[26] I. Mantellini, On the asymptotic behaviour of linear combinations of Mellin-Picard type operators, to appear in Math. Nachr. **286**(17-18) (2013), 1820-1832.

[27] J. Musielak, On some approximation problems in modular spaces, In *Constructive Function Theory,* Proc. Int. Conf. Varna, June 1-5, 1981, pp. 455-461, Publ. House Bulgarian Acad. Sci., Sofia 1983.

[28] J. Musielak, Orlicz Spaces and Modular Spaces, Lecture Notes in Math. **1034**, Springer-Verlag, Berlin, 1983.

[29] J. Musielak, Approximation by nonlinear singular integral operators in generalized Orlicz spaces, Comment. Math. **31** (1991), 79-88.

[30] J. Musielak, Nonlinear approximation in some modular function spaces I, Math. Japonica, **38** (1993), 83-90.

[31] J. Musielak, On approximation by nonlinear integral operators with generalized Lipschitz kernel over a compact abelian group, Comment. Math. **35** (1993), 99-104.

[32] T. Swiderski and E. Wachnicki, Nonlinear singular integrals depending on two parameters, Comment. Math. **40** (2000), 181-189.

[33] F. Ventriglia and G. Vinti, Nonlinear Kantorovich-type operators: a unified approach, submitted (2013).

[34] G. Vinti and L. Zampogni, A unified approach for the convergence of linear Kantorovich-type operators, submitted (2013)

[35] E.V. Voronovskaja, Determination of the asymptotic form of approximation of functions by the polynomials of S.N. Bernstein, Dokl. Akad. Nauk SSSR, A (1932), 79-85.

[36] A.H. Zemanian, Generalized Integral Transformation, Interscience, New York, 1968.

Carlo Bardaro
Department of Mathematics and Informatics, University of Perugia
Via Vanvitelli 1, 06123 Perugia, Italy
E-mail: `bardaro@unipg.it`

Ilaria Mantellini
Department of Mathematics and Informatics, University of Perugia
Via Vanvitelli 1, 06123 Perugia, Italy
E-mail: `mantell@dmi.unipg.it`

(Received: 19.08.2013)

COMMENTATIONES MATHEMATICAE Vol. 53, No. 2 (2013), 79-88

Wieslaw Pleśniak

Identity principles for Bernstein quasianalytic functions

In tribute to Julian Musielak on his 85th birthday

Abstract. We prove an identity principle for Bernstein quasianalytic functions on plane continua and study its extensions to the multivariate case.

2010 Mathematics Subject Classification: Primary 30D60, Secondary 41A17, 32U35, 32W20.

Key words and phrases: Bernstein quasianalytic functions, Bernstein-Walsh inequality, Siciak extremal function, Zakharyuta-Siciak theorem, Bedford-Taylor pluripotential theory.

1. Let f be a continuous function on the interval $I = [-1, 1]$, and let

$$E_n(f) = \operatorname{dist}(f, \mathbb{P}_n) := \inf\{\|f - p\|_I : \ p \in \mathbb{P}_n\},$$

where $\mathbb{P}_n$ stands for the space (of the restrictions to I) of the polynomials of degree $\leq n$ and $\|\cdot\|_I$ is the Chebyshev norm in the space $C(I)$ of all continuous functions on I. By a classical result of Bernstein ([2], [3]), if

$$E_{n_k}(f) \leq M\rho^{n_k} \tag{1}$$

for some strictly increasing subsequence $\{n_k\}$ of $\mathbb{N}$ and for positive constants M and $\rho \in (0, 1)$, and if f vanishes on a subinterval F of I, then f vanishes on the whole interval I. Such functions are called the Bernstein *quasianalytic functions* and their set is denoted by $B(I)$. Bernstein's identity principle was essentially strengthened by Szmuszkowiczówna [22] and independently by Lelong [9]. They proved that a function $f \in B(I)$ has to be the zero function whenever it vanishes on a subset F of I of positive logarithmic capacity $c(F)$. (For the definition of the logarithmic capacity of a subset F of the complex plane $\mathbb{C}$, see e.g. [18].) The proofs provided by both authors are based on properties of Green's functions. Given a compact set E in $\mathbb{C}$ with a positive capacity $c(E)$, we denote by G_E the (generalized) Green function with pole at ∞ for the unbounded connected component $D_\infty(E)$ of the set $\mathbb{C} \setminus E$, i.e. G_E is a (unique) function that is harmonic and strictly positive in $D_\infty(E)$ and such that

$$\text{(i)} \qquad \lim_{|z|\to\infty} (G_E(z) - \log|z|) = -\log c(E),$$

and

(ii) the set of points $\zeta \in \partial D_\infty(E)$ such that $G_E(z) \to 0$ as $D_\infty(E) \ni z \to \zeta$ is of positive logarithmic capacity.

