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An example of a topological group*

In constructing topological spaces, the product topology plays an important role. However, the classical concept of product topology can also be modified in several ways. We show how one of them leads to a simple example of a countable normal topological commutative group which does not satisfy the first axiom of countability or, equivalently, which is non-metrizable.

LEMMA. *If N is an infinite countable set, then there exists a cover C of N such that C is finitely additive (i.e. $U, V \in C$ implies $U \cup V \in C$) and C refines no countable subcollection of C (i.e. $C' \subset C$ and C' countable implies the existence of $U \in C$ satisfying $U \not\subset V$ for $V \in C'$).*

Proof. It is enough to construct a countable set N and a cover C of N which is finitely additive and refines no countable subcollection of C. To do this, let us choose, for each irrational t, a sequence of rationals $r_1(t), r_2(t), \ldots$ converging to t. We define C be the collection consisting of all finite unions of sets $\{r_1(t), r_2(t), \ldots\}$, where t are arbitrary irrationals. Let N be the union of all sets from C. Thus N is countable and, for each countable collection $C' \subset C$, there exist irrationals $t_1, t_2, \ldots$ such that every set belonging to C' is contained in a set

$$K_m = \bigcup_{i=1}^{m} \{r_1(t_i), r_2(t_i), \ldots\},$$

where $m = 1, 2, \ldots$ Now, let t_0 be an irrational such that $t_0 \neq t_i$ for $i = 1, 2, \ldots$ Then no set K_m contains the set $\{r_1(t_0), r_2(t_0), \ldots\} \in C$, and the proof of the lemma is complete.

EXAMPLE. Let A_n ($n = 1, 2, \ldots$) be a non-trivial finite commutative group and X be the direct sum of $A_1, A_2, \ldots$ We put $N = \{1, 2, \ldots\}$

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and use the collection $\boldsymbol{C}$ from our lemma to define a neighbourhood system $\boldsymbol{B}$ of the neutral element O in X. Namely, let $\boldsymbol{B} = \{G_U: U \in \boldsymbol{C}\}$, where G_U is the set of all elements $x = (x_1, x_2, \ldots) \in X$ such that $x_n = 0$ for $n \in U$.

Observe that each set G_U is a subgroup of the commutative group X. Moreover, since $\boldsymbol{C}$ is a cover of N, the intersection of all sets G_U $(U \in \boldsymbol{C})$ is $\{O\}$. Since $\boldsymbol{C}$ is finitely additive, we have

$$G_{U \cup V} \subset G_U \cap G_V$$

for $U, V \in \boldsymbol{C}$. Thus X with the topology generated by $\boldsymbol{B}$ is a topological group. This group is countable and, consequently, the underlying space is normal.

We prove that X does not satisfy the first axiom of countability at O. If $H_1, H_2, \ldots$ are neighbourhoods of O in X, then there exist sets $V_j \in \boldsymbol{C}$ such that

$$G_{V_j} \subset H_j$$

for $j = 1, 2, \ldots$ Since $\boldsymbol{C}$ does not refine the collection $\{V_1, V_2, \ldots\}$, there exists an element $U \in \boldsymbol{C}$ such that $U \not\subset V_j$ for $j = 1, 2, \ldots$ The latter condition implies that

$$G_{V_j} \not\subset G_U,$$

whence $H_j \not\subset G_U$ for $j = 1, 2, \ldots$, and we conclude that $\{H_1, H_2, \ldots\}$ cannot be a neighbourhood system of X at O.

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