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On some properties of the superposition operator in generalized Orlicz spaces of vector-valued functions

Abstract. In this paper we consider a superposition operator F defined by the formula

$$Fx(t) = f(t, x(t)),$$

where the function $f\colon T\times X \to X$ satisfies the Carathéodory condition, T is a measurable space, X is a separable, reflexive Banach space and x is a vector-valued function defined on T. Conditions are found under which the operator F, acting from some region of the generalized Orlicz space L_{M_1} into the space L_{M_2}, is continuous. Moreover, the autor formulates several propositions on the properties of the operator F.

1. Introduction to the theory of Orlicz spaces.

1.1. DEFINITION. Let $\mathscr{X}$ be a linear real space. A function $I\colon \mathscr{X} \to [0, \infty]$ is called a *modular on* $\mathscr{X}$, if for any $x, y \in \mathscr{X}$ we have

1° $I(x) = 0$ iff $x = 0$,

2° $I(-x) = I(x)$,

3° $I(\alpha x+\beta y) \leqslant \alpha I(x)+\beta I(v)$ for $\alpha, \beta \geqslant 0$ and $\alpha+\beta = 1$.

1.2. PROPERTIES (see [17]). (a) $I(\alpha x) \leqslant I(x)$ for $|\alpha| \leqslant 1$,

(b) $I(\sum_{i=1}^{n} \alpha_i x_i) \leqslant \sum_{i=1}^{n} \alpha_i I(x_i)$ for $\alpha_i \geqslant 0$ such that $\sum_{i=1}^{n} \alpha_i = 1$.

1.3. DEFINITION. The set

$$\mathscr{X}_I = \{x \in \mathscr{X}\colon \lim_{\lambda\to 0} I(\lambda x) = 0\}$$

is called a *modular space*.

1.4. THEOREM. *The functional* $\|\cdot\|$, *defined by the formula*

$$\|x\| = \inf\{\eta > 0\colon I(x/\eta) \leqslant 1\}$$

is a norm in $\mathscr{X}_I$. *This norm has the following properties*:

(a) $\|x\| \leqslant 1$ *iff* $I(x) \leqslant 1$,

(b) *if* $I(x) \leqslant 1$, *then* $I(x) \leqslant \|x\|$,

(c) *if* $I(x) > 1$, *then* $I(x) > \|x\|$.

1.5. THEOREM. *Let $x \in \mathscr{X}_I$ and $x_k \in \mathscr{X}_I$ for $k = 1, 2, \ldots$ Then the condition*

$$\|x_k - x\| \to 0 \quad \text{as } k \to \infty,$$

is equivalent to the condition

$$I\big(\lambda(x_k - x)\big) \to 0 \quad \text{as } k \to \infty \text{ for every } \lambda > 0.$$

The above theorem formulates convergence of the sequence $\{x_k\}$ to the element x with respect to the norm by means of modular. Apart from convergence in norm there is considered also a modular convergence on the space $\mathscr{X}_I$.

1.6. DEFINITION. A sequence $\{x_n\}$ of elements of the modular space $\mathscr{X}_I$ is said to be *convergent to* x with respect to the modular I (I-convergent) if there exists a constant $\lambda > 0$ such that

$$I\big(\lambda(x_k - x)\big) \to 0 \quad \text{as } k \to \infty.$$

It follows from Theorem 1.5 that every sequence $\{x_n\}$ which converges in the norm of $\mathscr{X}_I$ to some element x is also I-convergent to x.

1.7. DEFINITION. A sequence $\{x_n\}$ of elements of the modular space $\mathscr{X}_I$ is said to *satisfy the Cauchy condition* if for every $\varepsilon > 0$ and $\lambda > 0$ one can find an $N > 0$ such that

$$I\big(\lambda(x_k - x_l)\big) < \varepsilon,$$

provided $k, l > N$.

1.8. DEFINITION. The space $\mathscr{X}_I$ is called *complete* if each sequence $\{x_n\}$ satisfying the Cauchy condition is I-convergent to an element $x \in \mathscr{X}_I$.

More about modular spaces can be found in [15], [17], and [18].

Hereunder we shall consider generalized Orlicz space as a particular case of a modular space. We assume henceforth that T is a non-empty set, Σ is a σ-algebra of subsets of T, μ is a positive σ-finite complete measure on Σ and X is a separable Banach space with norm $\|\cdot\|_X$.

1.9. DEFINITION. A function $M\colon X \times T \to [0, \infty]$ is said to be an *N-function* if

(a) M is $\mathscr{B} \times \Sigma$-measurable, where $\mathscr{B}$ denotes the σ-algebra of Borel subsets of X,

(b) $M(\cdot, t)$ is even, convex and lower semicontinuous on X for almost every $t \in T$,

(c) $M(0, t) = 0$ a.e. in T,

(d) there exist two measurable functions $\alpha(\cdot)$, $\beta(\cdot)\colon T \to (0, \infty)$ such that implication

$$\|u\|_X > \beta(t) \Rightarrow M(u, t) > \alpha(t)$$

holds a.e. in T.

(e) $M(\cdot, t)$ is continuous at zero.

In the following we assume that measurable functions taking their values in a Banach space X are strongly measurable. By a Pettis theorem, if X is separable, then the strong measurability is equivalent to the weak one.

Let us denote the set of Σ-measurable functions from T into X by $\mathscr{X}_X$. At the same time two functions which differ only on a set of measure zero will be considered as equal. A composition $M(x(\cdot), \cdot)$ for $x \in \mathscr{X}_X$ is a measurable function (see [7]).