Let

$$V_E(z) = \begin{cases} G_E(z), \; z \in D_\infty(E); \\ 0, \; z \in \mathbb{C} \setminus D_\infty(E). \end{cases} \tag{2}$$

Then we have

Lemma 1.1 (Bernstein-Walsh Lemma) *If $P \in \mathbb{P}_n$, then*

$$|P(z)| \le \|P\|_E \exp\{nV_E(z)\} \quad \textit{for } z \in \mathbb{C}.$$

A similar identity principle to that of Szmuszkowiczówna and Lelong was also proved by Akutowicz [1] whose argument was based on a fundamental result of Leja [8] concerning the construction of the Green's function with the aid of Lagrange interpolation polynomials. In all these indentity principles the assumption of positive capacity of the zero set of $f \in B(I)$ is essential, in view of an example given by Mazurkiewicz and Szmuszkowiczówna [12] of a non-zero function $f \in B(I)$ that is infinitely differentiable and vanishes on an uncountable subset of I. We remark that such a function f cannot be quasianalytic in the sense of Denjoy-Carleman, since it must vanish together with all its derivatives at some point of I. We add that such functions are rare, since the intersection of the set $B(I)$ and the set of all Denjoy-Carleman functions is a residual subset of the space $C^\infty(I)$ of all infinitely differentiable functions on I endowed with its natural projective topology (*see* [17], Theorem 14.4).

Although Bernstein's quasianalytic functions have such a nice uniqueness property, in general, they do not posses any smoothness. Following Mazurkiewicz [11], the set $B(I)$ is residual in the Banach space $C(I)$. Hence, since by the well-known result of Banach the set of nowhere differentiable functions is also residual in $C(I)$, the subset of the set $B(I)$ consisting of nowhere differentiable functions is residual as well. As was already observed by Bernstein, the more lacunary is the sequence $\{n_k\}$ in (1), the worse is the regularity of f. On the other hand, if

$$\limsup_{k\to\infty} \frac{n_{k+1}}{n_k} < \infty,$$

then condition (1) is necessary and sufficient for f be analytically continuable in a neighborhood (in $\mathbb{C}$) of I. In a similar way, if

$$\lim_{k\to\infty} \frac{\log n_{k+1}}{n_k} = 0,$$

then condition (1) describes the infinitely differentiable functions on I. Let us also mention the following interesting property of $B(I)$ found by Markushevich [10]: any function $f \in C(I)$ can be written as the sum $f = f_1 + f_2$ with some $f_1, f_2 \in B(I)$. For a surprisingly short proof of the last property, based on a decomposition of a Baire topological group by means of its residual subset, see [16] and [17].

The notion of Bernstein quasianalytic functions naturally extends to a compact subset E of the complex plane $\mathbb{C}$. Now the point is to prove a counterpart of the Szmuszkowiczówna-Lelong result. However, this is not straightforward. The reason is that Szmuszkowiczówna and Lelong's argument (as well as that of Akutowicz) heavily relies on the fact that the closure of a union of subintervals of $I = [-1, 1]$ with non-void intersection is still a subinterval of I. In this note we propose to overcome this difficulty by applying the following known theorem of Janiszewski (*see e.g.* [7])

Theorem 1.2 *Let X be a continuum (a connected compact space), and let $A = \overline{A} \subsetneq X$. Then every connected component C of the set A has a non-void intersection with the set $\overline{X \setminus A}$.*

Now we are ready to prove the following

Theorem 1.3 (Identity Principle for the class) $\boldsymbol{B(E)}$) *Let $E \subset \mathbb{C}$ be a continuum not reducing to a single point and let $f \in B(E)$. Suppose that f vanishes on a subset F of E with a positive logarithmic capacity $c(F)$. Then $f = 0$ on E.*

