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On Hardy–Orlicz spaces, I

CONTENTS

INTRODUCTION

The purpose of this paper is to generalize the known Hardy spaces H^p $(p > 0)$ of analytic functions in the unit disc ([2], [4] and [5]) and to present properties of these generalized spaces. This is done using the methods and results of the general theory of modular spaces ([11] and [12]) as well as the theory of Orlicz spaces ([3], [6], [7], [8], [9] and [10]).

Some generalizations of the Hardy spaces H^p for $p \geqslant 1$ can be found in [13], [14], [15] and [16]. However, the investigations in these papers are not based systematically on the theory of modular spaces of analytic functions. The paper attempts to develop such a theory.

The paper consists of four chapters. The first chapter briefly outlines the already known theory of Orlicz spaces and also presents some properties of analytic functions in the unit disc, needed for further studies. Hardy–Orlicz classes and spaces are introduced in the second chapter and there the inclusion theorems are considered. The norm generated by the function φ is defined in the third chapter in which also one concerns the mutual relations of various kinds of convergence of the sequences in the Hardy–Orlicz spaces. The fourth chapter considers the problem of the existence of an s-homogeneous norm in Hardy–Orlicz spaces.

Theorems and definitions from other chapters are referred to by the number of the chapter.

The main results of this paper have been already published in the Bulletin de L'Academie Polonaise des Sciences 15 (1966).

Finally, I would like to express my warmest thanks to Professor W. Orlicz for his helpful criticism, valuable advices and inspirations.

I. INTRODUCTORY NOTIONS

1. φ-functions

1.1. A real function φ defined for $u \geqslant 0$ is called a *φ-function*, if it is non-decreasing, continuous for $u \geqslant 0$, equal to 0 only at $u = 0$ and tending to ∞ as $u \to \infty$.

1.2.1. Let φ_1 and φ_2 be two φ-functions. φ_1 is said to be *non-weaker* than φ_2, in writing $\varphi_2 \dashv \varphi_1$, if

$$\varphi_2(u) \leqslant a\varphi_1(bu) \qquad \text{for } u \geqslant u_0,$$

where a, $b > 0$ and $u_0 \geqslant 0$ are constants.

Since the relation $\dashv$ is reflexive and transitive, we may say that

1.2.2. φ-functions φ_1 and φ_2 are called *equivalent*, in writing $\varphi_1 \sim \varphi_2$, if $\varphi_1 \dashv \varphi_2$ and $\varphi_2 \dashv \varphi_1$, simultaneously.

It is clear that $\varphi_1 \sim \varphi_2$, if and only if, for some constants $a_1, a_2, b_1, b_2 > 0$ and $u_0 \geqslant 0$ the following inequality is satisfied:

$$a_1\varphi_1(b_1 u) \leqslant \varphi_2(u) \leqslant a_2\varphi_1(b_2 u) \qquad \text{for } u \geqslant u_0.$$

1.2.3. LEMMA. *A necessary and sufficient condition in order that a φ-function φ be equivalent to a convex φ-function is that for some constants a, $b > 0$ and $u_0 \geqslant 0$ the following inequality holds:*

$$u_2^{-1}\varphi(u_2) \geqslant a u_1^{-1}\varphi(bu_1) \qquad \textit{for } u_2 \geqslant u_1 > u_0.$$

Changing this inequality to the converse one we obtain a necessary and sufficient condition in order that the φ-function φ be equivalent to a concave φ-function ([6] and [7]).

1.3.1. A φ-function φ is said *to satisfy condition* (Δ_2), if for some constants $d > 1$ and $u_0 \geqslant 0$ there holds the inequality

$$\varphi(2u) \leqslant d\varphi(u) \quad \text{for } u \geqslant u_0.$$

1.3.2. We say that a φ-function φ *satisfies condition* (Δ_a) for $a > 1$, if for some constants $d_a > 1$ and $u_a \geqslant 0$ there holds the inequality

$$\varphi(au) \leqslant d_a \varphi(u) \quad \text{for } u \geqslant u_a.$$

1.3.3. LEMMA. *The following four conditions are equivalent for any φ-function:*

1° *φ satisfies condition (Δ_2),*

2° *φ satisfies condition (Δ_a) for some $a > 1$,*

3° *φ satisfies condition (Δ_a) for all $a > 1$,*

4° *there exist a concave φ-function χ and a number $s > 0$ such that $\varphi(u) \sim \chi(u^s)$* ([6] and [7]).

1.3.4. LEMMA. *If a φ-function φ_1 satisfies condition (Δ_2) and $\varphi_1 \sim \varphi_2$, then φ_2 satisfies also condition (Δ_2)* (see [7]).

1.4.1. We say that a φ-function φ *satisfies condition* (V_2), if for some *constants* $d > 1$ and $u_0 \geqslant 0$ there holds the inequality

$$2\varphi(u) \leqslant \varphi(du) \quad \text{for } u \geqslant u_0.$$

1.4.2. A φ-function φ is said *to satisfy condition* (V_a) for $a > 1$, if for some constants $d_a > 1$ and $u_a \geqslant 0$ the following inequality holds:

$$a\varphi(u) \leqslant \varphi(d_a u) \quad \text{for } u \geqslant u_a.$$

1.4.3. LEMMA. *The following four conditions are equivalent for any φ-function:*

1° *φ satisfies condition (V_2),*

2° *φ satisfies condition (V_a) for some $a > 1$,*

3° *φ satisfies condition (V_a) for all $a > 1$,*

4° *there exist a convex φ-function ψ and a number $s > 0$ such that $\varphi(u) \sim \psi(u^s)$* ([6] and [7]).

1.4.4. LEMMA *If a φ-function φ_1 satisfies condition (V_2) and $\varphi_1 \sim \varphi_2$, then φ_2 satisfies also condition (V_2)* (cf. [7]).

1.5. LEMMA. *Let ψ be a convex φ-function satisfying the following two conditions:*

$$(0_1) \qquad \lim_{u\to 0+} u^{-1}\psi(u) = 0$$

and

$$(\infty_1) \qquad \lim_{u\to\infty} u^{-1}\psi(u) = \infty.$$

Then the function

$$\psi'(v) = \sup\{uv-\psi(u)\,|\,u \geqslant 0\} \quad \textit{for } v \geqslant 0$$

is also a convex φ-function and satisfies conditions (0_1) *and* (∞_1). *Moreover,* $(\psi')' = \psi$ ([3], Chapter I, §2).

1.6.1. The fundamental notion applied in this paper will be that of a log-convex φ-function:

A φ-function φ is called a *log-convex φ-function,* if it may be written in the form

$$\varphi(u) = \Phi(\log u) \quad \text{for } u > 0,$$

where Φ is a convex function on the whole real axis, satisfying condition (∞_1).

1.6.2. LEMMA. *Each log-convex φ-function φ can be written in the form*

$$(*) \qquad \varphi(u) = \int_0^u t^{-1}p(t)\,dt \quad \textit{for } u \geqslant 0,$$

where p is a positive and non-decreasing function for $t > 0$, tending to ∞ as $t \to \infty$.

Conversely, every function φ finite for $u \geqslant 0$ which is of the form $(*)$, *is a log-convex φ-function.*

Proof. If φ is a log-convex φ-function, then the function $\Phi(x) = \varphi(e^x)$ is positive and convex on the whole real axis, tends to 0 as $x \to -\infty$, and satisfies condition (∞_1). As a convex function, Φ may be written ([3], Chapter I, Theorem 1.1) in the form

$$\Phi(x) = \Phi(x_0) + \int_{x_0}^{x} p_1(\tau)\,d\tau,$$

where p_1 is a non-decreasing function on the whole axis. But $\Phi(x_0) \to 0$ as $x_0 \to -\infty$. Hence we get

$$(**) \qquad \Phi(x) = \int_{-\infty}^{x} p_1(\tau)\,d\tau \quad (-\infty < x < \infty).$$

Now, from the fact that Φ is positive we deduce that p_1 is also a positive function. Since

$$x^{-1}\big(\Phi(x)-\Phi(0)\big) = x^{-1}\int_0^x p_1(\tau)\,d\tau \leqslant p_1(x) \qquad \text{for } x>0,$$

we get from condition (∞_1), $p_1(x)\to\infty$ as $x\to\infty$. Substituting $x=\log u$ in $(**)$ we obtain

$$\varphi(u) = \int_{-\infty}^{\log u} p_1(\tau)\,d\tau = \int_0^u t^{-1}p_1(\log t)\,dt = \int_0^u t^{-1}p(t)\,dt \qquad \text{for } u>0,$$

where $p(t)=p_1(\log t)$ is a positive and non-decreasing function for $t>0$, and $p(t)\to\infty$ as $t\to\infty$.