1.10. Remark. Elements of the set $\mathscr{X}_X$ will be denoted by $x(\cdot)$, $y(\cdot)$, $z(\cdot)$ or, in order to simplify the notation we will omit sometimes the brackets when it does not lead to a misunderstanding. Symbols u, v will be used for vectors from Banach space X.

We introduce the following functional I_M by formula

$$I_M(x) = \int_T M(x(t), t)\, d\mu(t).$$

Let

$$\operatorname{dom} I_M = \{x \in \mathscr{X}_X \colon I_M(x) < \infty\}$$

and let $\operatorname{lin}\operatorname{dom} I_M$ be the smallest linear space spanned on $\operatorname{dom} I_M$. From convexity of I_M we have that $\operatorname{dom} I_M$ is a convex set.

1.11. THEOREM. *The functional I_M is a modular on $\mathscr{X}_X$.* ∎

1.12. DEFINITION. A modular space defined by modular I_M is called *generalized Orlicz space* and is denoted by L_M. The norm defined as in Theorem 1.4, is called the *Luxemburg norm* and is denoted by $\|\cdot\|_M$.

1.13. THEOREM. *The following conditions are equivalent:*

(a) $x \in L_M$,

(b) *there exists a sequence* $\{x_n\}$ *of elements of* $\operatorname{lin}\operatorname{dom} I_M$ *such that*

$$\lim_{n\to\infty} I_M(\xi(x_n - x)) = 0$$

for every $\xi > 0$,

(c) *there exists a* $\xi_0 > 0$ *such that* $I_M(\xi_0 x) < \infty$. ∎

Hereunder, let the following condition for N-function M be satisfied:

B: there exist an increasing sequence of measurable sets T_n, $n = 1, 2, \ldots$, with $\mu(T_n) < \infty$, $\bigcup_{n=1}^{\infty} T_n = T$, and a sequence of μ-measurable, non-negative functions f_n, $n = 1, 2, \ldots$, such that $M(u, t) \leqslant f_n(t)$ for μ-a.e. $t \in T$ and $\|u\|_X \leqslant n$, where

$$\int_{T_i} f_n(t)\, d\mu(t) < \infty$$

for i, $n = 1, 2, \ldots$ (see [11]).

We shall denote by E_M the closure in L_M of the set of all simple functions from T into X vanishing outside a subset, which is included in T_i for some natural number i.

1.15. THEOREM. *If the condition B is satisfied, then E_M has the following properties*:

(a) *E_M is the largest linear subspace of* $\operatorname{dom} I_M$,

(b) *every measurable bounded function vanishing outside a finite number of T_i is an element of E_M*,

(c) *$x \in E_M$ if and only if $I_M(\xi x) < \infty$ for every $\xi > 0$*,

(d) *if $x \in E_M$, then for every $\varepsilon > 0$ a $\delta > 0$ can be found such that*

$$\|x\chi_A\|_M < \varepsilon,$$

provided $A \in \Sigma$ and $\mu(A) < \delta$,

(e) *if the measure μ is separable, then the space E_M is separable*,

(f) *for every $x \in E_M$ and for every $\varepsilon > 0$ there exists a set $V \subset T$ such that*

$$\mu(T \setminus V) < \infty \quad \text{and} \quad \|x\chi_V\|_M < \varepsilon.$$

For the proofs of (a), (b), (c) and (d) we refer to [11]. Property (e) is a consequence of Theorem 3.2 in [2]. We will prove Property (f).

Proof. Let $x \in E_M$. Then, in virtue of Theorem 1.15 (c), $I_M(\varepsilon^{-1} x) < \infty$ for every $\varepsilon > 0$. Denoting $V_i = T \setminus T_i$ $(i = 1, 2, \ldots)$, we have

$$\mu(T \setminus V_i) = \mu[T \setminus (T \setminus T_i)] = \mu(T_i) < \infty.$$

Moreover,

$$M(\varepsilon^{-1} x(t)\chi_{V_i}(t), t) \leqslant M(\varepsilon^{-1} x(t), t)$$

a.e. in T. Since $I_M(\varepsilon^{-1} x) < \infty$, so from the Lebesgue theorem we obtain

$$\lim_{i \to \infty} I_M(\varepsilon^{-1} x\chi_{V_i}) = \lim_{i \to \infty} \int_{V_i} M(\varepsilon^{-1} x(t), t)\, d\mu(t) = 0.$$

Hence, there exists i_0 such that

$$I_M(\varepsilon^{-1} x\chi_{V_{i_0}}) \leqslant 1.$$

Therefore, putting $V = V_{i_0}$, we have

$$\|x\chi_V\|_M < \varepsilon \quad \text{and} \quad \mu(T \setminus V) < \infty.$$

1.16. DEFINITION. As the distance between $x \in L_M$ and the space E_M we shall regard the number

$$d(x, E_M) = \inf\{\|x-y\|_M \colon y \in E_M\}.$$

We shall denote by $\Pi(E_M, r)$ the totality of functions $x \in L_M$ for which

$d(x, E_M) < r$. The closure of $\Pi(E_M, r)$ with respect to the modular convergence will be denoted by $\bar{\Pi}(E_M, r)$.