Proof Let $N := \{z \in E : f(z) = 0\}$. The set N is compact and it contains the set F. In particular, $c(N) > 0$. Suppose that $N \neq E$. By (ii) of the definition of Green's function, there exists a point $a \in N$ such that G_N is continuous at a. Let C be the connected component of the set N containing the point a. Then by Janiszewski's Theorem, $C \cap \overline{E \setminus N} \neq \emptyset$. Choose a point $b \in C \cap \overline{E \setminus N}$. The following two cases can occur:

1^o $b = a$. Since $f \in B(E)$, there exist a strongly increasing sequence $\{n_k\} \subset \mathbb{N}$, polynomials p_{n_k} of degree $\leq n_k$ $(k = 1, 2, \dots)$ and constants $M > 0$ and $\rho \in (0, 1)$ such that

$$\|f - p_{n_k}\|_E \leq M\rho^{n_k}.$$

Hence $|p_{n_k}(z)| \leq M\rho^{n_k}$ if $z \in N$, $k = 1, 2, \dots$. Choose now $R > 1$ such that $\rho R < 1$. Then, since the function G_N is continuous at the point a, one can find $\delta > 0$ such that $\exp V_N(z) < R$, if z belongs to the disk $B(a, \delta)$. Hence by the Bernstein-Walsh Lemma, the sequence of polynomials $\{p_{n_k}\}$ is uniformly convergent to zero in $B(a,\delta)$, whence $f(z)=0$, if $z\in E\cap B(a,\delta)$. This, however, leads to a contradiction, since the disk $B(a,\delta)$ contains points of the set $E\setminus N$.

2^o $b\neq a$. In such a case, the set C is a continuum that does not reduce to a single point. It is well-known (*see e.g.* [18]) that then the function V_N is continuous at every point of C, whence in particular at the point b. Now repeating the argument of point 1^0 (where the point a is replaced by b) gives again a contradiction. Thus we have proved that $N=E$, as claimed. ∎

Remark 1.4 Theorem 3 was stated without proof in [17]. To the author's knowledge no proof of it has ever appeared in the literature. Thus, the presented proof of Theorem 3 fulfils a gap in [17].

Besides the authors mentioned above, the functions of a one variable satisfying (1) were also studied e. g. by Beurling [4], Mergelyan [13], Borichev, Nazarov and Sodin, [5], and by no means is this list complete.

2. The study of multivariate Bernstein quasianalytic functions was begun in [14] and then developed in [17]. Skiba [21] studied such functions on algebraic sets in the space $\mathbb{C}^N$. In all these papers, an important role has been played by Siciak's extremal function Φ_E, associated with a compact subset E of $\mathbb{C}^N$, defined by

$$\Phi_E(z)=\sup_{n\in\mathbb{N}}\sup\{|p(z)|^{1/n}:\ p\in\mathbb{P}_n(\mathbb{C}^N),\ \|p\|_E\leq 1\}.\quad z\in\mathbb{C}^N.$$

Siciak introduced this function in [19] in order to extend the well-known Bernstein-Walsh theorem (*see e.g.* [18]) to the space $\mathbb{C}^N$. Siciak's result reads as follows.

Theorem 2.1 **1°**. *If f is a holomorphic function in an open neighbourhood of a polynomially convex compact subset E of $\mathbb{C}^N$ (i.e. $E=\hat{E}:=\{z\in\mathbb{C}^N:\ |p(z)|\leq\|p\|_E$ for all polynomials p in $\mathbb{C}^N\}$), then*

$$\limsup_{n\to\infty}\sqrt[n]{\operatorname{dist}(f,\mathbb{P}_n(\mathbb{C}^N))}<1. \tag{3}$$

2°. *Let E be a compact subset of $\mathbb{C}^N$ such that the function Φ_E is continuous in $\mathbb{C}^N$. If $f:E\mapsto\mathbb{C}$ is a continuous function satisfying* (3) *then it can be holomorphically extended to an open neighbourhood of E.*