Conversely, let a function φ finite for $u\geqslant 0$ be of the form $(*)$. It is seen directly that φ is a φ-function. Substituting $u=e^x$ in $(*)$ and writing $\varphi(e^x)=\Phi(x)$, the function Φ becames of the form $(**)$, where $p_1(\tau)=p(e^\tau)$ is a non-decreasing function on the whole axis and tends to ∞ as $\tau\to\infty$. Since p_1 is non-decreasing, we get for $x<y$

$$\begin{aligned}\Phi\big((x+y)/2\big) &= \int_{-\infty}^{(x+y)/2} p_1(\tau)\,d\tau\\ &\leqslant \int_{-\infty}^{x} p_1(\tau)\,d\tau + \tfrac{1}{2}\Big(\int_x^{(x+y)/2} p_1(\tau)\,d\tau + \int_{(x+y)/2}^{y} p_1(\tau)\,d\tau\Big)\\ &= \tfrac{1}{2}\Big(\int_{-\infty}^{x} p_1(\tau)\,d\tau + \int_{-\infty}^{y} p_1(\tau)\,d\tau\Big) = \tfrac{1}{2}\big(\Phi(x)+\Phi(y)\big).\end{aligned}$$

By the continuity of the function Φ, this means that Φ is convex. Now, since $p_1(\tau)\to\infty$ as $\tau\to\infty$, the inequality

$$x^{-1}\Phi(x) \geqslant x^{-1}\int_{x/2}^{x} p_1(\tau)\,d\tau \geqslant \tfrac{1}{2}p_1(x/2) \qquad \text{for } x>0$$

shows that Φ satisfies condition (∞_1), and the proof is concluded.

1.6.3. LEMMA. *Every log-convex φ-function is strictly increasing for $u\geqslant 0$.*

We deduce this directly from Lemma 1.6.2, since the function $p(t)$ in $(*)$ is positive for $t>0$.

From this lemma it follows at once that a log-convex φ-function φ possesses an inverse φ_{-1}. The function φ_{-1} is obviously a φ-function itself, but it does not need be log-convex. For example, $\varphi(u)=\exp(u^2)-1$ is a log-convex φ-function and possesses an inverse $\varphi_{-1}(u)=\log^{1/2}(1+u)$ which is not log-convex.

1.6.4. Lemma. *Every function $\varphi(u) = \psi(u^s)$, where $s>0$ and ψ is a convex φ-function, is a log-convex φ-function.*

Proof. Obviously, $\varphi(u) = \psi(u^s)$ is a φ-function. It is also easily seen that the function $\Phi(x) = \varphi(e^x) = \psi(e^{sx})$ is convex, because it is a superposition of convex functions ψ and e^{sx}. Now, by Jensen's inequality defining convex functions, we have for the convex φ-function ψ

$$\psi(1) = \psi\left(\frac{1}{v}\cdot v + \left(1-\frac{1}{v}\right)\cdot 0\right) \leqslant \frac{1}{v}\,\psi(v) + \left(1-\frac{1}{v}\right)\psi(0) = \frac{1}{v}\,\psi(v)$$

for $v>1$. Hence

$$x^{-1}\Phi(x) = x^{-1}\psi(e^{sx}) \geqslant x^{-1}e^{sx}\psi(1) \quad \text{for } x>0.$$

Since $x^{-1}e^{sx}\to\infty$ as $x\to\infty$, the function Φ satisfies condition (∞_1).

2. Orlicz spaces

2.1.1. Let φ be a φ-function. For any complex-valued function f defined and measurable in the interval $\langle 0, 2\pi)$ we define

$$\mathscr{I}_\varphi(f) = \int_0^{2\pi} \varphi(|f(t)|)\,dt.$$

2.1.2. Theorem. *The functional $\mathscr{I}_\varphi(\cdot)$ possesses the following properties:*

1° $\mathscr{I}_\varphi(f) = 0$, *if and only if*, $f=0$ ($f(t)=0$ *almost everywhere in* $\langle 0, 2\pi)$),

2° $\mathscr{I}_\varphi(af) = \mathscr{I}_\varphi(f)$ *for* $|a| = 1$,

3° $\mathscr{I}_\varphi(af_1+bf_2) \leqslant \mathscr{I}_\varphi(f_1)+\mathscr{I}_\varphi(f_2)$ *for real*, $a, b \geqslant 0$, $a+b = 1$,

4° *if* $\mathscr{I}_\varphi(f) < \infty$, *then* $\mathscr{I}_\varphi(af)\to 0$ *as* $a\to 0$ ([6] and [8]).

The above properties show that the functional $\mathscr{I}_\varphi(\cdot)$ is an example of a modular in the sense of Musielak and Orlicz [11].

2.1.3. Let us denote by L^φ the set of functions f measurable in $\langle 0, 2\pi)$ for which $\mathscr{I}_\varphi(f) < \infty$. The set L^φ is called an *Orlicz class* (see [3]).

Orlicz classes L^φ are convex sets, symmetric with respect to zero — this follows from 2.1.2, 3° and 2°, immediately — but in general they are not linear sets. Therefore the following notion is introduced.

By $L^{*\varphi}$ we denote the set of measurablef unctions f such that $af\in L^\varphi$ for some $a>0$ (depending on f). Clearly, the set $L^{*\varphi}$ is the linear hull of L^φ in the space of all measurable functions on $\langle 0, 2\pi)$. The set $L^{*\varphi}$ is called the *Orlicz space*. Moreover, we denote by M^φ the set of measurable functions f such that $af\in L^\varphi$ for each $a>0$. Applying 2.1.2 we verify easily that M^φ is the greatest linear subset of the space $L^{*\varphi}$, which is contained in L^{φ}. The set M^{φ} is called the *space of finite elements* in $L^{*\varphi}$ ([6]-[8] and [3]).

It is obvious that if $\hat{\varphi}_m(u) = \varphi(u/m)$ and $\check{\varphi}_m(u) = \varphi(mu)$, then

$$L^{*\varphi} = \bigcup_{m=1}^{\infty} L^{\hat{\varphi}_m} \quad \text{and} \quad M^{\varphi} = \bigcap_{m=1}^{\infty} L^{\check{\varphi}_m}.$$

In case $\varphi(u) = u^p$, $p > 0$, L^{φ} is the space of functions integrable with power p; we use then the usual symbol L^p in place of L^{φ}.

2.1.4. THEOREM. *The inclusion*

$$L^{\varphi} \subset \bigcup_{n=1}^{\infty} L^{\varphi_n}$$

holds if and only if there exist a positive integer m and numbers $d > 0$ and $u_0 \geqslant 0$ such that the following inequality is satisfied:

$$\varphi_m(u) \leqslant d\varphi(u) \quad \text{for } u \geqslant u_0 \quad ([6] \text{ and } [7]).$$

2.1.5. THEOREM. *The inclusion*

$$\bigcap_{n=1}^{\infty} L^{\varphi_n} \subset L^{\varphi}$$

holds if and only if there exist a positive integer m and numbers $d > 0$ and $u_0 \geqslant 0$ such that the following inequality is satisfied:

$$\varphi(u) \leqslant d \sup\{\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u)\} \quad \text{for } u \geqslant u_0 \quad ([6] \text{ and } [7]).$$

2.1.6. THEOREM. *The inclusion $L^{\varphi_1} \subset L^{\varphi_2}$ holds if and only if for some constants $d > 0$ and $u_0 \geqslant 0$ there is satisfied the inequality*

$$\varphi_2(u) \leqslant d\varphi_1(u) \quad \text{for } u \geqslant u_0.$$

Thus, the necessary and sufficient condition for the equality $L^{\varphi_1} = L^{\varphi_2}$ is the existence of constants $d_1, d_2 > 0$ and $u_0 \geqslant 0$ such that

$$d_1\varphi_1(u) \leqslant \varphi_2(u) \leqslant d_2\varphi_1(u) \quad \text{for } u \geqslant u_0 \quad ([6] \text{ and } [7]).$$

2.2.1. THEOREM. *If $L^{\varphi_1} \subset L^{\varphi_2}$, then $\mathscr{I}_{\varphi_1}(f_n) \to 0$ implies $\mathscr{I}_{\varphi_2}(f_n) \to 0$ for an arbitrary sequence (f_n) of functions from L^{φ_1}.*

2.2.2. A sequence (f_n), $f_n \in L^{*\varphi}$, is called *φ-convergent* or *modular convergent* to $f \in L^{*\varphi}$, in writing $f_n \xrightarrow{\varphi} f$, if $\mathscr{I}_{\varphi}(a(f_n - f)) \to 0$ for a constant $a > 0$ (depending on the sequence (f_n)).

2.3.1. We define for $f \in L^{*\varphi}$,

$$\|f\|_{\varphi}^{*} = \inf\{k > 0 \mid \mathscr{I}_{\varphi}(f/k) \leqslant k\}.$$

This functional in $L^{*\varphi}$ is called the *norm* generated by φ.