The connections between the set dom I_M and the space E_M are described well enough by the following theorem

1.17. THEOREM. *We have*

$$\Pi(E_M, 1) \subset \operatorname{dom} I_M \subset \bar{\Pi}(E_M, 1). \ \blacksquare$$

This result for the Orlicz norm goes back to Kozek [11]. The proof for the Luxemburg norm is similar. Moreover, for the Luxemburg norm the following theorem is true

1.18. THEOREM. *For every $x \in L_M$, we have*

$$d(x, E_M) = \lim_{n\to\infty} \|x - x_n\|_M,$$

where

$$x_n(t) = \begin{cases} x(t) & \text{if } \|x(t)\|_X \leqslant n \text{ and } t \in T_n, \\ 0 & \text{otherwise}. \ \blacksquare \end{cases}$$

The proof of this theorem is similar to that of the same theorem for the Orlicz norm (see [11]).

The Δ_2-condition which plays an important role in the theory of Orlicz spaces is here of the following form:

1.19. DEFINITION. We say that the N-function M *satisfies the Δ_2-condition* if there exist a constant $K > 1$ and a non-negative summable function h such that

$$M(2u, t) \leqslant KM(u, t) + h(t) \quad \text{a.e. in } T.$$

1.20. THEOREM. *The N-function M satisfies the Δ_2-condition if and only if*

$$E_M = \operatorname{dom} I_M = L_M.$$

This theorem follows from Corollary 1.7.4 in [10], immediately.

1.21. THEOREM. *If the N-function M satisfies the Δ_2-condition, then the modular convergence and the norm convergence are equivalent* (see [9]).

2. A superposition operator and its fundamental properties. Let M be an N-function satisfying condition B.

2.1. DEFINITION. Suppose the function f: $T \times X \to X$ satisfies the Carathéodory conditions, i.e., it is continuous in $u \in X$ for almost all $t \in T$ and measurable for every $u \in X$. The operator F, defined by the formula

$$[Fx](t) = f(t, x(t)),$$

where $x \in \mathscr{X}_X$, is called a *superposition operator*.

2.2. Corollary. *The operator F transforms measurable functions into measurable functions.*

Proof. Let x be a measurable function. In virtue of definition of measurability of vector functions (see [1]) a sequence of measurable simple functions $\{x_n\}$ converging to x in the norm $\|\cdot\|_X$ can be found. By continuity, for almost all $t \in T$, we have

$$f(t, x(t)) = \lim_{n \to \infty} f(t, x_n(t)).$$

Moreover, let x_n be given by

$$x_n(t) = \sum_{i=1}^{k} a_{i,n} \chi_{A_{i,n}}(t),$$

where, for every i, $A_{i,n}$ are pairwise disjoint measurable sets and $a_{i,n} \in X$. Since for $n = 1, 2, \ldots$ we obtain

$$f(t, x_n(t)) = \begin{cases} f(t, a_{1,n}) & \text{for } t \in A_{1,n}, \\ f(t, a_{2,n}) & \text{for } t \in A_{2,n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ f(t, a_{k,n}) & \text{for } t \in A_{k,n}, \end{cases}$$

so $f(\cdot, x_n(\cdot))$ are measurable, by measurability of each $f(t, a_{i,n})$ for $1 \leqslant i \leqslant k$ and $n = 1, 2, \ldots$ Thus $f(\cdot, x(\cdot))$ is measurable as a limit of a sequence of measurable functions.

2.3. Corollary. *The superposition operator has a partial additivity property, i.e., for functions $x_1, x_2, \ldots, x_n$ such that for $i \neq j$*

$$x_i(t) x_j(t) = 0 \quad (t \in T)$$

there holds the equality

$$F(x_1 + x_2 + \ldots + x_n) = Fx_1 + Fx_2 + \ldots + Fx_n - (n-1) F(O),$$

where O denotes a function equal to zero.

Proof of this theorem is obvious. ∎

2.4. Theorem. *Let M_1 and M_2 be N-functions satisfying condition B.*

(a) *If the operator F acts from a ball*

$$S_{M_1}(r) = \{x \in L_{M_1} \colon \|x\|_{M_1} < r\}$$

into the space L_{M_2} or E_{M_2}, then the operator F acts from $\Pi(E_{M_1}, r)$ into the space L_{M_2} or E_{M_2}, respectively.

If we suppose additionally $FO = 0$ and if $F[S_{M_1}(r)] \subset \operatorname{dom} I_{M_2}$, then F acts from $\Pi(E_{M_1}, r)$ into $\operatorname{dom} I_{M_2}$.

(b) *If the operator F acts from a ball*

$$S^E_{M_1}(r) = \{x \in E_{M_1}\colon \|x\|_{M_1} < r\}$$

into L_{M_2} or E_{M_2}, then it acts from all of E_{M_1} into L_{M_2} or E_{M_2}, respectively.

If $FO = 0$ and $F[S^E_{M_1}(r)] \subset \operatorname{dom} I_{M_2}$, then F acts from all of E_{M_1} into $\operatorname{dom} I_{M_2}$.

Proof. First we shall prove both parts (a) and (b) of theorem in the case of $[FO](t) = f(t, 0) = 0$ for every $t \in T$. Let $x \in \Pi(E_{M_1}, r)$. Then, in virtue of Theorem 1.18, we have

$$r > d(x, E_{M_1}) = \lim_{n\to\infty} \|x - x_n\|_{M_1},$$

where

$$x_n(t) = \begin{cases} x(t) & \text{if } \|x(t)\|_X \leqslant n \text{ and } t \in T_n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence there exists a natural number n_0 such that

$$\|x - x_{n_0}\|_{M_1} < r.$$

Obviously,

$$(x - x_{n_0})(t) = \begin{cases} 0 & \text{for } t \in T_{n_0} \text{ and } \|x(t)\|_X \leqslant n_0, \\ x(t) & \text{otherwise.} \end{cases}$$