Observe that $\Phi_E(z)\geq 1$ in $\mathbb{C}^N$ and $\Phi_E(z)=1$ iff $z\in\hat{E}$. By a result of Zakharyuta [23] and Siciak [20], the function $\log\Phi_E$ is equal to the Zakharyuta extremal function

$$V_E(z):=\sup\{u(z):\ u\in\mathcal{L}(\mathbb{C}^N),\ u\leq 0 \text{ on } E\},$$

where $\mathcal{L}(\mathbb{C}^N)$ is the Lelong class of plurisubharmonic functions in $\mathbb{C}^N$ such that $u(z) - \log\|z\| = \mathrm{O}(1)$ as $\|z\| \to \infty$. If K is a compact subset of $\mathbb{C}^N$, one can define the *logarithmic capacity* of K by

$$c(K) = \liminf_{\|z\|\to\infty} \frac{\|z\|}{\exp\{V_K^*(z)\}},$$

where $V_K^*(z) = \limsup_{w\to z} V_K(w)$ is the upper semicontinuous regularization of V_K. For an arbitrary subset F of $\mathbb{C}^N$, we set $c(F) := \sup\{c(K) : K \text{ is a compact subset of } F\}$. It can be shown that $c(E) = 0$ if and only if the set E is *pluripolar*, which means that there exists a plurisubharmonic function u in $\mathbb{C}^N$, $u(z) \not\equiv -\infty$, such that $E \subset \{u(z) = -\infty\}$. If $N = 1$, the capacity introduced above coincides with the planar logarithmic capacity of Theorem 3. We also note that a countable union (in particular, a finite union) of pluripolar sets is still pluripolar. By the developed in the eighties of the last century Bedford and Taylor's theory of the complex Monge-Ampère operator, if E is a non-pluripolar compact subset of $\mathbb{C}^N$, the function V_E^* is a solution of the homogeneous Monge-Ampère equation, which reduces in the one-dimensional case to the Laplace equation. Therefore the function V_E^* is a multivariate counterpart of the planar Green function and if $N = 1$, the function V_E is simply equal to the function defined by (2). In particular, by a Bedford-Taylor result (*see* [6]), the set $\{V_E(z) < V_E^*(z)\}$ is pluripolar, whence one gets a multivariate version of **Kellogg's Lemma:** *If* $c(E) > 0$, the set of points in E where the function Φ_E is discontinuous, is pluripolar. For more information about pluripotential theory, the reader is referred to the monograph of Klimek [6].

The central problem of the multivariate theory obviously consists in establishing a satisfactory uniqueness property for functions satisfying condition (1) in the several variables setting. An attempt at this direction was made in [14]. It could not be satisfactory, since at that time the pluripotential theory methods were not yet available. More successful are without doubt identity principles for the class $B(E)$ provided by Skiba [21].

Further on, the space $\mathbb{R}^N$ is treated as the real part of the space $\mathbb{C}^N$ so that $\mathbb{C}^N = \mathbb{R}^N + i\mathbb{R}^N$. We first observe that if $N > 1$, Theorem 3 is not valid. It is seen by the following

Example 2.2 Let $E_1 = [-2,-1] \times [-1,1] \subset \mathbb{R}^2$, $E_2 = [1,2] \times [-1,1] \subset \mathbb{R}^2$ and $I = [-1,1] \times \{0\} \subset \mathbb{R}^2$. Let $E = E_1 \cup I \cup E_2$. By the Stone-Weierstrass theorem every compact subset of the space $\mathbb{R}^N$ is polynomially convex (*see e.g.* [6, Lemma 5.4.1]). Hence by Theorem 4, there exist constants $M > 0$, $\rho \in (0,1)$ and polynomials $p_n \in \mathbb{P}_n(\mathbb{C}^2)$ such that

$$\|p_n - 1\|_{E_1} \le M\rho^n \quad \text{and} \quad \|p_n\|_{E_2} \le M\rho^n.$$

Set $W_n(x,y) = y p_n(x,y)$. Then

$$W_n(x,y) \to f(x,y) = \begin{cases} y, & (x,y) \in E_1; \\ 0, & (x,y) \in I \cup E_2. \end{cases}$$

Moreover, $\|f - W_n\|_E \leq M\rho^n$, $n = 1.2.\ldots$. Hence $f \in B(E)$, $f(x,y) \not\equiv 0$, but $N := \{f = 0\} \supset I \cup E_2$, and $c(I \cup E_2) > 0$, since no subset K of $\mathbb{R}^N$ with non-void interior (in $\mathbb{R}^N$) can be pluripolar in $\mathbb{C}^N$. (If I is a cube in $\mathbb{R}^N$, then by Siciak [19] the function V_E is continuous in $\mathbb{C}^N$ and consequently, $c(I) > 0$.) Note that we cannot now use the argument of the proof of Theorem 3, since the function V_N is discontinuous at every point of the set $I \setminus \{(1,0)\}$.