2.3.2. THEOREM. *$\|\cdot\|_\varphi^*$ possesses the following properties in $L^{*\varphi}$:*

1° *$\|\cdot\|_\varphi^*$ is an F-norm,*

2° *$L^{*\varphi}$ is a complete space with respect to $\|\cdot\|_\varphi^*$,*

3° *if $|f_1(t)| \leqslant |f_2(t)|$ for almost all $t \in \langle 0, 2\pi)$, then $\|f_1\|_\varphi^* \leqslant \|f_2\|_\varphi^*$,*

4° *$\mathscr{I}_\varphi(f) \leqslant \|f\|_\varphi^*$ if $\|f\|_\varphi^* \leqslant 1$; $\mathscr{I}_\varphi(f) \leqslant 1$ implies $\|f\|_\varphi^* \leqslant 1$,*

5° *if $\|f_n\|_\varphi^* \to 0$, then $f_n \xrightarrow{\varphi} 0$,*

6° *$\|f_n\|_\varphi^* \to 0$ if and only if $\mathscr{I}_\varphi(af_n) \to 0$ for every $a > 0$* ([6] and [8]).

The Orlicz space $L^{*\varphi}$ as a Fréchet space with norm $\|\cdot\|_\varphi^*$ will be denoted by $[L^{*\varphi}, \|\cdot\|_\varphi^*]$.

2.3.3. THEOREM. *The space M^φ is identical with the closed linear hull in $[L^{*\varphi}, \|\cdot\|_\varphi^*]$ of the set of bounded measurable functions on $\langle 0, 2\pi)$* ([6] and [8]).

2.3.4. THEOREM. *The space M^φ is separable in the norm $\|\cdot\|_\varphi^*$* ([6] and [8]).

2.3.5. THEOREM. *The following conditions are equivalent:*

1° *$\varphi_2 \dashv \varphi_1$,*

2° *$L^{*\varphi_1} \subset L^{*\varphi_2}$,*

3° *$M^{\varphi_1} \subset M^{\varphi_2}$,*

4° *$\|f_n\|_{\varphi_1}^* \to 0$ implies $\|f_n\|_{\varphi_2}^* \to 0$ for $f_n \in L^{*\varphi_1} \cap L^{*\varphi_2}$,*

5° *$f_n \xrightarrow{\varphi_1} 0$ implies $f_n \xrightarrow{\varphi_2} 0$ for $f_n \in L^{*\varphi_1} \cap L^{*\varphi_2}$* ([6] and [8]).

2.3.6. THEOREM. *The following conditions are equivalent:*

1° *φ satisfies condition (Δ_2),*

2° *$L^\varphi = L^{*\varphi}$,*

3° *$L^\varphi = M^\varphi$,*

4° *$L^{*\varphi}$ is a separable space in the norm $\|\cdot\|_\varphi^*$,*

5° *$f_n \xrightarrow{\varphi} 0$ implies $\|f_n\|_\varphi^* \to 0$ for $f_n \in L^{*\varphi}$* ([6] and [8]).

2.4.1. THEOREM. *If $\varphi(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and ψ is a convex φ-function, then an s-homogeneous norm may be defined in $L^{*\varphi}$ by the formula*

$$\|f\|_{s\varphi}^* = \inf\{k > 0 \mid \mathscr{I}_\varphi(f/k^{1/s}) \leqslant 1\}.$$

Norms $\|\cdot\|_\varphi^*$ and $\|\cdot\|_{s\varphi}^*$ are then equivalent in the sense that $\|f_n\|_\varphi^* \to 0$ if and only if $\|f_n\|_{s\varphi}^* \to 0$ for $f_n \in L^{*\varphi}$ ([6], [8] and [9]).

2.4.2. THEOREM. *If an s-homogeneous norm $\|\cdot\|^0$ is defined in $L^{*\varphi}$, $0 < s \leqslant 1$, such that the space $L^{*\varphi}$ is complete with respect to this norm, and convergence to 0 in this norm implies modular convergence to 0, then $\varphi(u) \sim \psi(u^s)$, where ψ is a convex φ-function* ([6] and [9]).

2.5.1. THEOREM. *If ψ is a convex φ-function satisfying conditions* (0_1) *and* (∞_1), *then a homogeneous norm may be defined in $L^{*\psi}$ by means of the formula*

$$\|f\|_{(\psi)}^{*} = \sup\left\{\int_0^{2\pi} |f(t)g(t)|\,dt \,\middle|\, \mathscr{I}_{\psi'}(g) \leqslant 1,\ g \in L^{\psi'}\right\}.$$

This norm is equivalent to the norms $\|\cdot\|_\psi^*$ and $\|\cdot\|_{1\psi}^*$; the equivalency of $\|\cdot\|_{(\psi)}^*$ and $\|\cdot\|_{1\psi}^*$ (which is also homogeneous) may be written in the form of the inequality $\|f\|_{1\psi}^* \leqslant \|f\|_{(\psi)}^* \leqslant 2\|f\|_{1(\psi)}^*$, *where* $f \in L^*_\psi$ ([3], Chapter II).

2.5.2. THEOREM. *If $f \in L^{*\psi}$, where ψ is a convex φ-function satisfying conditions* (0_1) *and* (∞_1), *then*

$$\|f\|_{(\psi)}^{*} = \inf\left\{\frac{1}{k}\big(1+\mathscr{I}_\psi(kf)\big) \,\middle|\, k > 0\right\} \qquad \text{([3], Chapter II).}$$

3. Classes N and N' of analytic functions in the unit disc

3.1.1. We denote by N the set of functions F analytic in the disc $D = \{z \mid |z| < 1\}$, for which

$$\sup\left\{\int_0^{2\pi} \log^+|F(re^{it})|\,dt \,\middle|\, 0 \leqslant r < 1\right\} < \infty,$$

where $\log^+ u = \log\sup\{1, u\}$ for $u \geqslant 0$.

3.1.2. THEOREM. *A function F analytic and not vanishing identically in the disc D belongs to N if and only if it can be written in the form*

$$(*) \qquad F(z) = B(z)\cdot\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\,dh(t)\right),$$

where h is a real-valued function of bounded total variation in $\langle 0, 2\pi\rangle$ and B is the Blaschke product

$$B(z) = e^{id}z^m \prod_n \frac{z-\zeta_n}{z-\zeta_n^*}\frac{1}{|\zeta_n|} \qquad (\zeta_n^* = 1/\bar{\zeta}_n).$$

Here, d is a real number, m — a positive integer, and ζ_n satisfy the inequalities $0 < |\zeta_n| < 1$ *and* $\sum_n (1-|\zeta_n|) < \infty$ ([17], Chapter VII, (7.30); [2]).

3.1.3. THEOREM. *If $F \in N$, then for almost every t there exists the limit*

$$\lim F(z) \overset{\mathrm{df}}{=} F(e^{it}),$$

if z tends to e^{it} between two chords of the disc D starting at the point e^{it}. Moreover, if the function F does not vanish identically in D, then $\log|F(e^{i\cdot})|\in L^1$ ([17], Chapter VII, (7.25); [2]).

3.2.1. We denote by N' the set of functions $F\in N$ for which the function h in 3.1.2 (*) is of absolutely continuous positive variation in the interval $\langle 0, 2\pi\rangle$. Moreover, we shall include in N' also the function identically equal to 0 in D.

3.2.2. THEOREM. *A function F analytic in the disc D belongs to the class N' if and only if the integrals*

$$\int_0^x \log^+|F(re^{it})|\,dt, \qquad 0\leqslant r<1,$$

are uniformly (with respect to r) absolutely continuous functions of the variable x ([17], Chapter VIII, (7.51)).

3.2.3. THEOREM. *A function F of the class N belongs to N' if and only if*

$$\lim_{r\to 1-}\int_0^{2\pi}\log^+|F(re^{it})|\,dt=\int_0^{2\pi}\log^+|F(e^{it})|\,dt$$

([17], Chapter VII, (7.53)).

3.2.4. THEOREM. *Let $F\in N'$ and let Φ be a non-negative, non-decreasing and convex function for $u\geqslant 0$. Then*

$$\int_0^{2\pi}\Phi\big(\log^+|F(re^{it})|\big)\,dt\leqslant\int_0^{2\pi}\Phi\big(\log^+|F(e^{it})|\big)\,dt$$

for every r, $0\leqslant r<1$ ([17], Chapter VII, (7.50)).

3.2.5. THEOREM. *Let f be a non-negative function on the interval $\langle 0, 2\pi)$, and let $\log f(\cdot)\in L^1$. Then there exists a function $F\in N'$ such that $|F(e^{it})|=f(t)$ for almost all t from the interval $\langle 0, 2\pi)$* ([17], Chapter VII (7.33)).