Denoting

$$V_0 = \{t \in T_{n_0}\colon \|x(t)\|_X \leqslant n_0\},$$

we obtain

$$(x - x_{n_0})(t) = x(t)\chi_{T\setminus V_0}(t) = y_0(t).$$

Therefore $\|y_0\|_{M_1} < r$. Moreover, from inclusion $V_0 \subset T_{n_0}$ there follows the inequality

$$\mu(V_0) \leqslant \mu(T_{n_0}) < \infty.$$

Thus $x\chi_{V_0} \in E_{M_1}$ as a measurable bounded function which vanishes outside a finite number of T_i (Theorem 1.15 (b)). Moreover, by Theorem 1.15(d) for $r > 0$ a $\delta > 0$ can be found such that for all $V \subset T$ we have

$$\|x\chi_V\|_{M_1} < r$$

provided $\mu(V) < \delta$. Let us suppose that $V_1, V_2, \ldots, V_k$ are pairwise disjoint measurable subsets of the set V_0 with $\mu(V_i) < \delta$ $(i = 1, 2, \ldots, k)$ and $\bigcup_{i=1}^{k} V_i = V_0$. We put $y_i = x\chi_{V_i}$. Then the function x can be written in the form

$$x = y_0 + y_1 + \ldots + y_k, \tag{2.1}$$

where $y_i \in S_{M_1}(r)$ $(i = 1, 2, \ldots, k)$ are pairwise disjoint. Applying Corollary 2.3 in case $FO = 0$ we obtain

$$Fx = Fy_0 + Fy_1 + \ldots + Fy_k. \tag{2.2}$$

If $F[S_{M_1}(r)] \subset L_{M_2}$, then each of the terms at the right-hand side of formula (2.2) is a function in L_{M_2}. Therefore, Fx also belongs to L_{M_2}.

If $F[S_{M_1}(r)] \subset \operatorname{dom} I_{M_2}$, then, in virtue of (2.2), we have

$$I_{M_2}(x) = \int_T M_2(Fx(t), t)\, d\mu(t) = \sum_{i=0}^{k} \int_{V_i} M_2(Fy_i(t), t)\, d\mu(t) < \infty,$$

i.e., Fx also belongs to $\operatorname{dom} I_{M_2}$.

If $F[S_{M_1}(r)] \subset E_{M_2}$, then all the terms at the right-hand side of (2.2) are functions in E_{M_2}. Therefore, Fx is also an element of E_{M_2}.

Now, we shall prove part (b) of this theorem. To this end, we suppose that $x \in E_{M_1}$. Then $r^{-1}x \in E_{M_1}$, so, by Theorem 1.15(c) and (a), it follows

$$I_{M_1}(r^{-1}x) = \int_T M_1(r^{-1}x(t), t)\, d\mu(t) < \infty.$$

Let, as above,

$$x_n(t) = \begin{cases} x(t) & \text{if } \|x(t)\|_X < n \text{ and } t \in T_n, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, x_n are measurable, bounded and $x_n(t)$ is convergent to $x(t)$ almost everywhere in T. Moreover,

$$M_1[r^{-1}(x(t) - x_n(t)), t] \leqslant M_1(r^{-1}x(t), t) \quad \text{a.e. in } T$$

and if

$$A_n = \{t \in T: x_n(t) = x(t)\},$$

then

$$\mu\Big(T \setminus \bigcup_{n=1}^{\infty} A_n\Big) = 0.$$

Hence and from the dominated convergence theorem we have

$$\lim_{n\to\infty} I_{M_1}[r^{-1}(x - x_n)] = \lim_{n\to\infty} \int_{T\setminus A_n} M_1(r^{-1}x(t), t)\, d\mu(t) = 0.$$

Therefore a natural number n_0 can be found such that

$$I_{M_1}[r^{-1}(x - x_{n_0})] = I_{M_1}[r^{-1}x\chi_{T\setminus A_{n_0}}] < 1,$$

i.e.,

$$\|x\chi_{T\setminus A_{n_0}}\|_{M_1} < r.$$

We put $V_0 = T \backslash A_{n_0}$ and $y_0 = x\chi_{V_0}$. Then $T \backslash V_0 \subset T_{n_0}$, so

$$\mu(T \backslash V_0) \leqslant \mu(T_{n_0}) < \infty .$$

From here, by Theorem 1.15(d) with $\varepsilon = 1$ we find a $\delta > 0$ and pairwise disjoint sets $V_1, V_2, \ldots, V_k$ such that $\bigcup_{i=1}^{k} V_i = T \backslash V_0$, $\mu(V_i) < \delta$ for $i \in \{1, 2, \ldots, k\}$ and

$$\|r^{-1} x\chi_{V_i}\|_{M_1} < 1.$$

Denoting $y_i = x\chi_{V_i}$, we obtain

$$\|y_i\|_{M_1} < r \quad (i = 1, 2, \ldots, k).$$

Next, we describe x in the form (2.1) and analogously as in part (a) of the theorem we obtain the assertion of part (b) for $f(t, 0) = 0$.