Remark 2.3 The example above also shows that the assumption of continuity of the function Φ_E in Theorem 4 (2^o) is necessary. Indeed, the function f of Example 5 fulfils condition (3) of Theorem 4 but it cannot be holomorphically extended to any open neighbourhood of E.

Following Skiba [21], we say that a compact set E in $\mathbb{C}^N$ is an NB-set, if for every function $f \in B(E)$ and every subset F of E with $c(F) > 0$, if $f = 0$ on F then $f = 0$ in E. From Theorem 3 one derives the following uniqueness property:

Corollary 2.4 *Any set $E = E_1 \times \cdots \times E_N \subset \mathbb{C}^N$, where each E_j is a continuum in $\mathbb{C}$ not reducing to a single point, is an NB-set.*

Proof Let $f \in B(E)$ be a function vanishing on a compact subset F of E with $c(F) > 0$. By the definition of the function Φ_F, for the polynomials p_{n_k} of (1) (in the multivariate setting) we have

$$|p_{n_k}(z)| \leq \|p_{n_k}\|_F [\Phi_F(z)]^{n_k}, \quad z \in \mathbb{C}^N. \tag{4}$$

By Kellogg's Lemma there is a point $a = (a_1 \ldots, a_N) \in F$ such that Φ_F is continuous at a. Hence, given $R > 1$, one can find a number $r > 1$ such that $\Phi_F(z) < R$ in the polydisk $P(a;r) = \{z = (z_1, \ldots, z_N) : \ |z_j| \leq r, \ j = 1, \ldots, N\}$. If we choose R so that $\eta := \rho R < 1$, then by (4) we get

$$|p_{n_k}(z)| \leq M\rho^{n_k} R^{n_k} = M\eta^{n_k}, \quad z \in P(a;r).$$

It follows that $f = 0$ in the set $E \cap P(a;r)$. In particular, $f(z) = 0$ on the set $C_1 \times \cdots \times C_N$, where C_j is the connected component of the set $E_j \cap \{z_j \in \mathbb{C} : |z_j| \leq r\}$ that contains the point a_j, $j = 1, \ldots, N$. Now, an application of Theorem 3 completes the proof. ∎

Example 2.5 Let $E_1 = \{z \in \mathbb{C} : \ \Im z = \sin(\frac{1}{\Re z})$ if $0 < |\Re z| \leq 1$, and $|\Im z| \leq 1$ if $\Re z = 0\}$. Let $E_2 = \{w \in \mathbb{C} : \ \Im w = 0$ and $|\Re w| \leq 1\}$. Then the set $E = E_1 \times E_2$ is an NB-set in $\mathbb{C}^2$.

In [21], Skiba proved the following identity principle

Theorem 2.6 . *Let E be a compact subset of $\mathbb{C}^N$ such that any two points $a, b \in E$ can be joined by an analytic arc $\Gamma_{ab} \subset E$, Then E is an NB-set.*

Note that the theorem above does not cover Corollary 7, since the set E of Example 8 is not arcwise connected. Roughly speaking, Skiba's result follows from the fact that analytic substitutions preserve the Bernstein quasianalyticity. This, in turn, follows from a "uniform"version of the Bernstein-Walsh-Siciak Theorem 4 that was first proved in [15]. It has appeared very useful in multivariate polynomial approximation. If E is the closure of a bounded domain (i.e. an open connected set), one can give a more convenient version of Theorem 8. We first start with the real case.