3.2.6. THEOREM. *Classes N and N' are linear sets in the space of functions analytic in the disc D.*

Proof. Let us remark that

$$\log(1+u)-\log 2\leqslant\log^+u\leqslant\log(1+u)\qquad\text{for } u\geqslant 0.$$

Now, let F and G analytic functions in D, and a and β be complex numbers. For an arbitrary measurable set E and arbitrary r, $0\leqslant r<1$, we have

$$\int_E \log^+ |\alpha F(re^{it}) + \beta G(re^{it})|\, dt \leqslant \int_E \log\bigl(1 + |\alpha F(re^{it}) + \beta G(re^{it})|\bigr)\, dt$$

$$\leqslant \int_E \log\bigl(1 + |\alpha|\,|F(re^{it})| + |\beta|\,|G(re^{it})|\bigr)\, dt$$

$$\leqslant \int_E \log\Bigl((1+|\alpha|)(1+|\beta|)\bigl(1+|F(re^{it})|\bigr)\bigl(1+|G(re^{it})|\bigr)\Bigr)\, dt$$

$$= \log\bigl((1+|\alpha|)(1+|\beta|)\bigr)\cdot \operatorname{mes} E + \int_E \log\bigl(1+|F(re^{it})|\bigr)\, dt +$$

$$+ \int_E \log\bigl(1+|G(re^{it})|\bigr)\, dt$$

$$\leqslant \log\bigl(4(1+|\alpha|)(1+|\beta|)\bigr)\cdot \operatorname{mes} E + \int_E \log^+ |F(re^{it})|\, dt +$$

$$+ \int_E \log^+ |G(re^{it})|\, dt.$$

Hence we deduce, by 3.2.2, that if $F, G \in N'$, then also $\alpha F + \beta G \in N'$. Taking in the above inequality $E = \langle 0, 2\pi)$ we see that if $F, G \in N$, then also $\alpha F + \beta G \in N$.

II. HARDY–ORLICZ CLASSES AND SPACES. COMPARISON OF CLASSES AND SPACES

1. The modular $\mu_\varphi(\cdot)$

To simplify the formulations of theorems and definitions we take here the convention that the letter φ will always mean a log-convex φ-function, because our considerations will concern only log-convex φ-function.

1.1.1. We define for any analytic function F in the disc $D = \{z \mid |z| < 1\}$

$$\mu_\varphi(r, F) = \mathscr{I}_\varphi\bigl(F(re^{i\cdot})\bigr) = \int_0^{2\pi} \varphi\bigl(|F(re^{it})|\bigr)\, dt \quad \text{for } 0 \leqslant r < 1$$

and

$$\mu_\varphi(F) = \sup\{\mu_\varphi(r, F) \mid 0 \leqslant r < 1\}.$$

1.1.2. Theorem. *Let F be an analytic function in the disc D. Then $\mu_\varphi(r, F)$ is a non-decreasing function for $0 \leqslant r < 1$, and so*

$$\mu_\varphi(F) = \lim_{r \to 1-} \mu_\varphi(r, F).$$

Proof. It is known ([17], Chapter VII, (7.11)) that a function F analytic in the disc D satisfies the inequality

$$\log|F(\varrho e^{i\tau})| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-\varrho^2}{r^2-2r\varrho\cos(t-\tau)+\varrho^2}\log|F(re^{it})|\,dt$$

for arbitrary $0 \leqslant \varrho < r < 1$. Since the function $\Phi(x) = \varphi(e^x)$ is non-decreasing and convex on the whole real axis we get, by the Jensen's integral inequality

$$\Phi\big(\log|F(\varrho e^{i\tau})|\big) \leqslant \Phi\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-\varrho^2}{r^2-2r\varrho\cos(t-\tau)+\varrho^2}\log|F(re^{it})|\,dt\right)$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-\varrho^2}{r^2-2r\varrho\cos(t-\tau)+\varrho^2}\Phi\big(\log|F(re^{it})|\big)\,dt,$$

i.e.

$$(*)\qquad \varphi\big(|F(\varrho e^{i\tau})|\big) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-\varrho^2}{r^2-2r\varrho\cos(t-\tau)+\varrho^2}\varphi\big(|F(re^{it})|\big)\,dt$$

for $0 \leqslant \varrho < r < 1$. Integrating this inequality with respect to τ, $0 \leqslant \tau < 2\pi$, and changing the order of integration at the right-hand side, we obtain

$$\int_0^{2\pi} \varphi\big(|F(\varrho e^{i\tau})|\big)\,d\tau \leqslant \int_0^{2\pi} \varphi\big(|F(re^{it})|\big)\,dt \qquad \text{for } 0 \leqslant \varrho < r < 1,$$

and this concludes the proof.

1.1.3. THEOREM. *Let F be an analytic function in the disc D. Then*

$$|F(z)| \leqslant \varphi_{-1}\left(\frac{\mu_\varphi(F)}{\pi(1-|z|)}\right) \qquad \textit{for } |z| < 1.$$

Proof. Since

$$\frac{r^2-\varrho^2}{r^2-2r\varrho\cos(t-\tau)+\varrho^2} \leqslant \frac{r+\varrho}{r-\varrho} \leqslant \frac{2}{r-\varrho} \qquad \text{for } 0 \leqslant \varrho < r < 1,$$

it follows from inequality $(*)$ in the proof of Theorem 1.1.2 that

$$\varphi\big(|F(\varrho e^{i\tau})|\big) \leqslant \frac{1}{\pi(r-\varrho)}\int_0^{2\pi} \varphi\big(|F(re^{it})|\big)\,dt \qquad \text{for } 0 \leqslant \varrho < r < 1.$$

Passing to the limit as $r \to 1-$, Theorem 1.1.2 yields

$$\varphi\big(|F(\varrho e^{i\tau})|\big) \leqslant \frac{\mu_\varphi(F)}{\pi(1-\varrho)} \qquad \text{for } 0 \leqslant \varrho < 1.$$

Since a log-convex φ-function φ possesses an inverse φ_{-1} which is a φ-function (see I.1.6.3), we obtain hence the inequality given in the theorem.

1.2.1. THEOREM. *If F is an analytic function in the disc D such that $\mu_\varphi(F) < \infty$, then $F \in N'$.*

Proof. The function $\Phi(x) = \varphi(e^x)$ is positive, non-decreasing, convex on the whole real axis, and satisfies condition (∞_1). Since it is convex and positive, we may apply Jensen's integral inequality for an arbitrary set E of positive measure and for an arbitrary r, $0 \leqslant r < 1$. We get

$$\Phi\left(\frac{1}{\operatorname{mes} E}\int_E \log^+|F(re^{it})|\,dt\right) \leqslant \frac{1}{\operatorname{mes} E}\int_E \Phi\big(\log^+|F(re^{it})|\big)\,dt$$

$$\leqslant \frac{1}{\operatorname{mes} E}\int_0^{2\pi} \Phi\big(\log^+|F(re^{it})|\big)\,dt \leqslant \frac{1}{\operatorname{mes} E}\left(\int_0^{2\pi} \Phi\big(\log|F(re^{it})|\big)\,dt + 2\pi\Phi(0)\right)$$

$$= \frac{1}{\operatorname{mes} E}\big(\mu_\varphi(r, F) + 2\pi\varphi(1)\big) \leqslant \frac{1}{\operatorname{mes} E}\big(\mu_\varphi(F) + 2\pi\varphi(1)\big)$$

and hence

$$(*)\qquad \Phi\left(\frac{1}{\operatorname{mes} E}\int_E \log^+|F(re^{it})|\,dt\right)\cdot \operatorname{mes} E \leqslant \mu_\varphi(F) + 2\pi\varphi(1).$$

Now, let us suppose there exist a sequence of measurable sets (E_n) such that $\operatorname{mes} E_n > 0$, $\operatorname{mes} E_n \to 0$, and a sequence (r_n) of numbers $0 \leqslant r_n < 1$ such that

$$\int_{E_n} \log^+|F(r_n e^{it})|\,dt \geqslant \eta > 0 \qquad \text{for } n = 1, 2, \ldots,$$

where η is a constant independent of n. Hence from the fact that the function Φ is non-decreasing and satisfies condition (∞_1) follows

$$\lim_{n\to\infty}\Phi\left(\frac{1}{\operatorname{mes} E_n}\int_{E_n} \log^+|F(r_n e^{it})|\,dt\right)\cdot \operatorname{mes} E_n \geqslant \lim_{n\to\infty}\Phi\left(\frac{\eta}{\operatorname{mes} E_n}\right)\cdot \operatorname{mes} E_n = \infty.$$

But this is a contradiction to inequality $(*)$, whose right-hand side has a constant finite value. Thus we conclude from $(*)$ that the integrals

$$\int_0^x \log^+|F(re^{it})|\,dt \qquad (0 \leqslant r < 1)$$

are uniformly (with respect to r) absolutely continuous functions of the variable x. By Theorem I.3.2.2, we obtain the thesis of the theorem.