We now proceed to the consideration of the general case. We define

$$F_1 x = f(\cdot, x(\cdot)) - f(\cdot, 0) = f_1(\cdot, x(\cdot))$$

for every $x \in X$ a.e. in T. Since $F_1 O = 0$, we have, by what has already been proved, that the operator F_1 acts from $\Pi(E_{M_1}, r)$ into the space L_{M_2} or E_{M_2} with assumptions of part (a) and F_1 acts from E_{M_1} into space L_{M_2} or E_{M_2} with assumptions of part (b). Since $f(\cdot, 0)$ is an element of L_{M_2} (E_{M_2}), therefore, by linearity of L_{M_2} (E_{M_2}),

$$Fx = f(\cdot, x(\cdot)) = F_1 x + f(\cdot, 0)$$

is an element of L_{M_2} (E_{M_2}) for every $x \in \Pi(E_{M_1}, r)$ [in part(a)] or for every $x \in E_{M_1}$ [in part(b)]. This implies the assertion of the theorem. ∎

The next property of the operator F will be concerned with preservation by this operator of the following condition of the family $\mathfrak{N}$ of functions,

(2.3) for every $\varepsilon > 0$ a $\delta > 0$ can be found such that

$$\|x\chi_A\|_M < \varepsilon,$$

for all functions of the family $\mathfrak{N}$, provided $\mu(A) < \delta$.

If $\mu(T) < \infty$, then condition (2.3) for the family $\mathfrak{N}$ is equivalent to the fact, that the family $\mathfrak{N}$ has equi-absolutely continuous norms. In general, condition (2.3) is a little weaker than condition of possession by the family $\mathfrak{N}$ equi-absolutely continuous norms. Therefore, it will be said that the family $\mathfrak{N}$ has almost equi-absolutely continuous norms.

2.5. THEOREM. *If the superposition operator F acts from $\Pi(E_{M_1}, r)$ into E_{M_2}, then the operator F transforms a family of functions with almost equi-absolutely continuous norms into family of functions with almost equi-absolutely continuous norms.*

Proof. We shall assume that a family $\mathfrak{N}$ has almost equi-absolutely continuous norms and show that the image $F(\mathfrak{N})$ has the same property. If we assume the contrary, there exist a sequence of functions $y_k \in \mathfrak{N}$ and a sequence of sets $A_k \subset T$ $(k = 1, 2, \ldots)$ such that

$$\lim_{k\to\infty} \mu(A_k) = 0$$

whereas

$$\|Fy_k \chi_{A_k}\|_{M_2} > a \quad (k = 1, 2, \ldots),$$

where a is some positive number. Without loss of generality it can be assumed that

$$\sum_{k=1}^{\infty} \mu(A_k) < \infty.$$

We put

$$B_k = \bigcup_{i=k}^{\infty} A_i \quad (k = 1, 2, \ldots).$$

By monotonicity of the norm, we have

$$\|Fy_k \chi_{B_k}\|_{M_2} > a \quad (k = 1, 2, \ldots),$$

from which

$$\int_{B_k} M_2\big(a^{-1} f(t, y_k(t)), t\big)\, d\mu(t) > 1 \quad (k = 1, 2, \ldots).$$

Since $\mu(B_k) \to 0$ as $k \to \infty$, then for y_1, by absolute continuity of integral, there exists a number $k_2 > 1 = k_1$ such that

$$\int_{B_{k_2}} M_2\big(a^{-1} f(t, y_1(t)), t\big)\, d\mu(t) < \int_{B_{k_1}} M_2\big(a^{-1} f(t, y_{k_1}(t)), t\big)\, d\mu(t) - 1.$$

It follows from inclusion $B_{k_2} \subset B_{k_1}$ that

$$\int_{B_{k_1} \setminus B_{k_2}} M_2\big(a^{-1} f(t, y_{k_1}(t)), t\big)\, d\mu(t) > 1,$$

in other words

$$\|Fy_1 \chi_{B_{k_1} \setminus B_{k_2}}\|_{M_2} > a.$$

Suppose, we have defined the number k_n in such a manner that

$$\|Fy_{k_{n-1}} \chi_{B_{k_{n-1}} \setminus B_{k_n}}\|_{M_2} > a.$$

From inclusion $A_{k_n} \subset B_{k_n}$ it follows that

$$\|Fy_{k_n} \chi_{B_{k_n}}\|_{M_2} \geqslant \|Fy_{k_n} \chi_{A_{k_n}}\|_{M_2} > a.$$

Then, repeating the argument as for k_2 and choosing k_n, a k_{n+1} can be found such that

$$\|Fy_{k_n}\chi_{B_{k_n}\setminus B_{k_{n+1}}}\|_{M_2} > a.$$

Now, let us denote

$$C_n = B_{k_n}\setminus B_{k_{n+1}} \quad (n = 1, 2, \ldots)$$

and

$$x_n(t) = y_{k_n}(t) \quad (n = 1, 2, \ldots).$$

Obviously, C_n are pairwise disjoint sets, $\mu(C_n) \to 0$ as $n \to \infty$ and

$$\|Fx_n\chi_{C_n}\|_{M_2} > a \quad (n = 1, 2, \ldots). \tag{2.4}$$

Since the functions x_n for $n = 1, 2, \ldots$ have almost equi-absolutely continuous norms, then without loss of generality one can assume that

$$\sum_{n=1}^{\infty} \|x_n\chi_{C_n}\|_{M_1} < \infty. \tag{2.5}$$

Let us define

$$z(t) = \begin{cases} x_n(t) & \text{for } t \in C_n, \\ 0 & \text{for } t \in T \setminus \bigcup\limits_{n=1}^{\infty} C_n. \end{cases}$$