THEOREM 2.7 *The closure of a bounded domain in* $\mathbb{R}^N$ *is an* NB*-set.*

PROOF Let f be a Bernstein quasianalytic function on $E = \overline{\Omega}$, where Ω is a bounded domain in $\mathbb{R}^N$. Suppose that f vanishes on a compact subset F of E with $c(F) > 0$. By an argument similar to that of the proof of Corollary 7, we can find a point $a \in \Omega$ and a number $\varepsilon > 0$ such that $f(x) = 0$ in the ball $B(a, \varepsilon) \subset \Omega$. Let b be an arbitrary point in Ω, $a \neq b$. Any connected open set in $\mathbb{R}^N$ is arcwise-connected. Therefore the points $a,\ b$ can be joined by an arc $\Gamma_{ab} \subset \Omega$ homeomorphic to the line-segment [0,1]. Let $\delta := \operatorname{dist}(\Gamma_{a,b}, \mathbb{R}^n \setminus \Omega)$. By the Weierstrass approximation theorem, for each $n \in \mathbb{N}$, one can find a polynomial map $p_n :\ [0,1] \ni t \mapsto (p_{n1}(t), \ldots, p_{nN}(t)) \in \mathbb{R}^N$ (of degree $\deg p_n = \max\{\deg p_{n1}, \ldots, \deg p_{nN}\}$ not necessarily $\leq n$) such that $\operatorname{dist}(\Gamma_{ab}, p_n([0,1])) < \frac{1}{n}$. Set $q_n(t) = p_n(t) - p_n(0) + a$ for $t \in [0.1]$. We have

$$\operatorname{dist}(q_n([0,1], \mathbb{R}^N \setminus \Omega) \geq \delta - \operatorname{dist}(q_n([0,1]), \Gamma_{a,b}) \geq \delta - \frac{2}{n} > 0$$

for $n \geq n_0(\delta)$. By [21], Lemma 3.3, the function $f \circ q_n$ is quasianalytic on the interval [0,1] and it vanishes on a subinterval $[0, t_0]$ such that $q_n([0, t_0]) \subset B(a, \varepsilon)$. Hence by the classical Bernstein theorem, $f \circ q_n$ vanishes on the whole interval [0,1]. In particular, if $b_n = q_n(1)$, we get $f(b_n) = f(q_n(1)) = 0$. But if $n \to \infty$, then $b_n \to b$ and consequently, $f(b) = 0$. Now, by continuity of f, $f = 0$ on the whole set E and the theorem follows. ∎

Theorem 10 can be easily extended to the complex case, since any compact set E in $\mathbb{C}^N$ can be treated as a compact subset of the space $\mathbb{R}^{2N}$, and any polynomial p of N complex variables $z = (z_1, \ldots, z_N)$ generates the polynomial $q(x_1, y_1, \ldots, x_N, y_N) := p(x_1 + iy_1, \ldots, x_N + iy_N)$ of $2N$ real variables. Hence, if a function f fulfils condition (1) on E with respect to polynomials in complex variables $z_1, \ldots, z_N$, it is also quasianalytic on E with respect to $2N$ variables $x_1, y_1, \ldots, x_N, y_N$. Moreover, if f vanishes on a compact set $F \subset E$ that is non-pluripolar in $\mathbb{C}^N$ then by the argument of the proof of Theorem 10 it vanishes on a ball $B \subset \operatorname{int} E$ (in $\mathbb{C}^N$) which (treated as a subset of the space $\mathbb{R}^{2N} \subset \mathbb{C}^{2N}$) is a non-pluripolar subset of the space $\mathbb{C}^{2N}$. Thus, from Theorem 10 we derive

THEOREM 2.8 *The closure of a bounded domain in* $\mathbb{C}^N$ *is an* NB*-set.*

Further examples of NB-sets can be provided by combining Theorems 10 and 11.

EXAMPLE 2.9 Let $P_1 = \{(z_1, z_2) \in \mathbb{C}^2 : \max\{|z_1 + 2|, |z_2|\} \leq 1\}$, let $P_2 = \{(z_1, z_2) \in \mathbb{C}^2 : \max\{|z_1 - 2|, |z_2|\} \leq 1\|$, and let $J = \{(x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1|, |x_2|\} \leq 1\}$. Let $E = E_1 \cup J \cup E_2$. Then E is an NB-set. Indeed, if F is a subset of E with $c(F) > 0$, then at least one of the sets $F \cap P_1$, $F \cap P_2$ and $F \cap J$ must be non-pluripolar. Since each of the extremal functions Φ_{P_1}, Φ_{P_2} and Φ_J is continuous in $\mathbb{C}^2$ (*see* [20] or [6]), applying Theorems 10 and/or 11 we prove our claim.