1.2.2. THEOREM. *If $F \in N'$, then*

$$\mu_\varphi(F) = \mathscr{I}_\varphi\big(F(e^{i\cdot})\big) = \int_0^{2\pi} \varphi\big(|F(e^{it})|\big)\,dt.$$

Proof. Since the function $\Phi(x) = \varphi(e^x)$ is non-negative, non-decreasing and convex, we conclude from Theorem I.3.2.4, that

$$\int_0^{2\pi} \Phi\big(\log^+|F(re^{it})|\big)\,dt \leqslant \int_0^{2\pi} \Phi\big(\log^+|F(e^{it})|\big)\,dt \quad \text{for } 0 \leqslant r < 1.$$

Hence, taking into account Theorem I.3.1.3, we get by Fatou's lemma

$$\lim_{r\to 1-} \int_0^{2\pi} \Phi\big(\log^+|F(re^{it})|\big)\,dt = \int_0^{2\pi} \Phi\big(\log^+|F(e^{it})|\big)\,dt.$$

Since

$$\Phi(\log^+ u) = \Phi(\log\sup\{1, u\}) = \sup\{\Phi(0), \Phi(\log u)\} = \sup\{\varphi(1), \varphi(u)\},$$

the above equality may be written in the form

$$(*) \qquad \lim_{r\to 1-} \int_0^{2\pi} \sup\big\{\varphi(1), \varphi\big(|F(re^{it})|\big)\big\}\,dt = \int_0^{2} \sup\big\{\varphi(1), \varphi\big(|F(e^{it})|\big)\big\}\,dt.$$

However, $0 \leqslant \inf\big\{\varphi(1), \varphi\big(|F(re^{it})|\big)\big\} \leqslant \varphi(1)$ for each r, $0 \leqslant r < 1$, and each t. Moreover, by Theorem I.3.1.3,

$$\lim_{r\to 1-} \inf\big\{\varphi(1), \varphi\big(|F(re^{it})|\big)\big\} = \inf\big\{\varphi(1), \varphi\big(|F(e^{it})|\big)\big\}$$

for almost all t. Hence

$$(**) \qquad \lim_{r\to 1-} \int_0^{2\pi} \inf\big\{\varphi(1), \varphi\big(|F(re^{it})|\big)\big\}\,dt = \int_0^{2\pi} \inf\big\{\varphi(1), \varphi\big(|F(e^{it})|\big)\big\}\,dt.$$

Adding both sides of equalities $(*)$ and $(**)$, and taking into account the identity $\sup\{a, b\} + \inf\{a, b\} = a + b$ valid for any real a, b, we get

$$\lim_{r\to 1-} \int_0^{2\pi} \Big(\varphi(1) + \varphi\big(|F(re^{it})|\big)\Big)\,dt = \int_0^{2\pi} \Big(\varphi(1) + \varphi\big(F(e^{it})|\big)\Big)\,dt.$$

Subtracting on both sides $2\pi\varphi(1)$ we obtain the required equality.

1.2.3. Remark. The assumption $F \in N'$ in Theorem 1.2.2 cannot be replaced by the weaker one $F \in N$.

This will be shown by the example of the function

$$F(z) = \exp\left(\frac{1+z}{1-z}\right) \qquad (|z| < 1).$$

We have for this function

$$|F(re^{it})| = \exp\left(\operatorname{re}\frac{1+re^{it}}{1-re^{it}}\right) = \exp\left(\frac{1-r^2}{1-2r\cos t+r^2}\right) \quad \text{for } 0 \leqslant r < 1.$$

Hence

$$\int_0^{2\pi} \log^+ |F(re^{it})|\,dt = \int_0^{2\pi} \frac{1-r^2}{1-2r\cos t+r^2}\,dt = 2\pi \quad \text{for } 0 \leqslant r < 1,$$

and this means that $F \epsilon N$. Now, we show that $\mu_\varphi(F) = \infty$ for every log-convex φ-function φ. By the inequality $|\sin x| \leqslant |x|$,

$$1-\cos(1-r) = 2\cdot \sin^2\frac{1-r}{2} \leqslant 2\left(\frac{1-r}{2}\right)^2 = \tfrac{1}{2}(1-r)^2.$$

Hence we get for $|t| \leqslant 1-r$

$$\frac{1-r^2}{1-2r\cos t+r^2} \geqslant \frac{1-r^2}{1-2r\cos(1-r)+r^2} = \frac{1-r^2}{(1-r)^2+2r\big(1-\cos(1-r)\big)}$$
$$\geqslant \frac{1-r^2}{(1-r)^2+r(1-r)^2} = \frac{1}{1-r}.$$

Thus, we obtain for $\varphi(u) = \Phi(\log u)$

$$\mu_\varphi(r, F) = \int_0^{2\pi} \varphi\big(|F(re^{it})|\big)\,dt = \int_0^{2\pi} \Phi\big(\log|F(re^{it})|\big)\,dt$$
$$= \int_0^{2\pi} \Phi\left(\frac{1-r^2}{1-2r\cos t+r^2}\right) dt \geqslant \int_{\{|t|\leqslant 1-r\}} \Phi\left(\frac{1-r^2}{1-2r\cos t-r^2}\right) dt$$
$$\geqslant 2\Phi\left(\frac{1}{1-r}\right)(1-r).$$

Now, condition (∞_1) for the function Φ yields

$$\lim_{r\to 1-} \mu_\varphi(r, F) \geqslant 2\lim_{r\to 1-} \Phi\left(\frac{1}{1-r}\right)(1-r) = \infty,$$

and this means that $\mu_\varphi(F) = \infty$. On the other hand, let us remark that

$$|F(e^{it})| = \lim_{r\to 1-} \exp\left(\frac{1-r^2}{1-2r\cos t+r^2}\right) = e^0 = 1$$

for $0 < t < 2\pi$, and so

$$\mathscr{I}_\varphi\big(F(e^{i\cdot})\big) = \int_0^{2\pi} \varphi(1)\,dt = 2\pi\varphi(1).$$

1.3. Theorem. *The non-negative functional $\mu_\varphi(\cdot)$ possesses the following properties on the set of analytic functions in D:*

1° $\mu_\varphi(F) = 0$ *if and only if* $F = 0$ *(i.e.* $F(z) = 0$ *identically in* D*),*

2° $\mu_\varphi(aF) = \mu_\varphi(F)$ *for numbers* a *with absolute value* $|a| = 1$,

3° $\mu_\varphi(aF_1 + bF_2) \leqslant \mu_\varphi(F_1) + \mu_\varphi(F_2)$ *for real* $a, b \geqslant 0$, $a+b = 1$,

4° *if* $\mu_\varphi(F) < \infty$, *then* $\mu_\varphi(aF) \to 0$ *as* $a \to 0$.

Proof. If $F = 0$, then obviously $\mu_\varphi(F) = 0$. Conversely, if $\mu_\varphi(F) = 0$, then $F = 0$, by Theorem 1.1.3. Properties 2° and 3° are obtained from the corresponding properties of $\mathscr{I}_\varphi(\cdot)$ (see I.2.1.2), immediately. Finally, property 4° is deduced from Theorem 1.2.1 and 1.2.2, and from the analogous property of $\mathscr{I}_\varphi(\cdot)$.

Similary as the functional $\mathscr{I}_\varphi(\cdot)$ for measurable functions of a real variable, the functional $\mu_\varphi(\cdot)$ for analytic functions is an example of a modular in the sense of Musielak and Orlicz [11].

1.4.1. We define two simple operators for analytic functions in the disc D.

Let F be an analytic function in the disc D, and let r and h be real numbers, $0 \leqslant r < 1$. We denote by T_rF and S_hF functions defined by formulae

$$T_rF(z) = F(rz) \quad \text{and} \quad S_hF(z) = F(ze^{ih}) \quad \text{for } z \in D.$$

Cleary, operators T_r and S_h are distributive and transform analytic functions in D into analytic function in D.

1.4.2. Lemma. *Let F be an analytic function in D. Then we have for every r, $0 \leqslant r < 1$,*

$$\mu_\varphi(r, F) = \mu_\varphi(T_rF).$$

Proof. Let us remark that for an arbitrary fixed r, $0 \leqslant r < 1$, the function T_rF is bounded in D, and $T_rF(e^{it}) = F(re^{it})$ for all t. Since T_rF is bounded, it belongs to N'. Hence, by Theorem 1.2.2,

$$\mu_\varphi(T_rF) = \int_0^{2\pi} \varphi\big(|T_rF(e^{it})|\big)\,dt = \int_0^{2\pi} \varphi\big(|F(re^{it})|\big)\,dt = \mu_\varphi(r, F).$$

1.4.3. Lemma. *If F is an analytic function in the disc D, then we have for an arbitrary real number h*

$$\mu_\varphi(S_hF) = \mu_\varphi(F).$$

This follows immediately, from the fact that the functions $\varphi\big(|F(re^{it})|\big)$, $0 \leqslant r < 1$, are 2π-periodic.

2. The definition of Hardy–Orlicz classes and spaces

2.1.1. Let us denote by H^{φ} the set of functions F analytic in D for which $\mu_{\varphi}(F) < \infty$. In the sequel the set H^{φ} will be called the *Hardy–Orlicz class.*

It follows directly from Theorems 1.2.1 and 1.3 that H^{φ} is a convex set symetric with respect to zero in the class N'. In general, H^{φ} are not linear sets. Therefore we define, just as in the case of functions of a real variable:

We denote by $H^{*\varphi}$ the set of analytic functions F such that $aF \epsilon H^{\varphi}$ for an $a > 0$ (depending in general on F). Obviously, the set $H^{*\varphi}$ is the linear hull of H^{φ} in N'. The set $H^{*\varphi}$ will be called the *Hardy–Orlicz space.*

Moreover, we shall denote by K^{φ} the set of analytic functions F such that $aF \epsilon H^{\varphi}$ for every $a > 0$. It is easily shown that K^{φ} is the greatest linear subset of the Hardy–Orlicz space $H^{*\varphi}$ contained in H^{φ}. The set K^{φ} will be called the *space of finite elements in* $H^{*\varphi}$.