We shall show that $z \in \Pi(E_{M_1}, r)$ for each $r > 0$. From convergence of series (2.5) there follows the existence of a natural number n_0 such that

$$\sum_{n=n_0+1}^{\infty} \|x_n\chi_{C_n}\|_{M_1} < r.$$

Moreover, for every n the function $x_n\chi_{C_n}$ has an absolutely continuous norm. Therefore, in virtue of finiteness of $\mu(C_n)$, $x_n\chi_{C_n}$ is an element of the space E_{M_1}, then also

$$\sum_{n=1}^{n_0} x_n\chi_{C_n} \in E_{M_1}.$$

Thus,

$$d(z, E_{M_1}) \leqslant \|z - \sum_{n=1}^{n_0} x_n\chi_{C_n}\|_{M_1} = \|\sum_{n=n_0+1}^{\infty} x_n\chi_{C_n}\|_{M_1} \leqslant \sum_{n=n_0+1}^{\infty} \|x_n\chi_{C_n}\|_{M_1} < r,$$

from which it follows that $z \in \Pi(E_{M_1}, r)$. Let us note that

$$Fz = \sum_{n=1}^{\infty} (Fx_n)\chi_{C_n} + (FO)\chi_{T_0},$$

where $T_0 = T \setminus \bigcup_{n=1}^{\infty} C_n$. In view of assumption of the theorem we have $Fz \in E_{M_2}$. On the other hand, using the definition of Luxemburg norm for (2.4), we obtain

$$\begin{aligned}\int_T M_2(a^{-1}Fz(t), t)\,d\mu(t) &= \int_T M_2\Big[a^{-1}\Big(\sum_{n=1}^{\infty} Fx_n(t)\chi_{C_n}(t) + FO\chi_{T_0}(t)\Big), t\Big]\,d\mu(t) \\ &= \sum_{n=1}^{\infty} \int_T M_2(a^{-1}Fx_n(t)\chi_{C_n}(t), t)\,d\mu(t) + \int_T M_2(a^{-1}FO(t)\chi_{T_0}(t), t)\,d\mu(t) \\ &\geqslant \sum_{n=1}^{\infty} \int_T M_2(a^{-1}Fx_n(t)\chi_{C_n}(t), t)\,d\mu(t) \geqslant \sum_{n=1}^{\infty} 1 = \infty.\end{aligned}$$

This means that $a^{-1}Fz \notin \operatorname{dom} I_{M_2}$, so $Fz \notin E_{M_2}$. We have thus arrived to a contradiction. ∎

To end this section we shall show yet a simple property of superposition operator.

2.6. THEOREM. *If $\mu(T) < \infty$, then a superposition operator transforms sequences of functions which are convergent in measure into sequences of functions which are convergent in measure also.*

Proof. Let $x_n \to x$ in measure μ. Then for each subsequence $\{x_{n_k}\}$ of sequence $\{x_n\}$ one can find a subsequence $\{x_{n_{k_i}}\}$ convergent to x everywhere. In virtue of continuity of $f(t, u)$ with respect to u, we deduce convergence of subsequence $\{Fx_{n_{k_i}}\}$ to Fx almost everywhere, from which we have that $Fx_n \to Fx$ in measure. ∎

3. Continuity of superposition operator.

3.1. THEOREM. *If the operator F acts from $\Pi(E_{M_1}, r)$ into E_{M_2}, then F is continuous at every point of $\Pi(E_{M_1}, r)$.*

Proof. For clarity of proof, we divide it into three parts:

I. We shall show continuity of the operator F in the case $\mu(T) < \infty$ and $FO = 0$.

II. We extend the result from part I to the case of the set T of σ-finite measure.

III. We proceed to the consideration of the general case, i.e., show continuity of the operator F without any additional assumptions.

I. We shall first assume that $FO = 0$ and $\mu(T) < \infty$. If we assume the contrary, there exists a sequence of functions $x_n \in \Pi(E_{M_1}, r)$ $(n = 1, 2, \ldots)$ which is convergent in the norm to 0, whereas

$$\|Fx_n\|_{M_2} > a \quad (n = 1, 2, \ldots), \tag{3.1}$$

where a is some positive number. Without loss of generality it can be assumed that

$$\sum_{n=1}^{\infty} \|x_n\|_{M_1} < r. \tag{3.2}$$

Hereinafter, we shall construct sequences of numbers $\{\varepsilon_k\}$, of functions $\{x_{n_k}\}$ and of sets $A_k \subset T$ $(k = 1, 2, \ldots)$, such that the following conditions are satisfied:

(a) $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$,

(b) $\mu(A_k) \leqslant \varepsilon_k$,

(c) $\|Fx_{n_k}\chi_{A_k}\|_{M_2} > \frac{2}{3}a$,

(d) if $\mu(E) < 2\varepsilon_{k+1}$ for every set $E \subset T$, then

$$\|Fx_{n_k}\chi_E\|_{M_2} < \tfrac{1}{3}a.$$

Let us assume that $\varepsilon_1 = \mu(T)$, $x_{n_1}(t) = x_1(t)$, $A_1 = T$. In virtue of absolute continuity of the norm of the function Fx_1 and of condition (3.1), it is easy to verify that there exists an ε_2 such that conditions (a), (b), (c) and (d) are satisfied. Let us suppose that ε_k, x_{n_k} and A_k are already defined. We assume that ε_{k+1} is a real number such that condition (d) will be fulfilled. The existence of this number is assured in view of the assumption $Fx_{n_k} \in E_{M_2}$ for each natural number k. Obviously, ε_{k+1} satisfies condition (a). Since $x_n \to 0$ in the norm, then it is also convergent to zero in measure. Therefore, by Theorem 2.6, Fx_n is convergent to zero in measure. Thus Fx_n cannot have equi-absolutely continuous norms, because it would be convergent in norm, i.e., continuous at zero in contradiction to the assumption (see [6]). Hence there exist a set A_{k+1} and a function $x_{n_{k+1}}$ such that $\mu(A_{k+1}) < \varepsilon_{k+1}$ and

$$\|Fx_{n_{k+1}}\chi_{A_{k+1}}\|_{M_2} > \tfrac{2}{3}a.$$

In virtue of the principle of mathematical induction we conclude that conditions (a), (b), (c) and (d) are satisfied for $k = 1, 2, \ldots$