References

[1] E.J. Akutowicz, *Extrapolation and approximation of functions given on linear sets of positive capacity*, Math. Zeitschr. **89** (1965), 414-419.

[2] S. Bernstein, *Sur la définition et les propriétés des fonctions analytiques d'une variable réelle*, Math. Ann **75** (1914), 449-468.

[3] S. Bernstein, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle*, Paris, Gauthier Villars, 1926.

[4] A. Beurling, *Quasianalyticity.* Mittag-Leffler Lectures on Complex Analysis, 1977-1978, Collected Works, vol. **I**, Birkhäuser, Boston 1989.

[5] A. Borichev, F. Nazarov and M. Sodin, *Lower bounds for quasianalytic functions.II. The Bernstein quasianalytic functions*, Math. Scand. **95** (2004), 44-58.

[6] M. Klimek, *Pluripotential Theory*, Oxford Univ. Press, London 1991.

[7] K. Kuratowski, *Wstęp do teorii mnogości i topologii*, PWN Warszawa 1955 (Polish).

[8] F. Leja, *Sur les suites de polynômes, les ensembles fermés et la fonction de Green*, Ann. Soc. Polon. Math. **12** (1934), 57-71.

[9] P. Lelong, *Sur une propriété simple des polynômes*, C. R. Acad. Sci. Paris **224** (1947), 883-885.

[10] A.I. Markushevich, *On best approximation*, Dokl. AN SSSR **44** (1944), 290-292 (Russian).

[11] S. Mazurkiewicz, *Les fonctions quasi-analytiques dans l'espace fonctionnel*, Mathematica (Cluj) **13** (1937), 16-21.

[12] S. Mazurkiewicz and H. Szmuszkowiczówna, *Sur les zéros des fonctions quasi-analytiques (B)*, Bull. Acad. Polon. Sci. A (1937), 1-6

[13] S.M. Mergelyan, *Some questions of the constructive function theory*, Proc. Steklov Inst. Math. **37**, Moscow 1951 (Russian).

[14] W. Pleśniak, *Quasianalytic functions of several complex variables*, Zeszyty Nauk. Uniw. Jagiello. (Prace Matematyczne) **15** (1971), 135-145

[15] W. Pleśniak *On superposition of quasianalytic functions*, Ann. Polon. Math. **26** (1972), 73-84.

[16] W. Pleśniak, *Some decompositions of functional spaces by means of quasianalytic functions of several variables*, Bull. Acad. Polon. Sci., sér. sci. math. astronom. phys. **20** (1972), 827-831.

[17] W. Pleśniak, *Quasianalytic functions in the sense of Bernstein*, Dissertationes Math. **147** (1977), 1-66.

[18] T. Ransford. *Potential theory in the complex plane*, Cambridge Univ. Press 1995.

[19] J. Siciak, *On some extremal functions and their applications in the theory of analytic functions of several complex variables*, Trans. Amer. Math. Soc. **105** (1962), 322-357.

[20] J. Siciak, *Extremal plurisubharmonic functions in* $\mathbb{C}^N$, Ann. Polon. Math. 39 (1981), 175-211.

[21] A. Skiba, *Bernstein quasianalytic functions on algebraic sets*, Univ. Iagello. Acta Math. **41** (2003), 215-223.

[22] H. Szmuszkowiczówna, *Un théorème sur les polynômes et son application à la théorie des fonctions quasi-analytiques*, C.R. Acad. Sci. Paris **198** (1934), 1119-1120.

[23] V.P. Zakharyuta, *Extremal plurisubharmonic functions, orthogonal polynomials and Bernstein-Walsh theorem for analytic functions of several complex variables*, Ann. Polon. Math. **33** (1976), 137-148 (Russian).

Wieslaw Pleśniak
Wydział Matematyki i Informatyki, Uniwersytet Jagielloński
ul. Stanisława Łojasiewicza 6, 30-348 Kraków, Poland
E-mail: Wieslaw.Plesniak@im.uj.edu.pl

(Received: 14.09.2013)