Obviously, if $\varphi_m(u) = \varphi(u/m)$ and $\tilde{\varphi}_m(u) = \varphi(mu)$, then

$$H^{*\varphi} = \bigcup_{m=1}^{\infty} H^{\varphi_m} \quad \text{and} \quad K^{\varphi} = \bigcap_{m=1}^{\infty} H^{\tilde{\varphi}_m}.$$

Let us denote yet by K the set of functions F analytic in the disc D and continuous in the closed disc $\bar{D} = \{z|\ |z| \leqslant 1\}$ and by H^{∞} the set of functions F analytic and bounded in the disc D. The following inclusions are evident:

$$K \subset H^{\infty} \subset K^{\varphi} \subset H^{\varphi} \subset H^{*\varphi} \subset N' \subset N.$$

In case $\varphi(u) = u^p$, $p > 0$, H^{φ} is the Hardy space for the power p; then we write H^p in place of H^{φ}.

2.1.2. Theorem. *A function F analytic in the disc D belongs to H^{φ} (to $H^{*\varphi}$, K^{φ}, respectively) if and only if it belongs to N' and its limit function $F(e^{i\cdot})$ belongs to L^{φ} (to $L^{*\varphi}$, M^{φ}, respectively).*

This follows at once from Theorem 1.3.1 and 1.3.2.

Let us turn to Theorems 3.2.3 and I.3.2.6 and let us remark, that the correspondence between an analytic function F from the class N and its limit function $F(e^{i\cdot})$ is an isomorphism of the class N onto the set of measurable fuctions f of a real variable in $\langle 0, 2\pi)$ for which there exists a function $F \epsilon N$ such that $f(t) = \lim_{r\to 1-} F(re^{it})$ for almost all t from the interval $\langle 0, 2\pi)$. Thus, if we neglect the difference between isomorphic spaces, we may write Theorem 2.1.2 in the form

$$H^{\varphi} = N' \cap L^{\varphi}, \quad H^{*\varphi} = N' \cap L^{*\varphi}, \quad K^{\varphi} = N' \cap M^{\varphi}.$$

2.1.3. THEOREM. *If F belongs to $H^{*\varphi_1}$ and $F(e^{i\cdot})$ belongs to L^{φ_2} (to $L^{*\varphi_2}$, M^{φ_2}, respectively), then F belongs to H^{φ_2} (to $H^{*\varphi_2}$, K^{φ_2}, respectively).*

This follows from Theorem 2.1.2, immediately.

The above theorem is more general than an analogous theorem given by Safronova [14] for convex φ-functions, because we suppose φ to be only a log-convex φ-function.

2.2. THEOREM. *Every function F from the class N' belongs to a Hardy–Orlicz class H^φ.*

Proof. We denote for a function $F \in N'$

$$E_n = \{t \in \langle 0, 2\pi)\,|\; n-1 \leqslant |F(e^{it})| < n\} \quad \text{for } n = 1, 2, \ldots$$

Applying the inequality $\log(1+u) \leqslant \log 2 + \log^+ u$ for $u \geqslant 0$ we have

$$\sum_{n=2}^{\infty} \log n \cdot \operatorname{mes} E_n \leqslant \int_0^{2\pi} \log\big(1+|F(e^{it})|\big)\,dt \leqslant 2\pi \log 2 + \int_0^{2\pi} \log^+ |F(e^{it})|\,dt < \infty.$$

It is known that one may choose a non-decreasing and tending to ∞ sequence of real numbers a_n such that still

$$\sum_{n=2}^{\infty} a_n \cdot \log n \cdot \operatorname{mes} E_n < \infty.$$

Here we may suppose additionally that $0 < a_2 < a_3 \log 2$; we construct a function

$$p(t) = \begin{cases} \frac{1}{2} a_2 t & \text{for } 0 < t < 2, \\ a_n & \text{for } n-1 \leqslant t < n,\ n = 3, 4, \ldots \end{cases}$$

Since the function p is positive, non-decreasing for $t > 0$ and tends to ∞ as $t \to \infty$, the function

$$\varphi(u) = \int_0^u t^{-1} p(t)\,dt \quad \text{for } u \geqslant 0$$

is a log-convex φ-function, by Lemma I.1.6.2, Since

$$\begin{aligned} \varphi(n) &= \int_0^n t^{-1} p(t)\,dt = a_2 + \sum_{k=3}^{n} a_k \big(\log k - \log(k-1)\big) \\ &\leqslant a_n \Big(\log 2 + \sum_{k=3}^{n} \big(\log k - \log(k-1)\big)\Big) = a_n \log n \quad \text{for } n = 3, 4, \ldots, \end{aligned}$$

Theorem 1.3.2 gives

$$\mu_\varphi(F) = \int_0^{2\pi} \varphi\big(|F(e^{it})|\big)\,dt \leqslant \varphi(2)\cdot(\operatorname{mes} E_1 + \operatorname{mes} E_2) + \sum_{n=3}^{\infty} \varphi(n) \operatorname{mes} E_n$$
$$\leqslant 2\pi\varphi(2) + \sum_{n=3}^{\infty} a_n \cdot \log n \cdot \operatorname{mes} E_n < \infty.$$

This proves $F \in H^\varphi$.

3. Comparison of classes and spaces

3.1.1. LEMMA. *If f is a real function belonging to L^φ and satisfying the inequality $f(t) \geqslant c$ for almost all t from the interval $\langle 0, 2\pi)$, where c is a positive constant, then* $\log f(\cdot) \in L^1$.

Proof. The function $\Phi(x) = \varphi(e^x)$ satisfies condition (∞_1). Hence there exists a real number $x_0 \geqslant 0$ such that $x \leqslant \Phi(x)$ for $x \geqslant x_0$, and $x \leqslant x_0 + \Phi(x)$ for all real x. Substituting $x = \log u$ we obtain

$$\log u \leqslant x_0 + \varphi(u) \qquad \text{for } u > 0.$$

Denoting

$$E = \{t \in \langle 0, 2\pi) \mid f(t) \geqslant 1\}$$

we get

$$\int_0^{2\pi} |\log f(t)|\,dt = \int_E \log f(t)\,dt - \int_{\langle 0, 2\pi)\setminus E} \log f(t)\,dt$$
$$\leqslant 2\pi x_0 + \int_E \varphi\big(f(t)\big)\,dt - 2\pi \log\inf\{1, c\}$$
$$\leqslant 2\pi(x_0 - \log\inf\{1, c\}) + \mathscr{I}_\varphi(f) < \infty.$$

3.1.2. THEOREM. *The inclusion*

$$\bigcap_{\nu=1}^{\infty} H^{\varphi_\nu} \subset H^\varphi$$

holds if and only if for a positive integer m and for some constants $d > 0$ the following inequality is satisfied:

$$(*) \qquad \varphi(u) \leqslant d\cdot\sup\{\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u)\} \quad \textit{for } u \geqslant u_0.$$

Proof. If $(*)$ holds, then applying Theorem I.2.1.5, we have the following inclusion for Orlicz classes:

$$(**) \qquad \bigcap_{\nu=1}^{\infty} L^{\varphi_\nu} \subset L^\varphi.$$

We multiply this inclusion by N'. By 2.1.2, we get the inclusion for Hardy–Orlicz classes given in the theorem.

Conversely, if inequality (*) does not hold, then according to Theorem I.2.1.5, inclusion (**) also does not hold. Hence there exists a measurable function g such that $g \in L^{\varphi_\nu}$ for each ν and $g \notin L^{\varphi}$. We take the function

$$f(t) = \begin{cases} |g(t)| & \text{if } |g(t)| \geqslant 1, \\ 1 & \text{elsewhere in } \langle 0, 2\pi). \end{cases}$$

Since

$$\mathscr{I}_{\varphi_\nu}(f) \leqslant \mathscr{I}_{\varphi_\nu}(g) + 2\pi\varphi_\nu(1) \quad \text{and} \quad \mathscr{I}_{\varphi}(f) \geqslant \mathscr{I}_{\varphi}(g),$$

we have also $f \in L^{\varphi_\nu}$ for each ν and $f \notin L^{\varphi}$. Applying Lemma 3.1.1 we deduce from $f \in L^{\varphi_\nu}$ and $f(t) \geqslant 1$ for $t \in \langle 0, 2\pi)$ that $\log f(\cdot) \in L^1$. Hence, by Theorem I.3.2.5, there exists a function $F \in N'$ such that $|F(e^{it})| = f(t)$ for almost all t from the interval $\langle 0, 2\pi)$. Applying Theorem 1.3.2 we get $F \in H^{\varphi_\nu}$ for each ν and $F \notin H^{\varphi}$.