Now, let us define a function y by formula

$$y(t) = \begin{cases} x_{n_k}(t) & \text{for } t \in B_k \ (k = 1, 2, \ldots), \\ 0 & \text{for } t \notin \bigcup\limits_{k=1}^{\infty} B_k, \end{cases} \tag{3.3}$$

where $B_k = A_k \setminus \bigcup\limits_{i=k+1}^{\infty} A_i$, $(k = 1, 2, \ldots)$. Obviously, for $i \neq j$ we have $B_i \cap B_j = \emptyset$. Since

$$\mu\Big(\bigcup_{i=k+1}^{\infty} A_i\Big) \leqslant \sum_{i=k+1}^{\infty} \varepsilon_i < 2\varepsilon_{k+1},$$

then from (c) and (d) there follows the inequality

$$\|Fy\chi_{B_k}\|_{M_2} = \|Fx_{n_k}\chi_{A_k\setminus\bigcup_{i=k+1}^{\infty}A_i}\|_{M_2}$$

$$\geqslant \|Fx_{n_k}\chi_{A_k}\|_{M_2} - \|Fx_{n_k}\chi_{\bigcup_{i=k+1}^{\infty}A_i}\|_{M_2} \geqslant \tfrac{2}{3}a - \tfrac{1}{3}a = \tfrac{1}{3}a,$$

i.e.,

(3.4) $$\|Fy\chi_{B_k}\|_{M_2} > \tfrac{1}{3}a \quad \text{for } k = 1, 2, \ldots$$

Moreover, by (3.2) we have

$$\|y\|_{M_1} \leqslant \sum_{k=1}^{\infty} \|x_{n_k}\chi_{B_k}\|_{M_1} \leqslant \sum_{k=1}^{\infty} \|x_{n_k}\|_{M_1} \leqslant \sum_{n=1}^{\infty} \|x_n\|_{M_1} < r,$$

whence $y \in \Pi(E_{M_1}, r)$. Applying the assumption of the theorem we obtain that $Fy \in E_{M_2}$. On the other hand, from (3.4) and from the fact that B_k $(k = 1, 2, \ldots)$ are pairwise disjoint, we have

$$\int_T M_2(3a^{-1}Fy(t), t)\,d\mu(t) = \int_T M_2\Big(3a^{-1}\sum_{i=1}^{\infty} Fy(t)\chi_{B_i}(t), t\Big)d\mu(t)$$

$$= \sum_{i=1}^{\infty} \int_T M_2(3a^{-1}Fy(t)\chi_{B_i}(t), t)\,d\mu(t) \geqslant \sum_{i=1}^{\infty} 1 = \infty,$$

and so $3a^{-1}Fy \notin \operatorname{dom} I_{M_2}$. Consequently, $Fy \notin E_{M_2}$. We have thus arrived to a contradiction.

II. Let us drop the assumption $\mu(T) < \infty$. Let us suppose that the measure μ is σ-finite and that the superposition operator F is not continuous at zero in L_{M_1}. Then there exists a sequence $\{x_n\}$ of elements of the space L_{M_1} such that conditions (3.1) and (3.2) are satisfied. We shall construct sequences of functions $\{x_{n_k}\}$ and of sets $\{B_k\}$ such that $\mu(B_k) < \infty$, $B_i \cap B_j = \emptyset$ for $i \neq j$ and

(3.5) $$\|Fx_{n_k}\chi_{B_k}\|_{M_2} > \tfrac{1}{2}a \quad (k = 1, 2, \ldots).$$

Let $x_{n_1} = x_1$. Applying Theorem 1.15 (f) and (3.1), a set B_1 can be chosen equal to $T\setminus V$, where V is as in Theorem 1.15 (f) for $\varepsilon = \frac{1}{2}a$. Since $\mu(B_1) < \infty$ and

$$\|Fx_{n_1}\chi_{T\setminus B_1}\|_{M_2} < \tfrac{1}{2}a,$$

we have

$$\|Fx_{n_1}\chi_{B_1}\|_{M_2} \geqslant \|Fx_{n_1}\|_{M_2} - \|Fx_{n_1}\chi_{T\setminus B_1}\|_{M_2} > a - \tfrac{1}{2}a = \tfrac{1}{2}a.$$

Let us suppose that x_{n_k} and B_k is already defined, at the same time $\mu(B_k) < \infty$ and (3.5) is satisfied. By continuity of the operator F at zero in the case o finite measure, a natural number n_{k+1} can be found such that

$$\|Fx_{n_{k+1}}\chi_{\bigcup\limits_{i=1}^{k} B_i}\|_{M_2} < \tfrac{1}{2}a.$$

Moreover, in virtue of (3.1) and of Theorem 1.15 (f) there exists a set C_{k+1} of finite measure such that

$$\|Fx_{n_{k+1}}\chi_{C_{k+1}}\|_{M_2} > a.$$

We put

$$B_{k+1} = C_{k+1}\setminus \bigcup_{i=1}^{k} B_i.$$

Hence

$$\|Fx_{n_{k+1}}\chi_{B_{k+1}}\|_{M_2} \geqslant \|Fx_{n_{k+1}}\chi_{C_{k+1}}\|_{M_2} - \|Fx_{n_{k+1}}\chi_{\bigcup\limits_{i=1}^{k} B_i}\|_{M_2} \geqslant a-\tfrac{1}{2}a = \tfrac{1}{2}a.$$

