(t)\, dt$ which, in view of (12) implies (13).

It can be readily verified (by expanding the determinant appearing in (13)) that (13) implies

$$\begin{vmatrix} f(b) & \cdots & 1 \\ \cdot & \cdots & \cdot \\ f''(x) & \ldots & 0 \end{vmatrix}$$

$$= 2!\left(f(b) - f(x) - \frac{f'(x)}{1}(b-x) - \frac{f''(x)}{2!}(b-x)^2\right) = \int_x^b (b-t)^2 f'''(t)\, dt$$

and therefore,

$$f(b) = f(x) + \frac{f'(x)}{1}(b-x) + \frac{f''(x)}{2!}(b-x)^2 + \frac{1}{2!}\int_x^b f'''(t)(b-t)^2\, dt,$$

where upon replacing x by a yields

$$f(b) = f(a) + \frac{f'(a)}{1}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \frac{1}{2!}\int_x^b f'''(t)(b-t)^2\, dt$$

which is the classical formulation of Taylor's theorem (in its integral remainder form) for the case of $n = 2$.

Based on the above, it is obvious how to give the determinant formulation and proof of Taylor's theorem (in its integral remainder form) for the general case. Without giving the proof we state below (using (12)) the determinant version of that theorem.

THEOREM 5. *Let f be a function whose $(n+1)$-st derivative $f^{(n+1)}$ is continuous on the closed interval $[a, b]$ with $a < b$. Then*

$$\begin{vmatrix} f(b) & b^n & b^{n-1} & \dots & b & 1 \\ f(a) & a^n & a^{n-1} & \dots & a & 1 \\ f'(a) & na^{n-1} & (n-1)a^{n-2} & \dots & 1 & 0 \\ f''(a) & (n-1)na^{n-2} & (n-2)(n-1)a^{n-3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ f^{(n)}(a) & n! & 0 & \dots & 0 & 0 \end{vmatrix} = \int_a^b f^{(n+1)}(t)(b-t)^n\,dt.$$

Obviously, as expected, the Fundamental Theorem of Calculus, i.e., $f(b)-f(a) = \int_a^b f'(t)\,dt$ is the conclusion of Theorem 5 for the special case of $n = 0$.

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