3.1.3. THEOREM. *The inclusion*

$$H^{\varphi} \subset \bigcup_{\nu=1}^{\infty} H^{\varphi_\nu}$$

holds if and only if for a positive integer m and for some constants $d > 0$ and $u_0 \geqslant 0$ the following inequality is satisfied:

$$(*) \qquad \varphi_m(u) \leqslant d\varphi(u) \quad \textit{for } u \geqslant u_0.$$

Proof is performed similarly as in case 3.1.2. Namely, if (*) holds, then Theorem I.2.1.4 implies the inclusion

$$(**) \qquad L^{\varphi} \subset \bigcup_{\nu=1}^{\infty} L^{\varphi_\nu}.$$

We multiply this inclusion by N'. By 2.1.2, we get the required inclusion. Now, if (*) does not hold, then (**) does not hold, too. Hence there exists a measurable function g such that $g \in L^{\varphi}$ and $g \notin L^{\varphi_\nu}$ for each ν. We define the function f as in the proof of 3.1.2. Since

$$\mathscr{I}_{\varphi}(f) \leqslant \mathscr{I}_{\varphi}(g) + 2\pi\varphi(1) \quad \text{and} \quad \mathscr{I}_{\varphi_\nu}(f) \geqslant \mathscr{I}_{\varphi_\nu}(g),$$

we have $f \in L^{\varphi}$ and $f \notin L^{\varphi_\nu}$ for each ν. But $f \in L^{\varphi}$ and $f(t) \geqslant 1$ whence, by Lemma 3.1.1, $\log f(\cdot) \in L^1$. Applying Theorem I.3.2.5, we see that there exists a function $F \in N'$ such that $|F(e^{it})| = f(t)$ for almost all t from the interval $\langle 0, 2\pi)$. According to Theorem 1.3.2, $F \in H^{\varphi}$ and $F \notin H^{\varphi_\nu}$ for each ν.

3.1.4. THEOREM. *The inclusion $H^{\varphi_1} \subset H^{\varphi_2}$ holds if and only if for some constants $d > 0$ and $u_0 \geqslant 0$ the following inequality is satisfied:*

$$\varphi_2(u) \leqslant d\varphi_1(u) \quad \textit{for } u \geqslant u_0.$$

Thus, the equality $H^{\varphi_1} = H^{\varphi_2}$ *holds if and only if for some constants* $d_1, d_2 > 0$ *and* $u_0 \geqslant 0$ *we have*

$$d_1 \cdot \varphi_1(u) \leqslant \varphi_2(u) \leqslant d_2 \cdot \varphi_1(u) \quad \textit{for } u \geqslant u_0.$$

This Theorem is a special case of Theorem 3.1.2 (and also Theorem 3.1.3).

3.1.5. The inclusion

$$H^{\varphi} \subset \bigcap_{\nu=1}^{\infty} H^{\varphi_\nu}$$

holds if and only if for each positive integer m there exists constants $d_m > 0$ and $u_m \geqslant 0$ such that the inequalities

$$\varphi_m(u) \leqslant d_m \cdot \varphi(u) \quad \text{for } u \geqslant u_m$$

are satisfied for $m = 1, 2, \ldots$

This follows from Theorem 3.1.4, immediately.

3.1.6. THEOREM. *The inclusion*

$$\bigcup_{\nu=1}^{\infty} H^{\varphi_\nu} \subset H^{\varphi}$$

holds if and only if for each positive integer m there exists constants $d_m > 0$ *and* $u_m \geqslant 0$ *such that the inequalities*

$$\varphi(u) \leqslant d_m \varphi_m(u) \quad \textit{for } u \geqslant u_m.$$

are satisfied for $m = 1, 2, \ldots$

This follows from Theorem 1.3.4, immediately.

3.1.7. THEOREM. *The identity*

$$H^{\varphi} = \bigcup_{\nu=1}^{\infty} H^{\varphi_\nu} \tag{*}$$

holds if and only if there exists a positive integer m for which

$$H^{\varphi_n} \subset H^{\varphi_m} = H^{\varphi} \quad \textit{for } n = 1, 2, \ldots \tag{**}$$

Proof. If (*) holds, then according to Theorem 3.1.3 there exists a positive integer m such that $\varphi_m(u) \leqslant d \cdot \varphi(u)$ for $u \geqslant u_0$, where $d > 0$ and $u_0 \geqslant 0$. Thus, applying Theorem 3.1.4, we get $H^{\varphi} \subset H^{\varphi_m}$. Hence we have

$$H^{\varphi_n} \subset \bigcup_{\nu=1}^{\infty} H^{\varphi_\nu} = H^{\varphi} \subset H^{\varphi_m} \quad \text{for } n = 1, 2, \ldots,$$

and we obtain (**). Conversely, it is obvious that (**) implies (*).

3.1.8. THEOREM. *The identity*

$$(*) \qquad H^{\varphi} = \bigcap_{\nu=1}^{\infty} H^{\varphi_\nu}$$

holds if and only if there exists a positive integer m such that

$$(**) \qquad H^{\varphi_n} \subset \bigcap_{\nu=1}^{m} H^{\varphi_\nu} = H^{\varphi} \quad \text{for } n = 1, 2, \ldots$$

Proof. If $(*)$ holds, then we deduce from Theorem 3.1.2 the existence of a positive integer m for which

$$(***) \qquad \varphi(u) \leqslant d \cdot \sup\{\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u)\} \quad \text{for } u \geqslant u_0,$$

where $d > 0$ and $u_0 \geqslant 0$. We define $\tilde{\varphi}_n(u) = \varphi_n(u)$ for $n = 1, 2, \ldots, m$, and $\tilde{\varphi}_n(u) = \varphi_m(u)$ for $n = m+1, \ldots$ By Theorem 3.1.2, we obtain from inequality $(***)$

$$\bigcap_{\nu=1}^{\infty} H^{\varphi_\nu} = H^{\varphi} \supset \bigcap_{\nu=1}^{\infty} H^{\tilde{\varphi}_\nu} = \bigcap_{\nu=1}^{m} H^{\varphi_\nu}.$$

Hence follows $(**)$. The converse implication is obvious.

3.1.9. THEOREM. (α) *If $H^{\varphi_n} \subset H^{\varphi_{n+1}}$ and $H^{\varphi_{n+1}} \neq H^{\varphi_n}$ for $n = 1, 2, \ldots$, then*

$$H^{\varphi} \neq \bigcup_{\nu=1}^{\infty} H^{\varphi_\nu}$$

for each φ.

(β) *If $H^{\varphi_n} \supset H^{\varphi_{n+1}}$ and $H^{\varphi_{n+1}} \neq H^{\varphi_n}$ for $n = =1, 2, \ldots$, then*

$$H^{\varphi} \neq \bigcap_{\nu=1}^{\infty} H^{\varphi_\nu}$$

for each φ.

This follows from Theorems 3.1.7 and 3.1.8, immediately.

3.2.1. THEOREM. *The necessary and sufficient condition for the inclusion $H^{*\varphi_1} \subset H^{*\varphi_2}$ is $\varphi_2 \dashv \varphi_1$. Thus, the equality $H^{*\varphi_1} = H^{*\varphi_2}$ holds if and only if $\varphi_1 \sim \varphi_2$.*

Proof. If $\varphi_2 \dashv \varphi_1$, then we have $L^{*\varphi_1} \subset L^{*\varphi_2}$, by Theorem I.2.3.5. We multiply this inclusion by N'. By 2.1.2, we get $H^{*\varphi_1} \subset H^{*\varphi_2}$. Conversely, if the inclusion $H^{*\varphi_1} \subset H^{*\varphi_2}$ holds, then

$$H^{\varphi_1} \subset \bigcup_{m=1}^{\infty} H^{\tilde{\varphi}_m} = H^{*\varphi_2},$$

where $\tilde{\varphi}_m(u) = \varphi_2(u/m)$. Applying Theorem 3.1.3 we obtain that for positive integer m and for some constants $d > 0$ and $u_0 \geqslant 0$ there holds the inequality

$$\varphi_2(u/m) \leqslant d\cdot\varphi_1(u) \quad \text{for } u \geqslant u_0.$$

But this means $\varphi_2 \dashv \varphi_1$.

3.2.2. THEOREM. *The necessary and sufficient condition for the inclusion $K^{\varphi_1} \subset H^{*\varphi_2}$ is $\varphi_2 \dashv \varphi_1$.*

Proof. If $\varphi_2 \dashv \varphi_1$, then we deduce from Theorem 3.2.1 at once that $K^{\varphi_1} \subset H^{*\varphi_1} \subset H^{*\varphi_2}$.