Let a function y be defined by formula (3.3). In virtue of (3.2) we have

$$\|y\|_{M_1} \leqslant \sum_{k=1}^{\infty} \|x_{n_k}\chi_{B_k}\|_{M_1} \leqslant \sum_{k=1}^{\infty} \|x_n\|_{M_1} < r,$$

so $y \in \Pi(E_{M_1}, r)$. Therefore $Fy \in E_{M_2}$. On the other hand, from (3.5) it follows that

$$\int_T M_2(a^{-1}2Fy(t), t)\,d\mu(t)$$
$$= \sum_{k=1}^{\infty} \int_T M_2(2a^{-1}Fx_{n_k}(t)\chi_{B_k}(t), t)\,d\mu(t) \geqslant \sum_{k=1}^{\infty} 1 = \infty,$$

so $2a^{-1}Fy \notin \operatorname{dom} I_{M_2}$. This means that $Fy \notin E_{M_2}$. A contradiction, thus F is continuous at zero.

III. We now proceed to the consideration of the general case: we shall show, without any additional assumptions, that the operator F is continuous at an arbitrary point x_0 of the set $\Pi(E_{M_1}, r)$. Let $d = d(x_0, E_{M_1})$. Clearly, $d < r$. The continuity of the operator F at the point x_0 is equivalent to the continuity of the operator

$$F_1 x = F(x_0+x) - Fx_0$$

at zero in L_{M_1}. The operator F_1 acts from the ball $S_{M_1}(r-d)$ into E_{M_2}. In virtue of Theorem 2.4, it acts from $\Pi(E_{M_1}, r-d)$ into E_{M_2}. Since $F_1 O = 0$, we have, by what has already been proved, that the operator F_1 is continuous at zero in L_{M_1}. ∎

From Theorem 3.1 we can deduce the corollary on the boundedness of the set of values of the operator F on a ball in the space L_{M_1}.

3.2. Corollary. *If an N-function M_1 satisfies the Δ_2-condition and F is a superposition operator from L_{M_1} into E_{M_2}, then F is bounded on any ball $S_{M_1}(r)$ $(r > 0)$, i.e.,*

$$\sup\{\|Fx\|_{M_2}\colon x \in S_{M_1}(r)\} < \infty .$$

Proof. Without loss of generality of the theorem we can assume that $FO = 0$. Let us suppose that the theorem is not true. Then there exists a sequence $\{x_n\}$ of elements of the space L_{M_1} such that

$$\|x_n\|_{M_1} < \beta \qquad (n = 1, 2, \ldots)$$

and

$$\|Fx_n\|_{M_2} > n \qquad (n = 1, 2, \ldots).$$

The set T can be decomposed into n pairwise disjoint measurable sets $T_1^{(n)}$, $T_2^{(n)}, \ldots, T_n^{(n)}$ such that

$$\int_{T_i^{(n)}} M_1(\beta^{-1} x_n(t), t)\, d\mu(t) < 1/n \qquad (n = 1, 2, \ldots). \tag{3.6}$$

From the negation of the assertion, we have

$$\int_T M_2(n^{-1} Fx_n(t), t)\, d\mu(t) > 1 \qquad (n = 1, 2, \ldots),$$

then there exists at least one set $T_{i_0}^{(n)}$ such that

$$\int_{T_{i_0}^{(n)}} M_2(n^{-1} Fx_n(t), t)\, d\mu(t) > 1/n.$$

Hence, in view of the convexity of the N-function M_2, we obtain

$$\begin{aligned}\int_T M_2(Fx_n(t)\chi_{T_{i_0}^{(n)}}(t), t)\, d\mu(t) &= \int_{T_{i_0}^{(n)}} M_2(n \cdot n^{-1} Fx_n(t), t)\, d\mu(t) \\ &\geqslant n \int_{T_{i_0}^{(n)}} M_2(n^{-1} Fx_n(t), t)\, d\mu(t) > 1\end{aligned}$$

for every natural number n. This means that

$$\|Fx_n \chi_{T_{i_0}^{(n)}}\|_{M_2} > 1. \tag{3.7}$$

Now, we shall define a new sequence of functions $\{y_n\}$ by the formula

$$y_n(t) = \begin{cases} x_n(t) & \text{for } t \in T_{i_0}^{(n)}, \\ 0 & \text{for } t \notin T_{i_0}^{(n)}. \end{cases}$$

In virtue of (3.6), the sequence $\{\beta^{-1} y_n\}$ is modular convergent to zero. Since M_1 satisfies the Δ_2-condition, then the modular convergence is equivalent to the convergence in the norm. Therefore

$$\|y_n\|_{M_1} \to 0 \quad \text{as } n \to \infty.$$

Hence, applying Theorem 3.1, we obtain

$$\|Fy_n\|_{M_2} \to 0 \quad \text{as } n \to \infty,$$

contrary to (3.7). Thus, the theorem in case $FO = 0$ is proved. If $FO \neq 0$, then we can put

$$F_1 x = Fx - FO.$$

Clearly, $F_1 O = 0$, so norms of values of the operator F_1 on any ball are bounded. From which it follows that

$$\|Fx\|_{M_2} \leqslant \|F_1 x\|_{M_2} + \|FO\|_{M_2} < \infty$$

for every $x \in S_{M_1}(\beta)$. Thus the proof is concluded. ■

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