Conversely, let us suppose $\varphi_2 \dashv \varphi_1$ does not hold. Then there exists a sequence (u_n) increasing to ∞ such that

$$u_1 \geqslant 1 \quad \text{and} \quad \varphi_2(u_n) > 2^n\varphi_1(n^2u_n) \quad \text{for } n = 1, 2, \ldots$$

We define disjoint sets $E_n \subset \langle 0, 2\pi)$ of measures

$$\operatorname{mes} E_n = \frac{2\pi\varphi_1(1)}{2^n\varphi_1(n^2u_n)},$$

and a function

$$f(t) = \begin{cases} n\cdot u_n & \text{for } t\in E_n,\ n = 1, 2, \ldots, \\ 1 & \text{elsewhere in } \langle 0, 2\pi). \end{cases}$$

We have for an arbitrary positive integer m

$$\begin{aligned} \mathscr{I}_{\varphi_1}(mf) &= \int_0^{2\pi} \varphi_1\big(mf(t)\big)\,dt \\ &\leqslant 2\pi\varphi_1(1) + \sum_{n=1}^{m-1} \varphi_1(mnu_n)\frac{2\pi\varphi_1(1)}{2^n\varphi_1(n^2u_n)} + \sum_{n=m}^{\infty}\varphi_1(mnu_n)\frac{2\pi\varphi_1(1)}{2^n\varphi_1(n^2u_n)} \\ &\leqslant 2\pi\varphi_1(1) + \varphi_1(m^2u_m)\cdot 2\pi + 2\pi\varphi_1(1) < \infty. \end{aligned}$$

and

$$\begin{aligned} \mathscr{I}_{\varphi_2}\left(\frac{1}{m}f\right) &= \int_0^{2\pi}\varphi_2\left(\frac{1}{m}f(t)\right)dt \geqslant \sum_{n=m}^{\infty}\varphi_2\left(\frac{n}{m}u_n\right)\frac{2\pi\varphi_1(1)}{2^n\varphi_1(n^2u_n)} \\ &\geqslant \sum_{n=m}^{\infty}\varphi_2(u_n)\frac{2\pi\varphi_1(1)}{2^n\varphi_1(n^2u_n)} \geqslant \sum_{n=m}^{\infty} 2\pi\varphi_1(1) = \infty. \end{aligned}$$

This means that $f\in M^{\varphi_1}$ and $f\notin L^{*\varphi_2}$. From that $f\in M^{\varphi_1} \subset L^{\varphi_1}$ and $f(t) \geqslant 1$ for $t\in\langle 0, 2\pi)$ we obtain $\log f(\cdot)\in L^1$, by Lemma 3.1.1. Now, by Theorem I.3.2.5, there exists a function $F\in N'$ such that $|F(e^{it})| = f(t)$ for almost all t from the interval $\langle 0, 2\pi)$. Applying Theorem 1.3.2, we get here $F\in K^{\varphi_1}$ and $F\notin H^{*\varphi_2}$.

3.2.3. THEOREM. *The inclusion $K^{\varphi_1} \subset K^{\varphi_2}$ holds if and only if $\varphi_2 \dashv \varphi_1$.*

Proof. If $\varphi_2 \dashv \varphi_1$, then $M^{\varphi_1} \subset M^{\varphi_2}$, by Theorem I.2.3.5. Multiplying this inclusion by N' we obtain $K^{\varphi_1} \subset K^{\varphi_2}$. Conversely, if $K^{\varphi_1} \subset K^{\varphi_2}$, then also $K^{\varphi_1} \subset H^{*\varphi_2}$. By Theorem 3.2.2, we conclude $\varphi_2 \dashv \varphi_1$.

3.2.4. *The inclusion $H^{*\varphi_1} \subset K^{\varphi_2}$ holds if and only if for every positive integer m there exists constants $d_m > 0$ and $u_m \geqslant 0$ such that the inequalities*

$$(*) \qquad \varphi_2(mu) \leqslant d_m \cdot \varphi_1(u) \quad \textit{for } u \geqslant u_m$$

are satisfied for $m = 1, 2, \ldots$

Proof. Supposing $(*)$, Theorem 3.1.5 yields

$$(**) \qquad H^{\varphi_1} \subset \bigcap_{\nu=1}^{\infty} H^{\tilde{\varphi}_\nu} = K^{\varphi_2},$$

where $\tilde{\varphi}_m(u) = \varphi_2(mu)$. Since K^{φ_2} is a linear set and $H^{*\varphi_1}$ is the linear hull of H^{φ_1} in the class N', we get $H^{*\varphi_1} \subset K^{\varphi_2}$. Conversely, if the inclusion $H^{*\varphi_1} \subset K^{\varphi_2}$ holds, then there holds also inclusion $(**)$. Applying Theorem 3.1.5, we get $(*)$.

3.3.1. THEOREM. *The necessary and sufficient condition for the equality $K^{\varphi_1} = H^{*\varphi_2}$ is that $\varphi_1 \sim \varphi_2$ and φ_2 satisfies condition (Δ_2).*

Proof. If $K^{\varphi_1} = H^{*\varphi_2}$, then we deduce from Theorem 3.2.2 that $\varphi_2 \dashv \varphi_1$, i.e.

$$(*) \qquad \varphi_2(u) \leqslant a\varphi_1(bu) \quad \text{for } u \geqslant u_0 \ (a, b > 0, u_0 \geqslant 0),$$

and from Theorem 3.2.4,

$$(**) \qquad \varphi_1(mu) \leqslant d_m\varphi_2(u) \quad \text{for } u \geqslant u_m \ (d_m > 0, u_m \geqslant 0)$$

for $m = 1, 2, \ldots$ We may suppose that the constant b in $(*)$ is a positive integer; in other case we could take the least positive integer greater then b in place of b. From $(*)$ and $(**)$ we get

$$\varphi_2(2u) \leqslant a \cdot \varphi_1(b \cdot 2u) \leqslant ad_{2b}\varphi_2(u) \quad \text{for } u \geqslant \sup\left\{\frac{u_0}{2}, u_{2b}\right\}.$$

Hence we obtain that $\varphi_1 \sim \varphi_2$ and φ_2 satisfies condition (Δ_2).

Conversely, let φ_2 satisfy condition (Δ_2), and let $\varphi_1 \sim \varphi_2$. Then $K^{\varphi_1} = K^{\varphi_2}$, by Theorem 3.2.3. Moreover, since φ_2 satisfies (Δ_2), we conclude from Theorem I.2.3.6, that $M^{\varphi_2} = L^{*\varphi_2}$. Multiplying this equality by N' we obtain $K^{\varphi_2} = H^{*\varphi_2}$. Thus, $K^{\varphi_1} = H^{*\varphi_2}$.

3.3.2. THEOREM. *The following four conditions are mutually equivalent:*

1° *φ satisfies condition (Δ_2),*

2° $H^{\varphi} = H^{*\varphi}$,

3° $H^{\varphi} = K^{\varphi}$,

4° $K^{\varphi} = H^{*\varphi}$.

Proof. From the preceding Theorem the equivalency of 1° and 4° follows, immediately. Now, if 2° or 3° holds, then H^{φ} is a linear set and this implies 4°. Conversely, 4° implies 2° and 3°, because if 4° holds, then $K^{\varphi} \subset H^{\varphi} \subset H^{*\varphi}$.

3.3.3. THEOREM. *If φ does not satisfy condition (Δ_2), then there exists functions $F_1, F_2 \in H^{*\varphi}$ such that*

(α) $\mu_{\varphi}(F_1) < \infty$ *and* $\mu_{\varphi}(aF_1) = \infty$ *for* $a > 1$,

(β) $\mu_{\varphi}(aF_2) < \infty$ *for* $0 < a < 1$ *and* $\mu_{\varphi}(F_2) = \infty$.

Proof. Let $\varphi_n(u) = \varphi\big((1+1/n)u\big)$. If ($\alpha$) does not hold, then

$$H^{\varphi} \subset \bigcup_{\nu=1}^{\infty} H^{\varphi_{\nu}}.$$

By Theorem 3.1.3, we have then for a positive integer and for some constants $d > 0$ and $u_0 \geqslant 0$ the inequality

$$\varphi\big((1+1/n)u\big) \leqslant d\cdot\varphi(u) \quad \text{ for } u \geqslant u_0.$$

By Lemma I.1.3.3, φ satisfies condition (Δ_2).

Let $\tilde{\varphi}_n(u) = \varphi\big(\big(1-1/(n+1)\big)u\big)$. If ($\beta$) does not hold, then

$$\bigcap_{\nu=1}^{\infty} H^{\tilde{\varphi}_{\nu}} \subset H^{\varphi}.$$

But then we conclude from Theorem 3.1.2 that for a positive integer m and for some $d > 0$ and $u_0 \geqslant 0$ there holds the inequality

$$\varphi(u) \leqslant d\cdot\sup\{\tilde{\varphi}_1(u), \tilde{\varphi}_2(u), \ldots, \tilde{\varphi}_m(u)\} = d\varphi\left(\left(1-\frac{1}{m+1}\right)u\right)$$

for $u \geqslant u_0$.

Replacing in this inequality u by $(1+1/m)u$, we get

$$\varphi\big((1+1/m)u\big) \leqslant d\varphi(u) \quad \text{ for } u \geqslant u_0.$$

Hence, φ satisfies condition (Δ_2).

III. SPACES $H^{*\varphi}$ WITH NORM GENERATED BY φ. COMPARISON OF CONVERGENCE OF SEQUENCES

1. Spaces $H^{*\varphi}$ with norm generated by φ

1.1.1. The space $H^{*\varphi}$ is algebraically isomorphic with the subspace of the space $L^{*\varphi}$, consisting of limit function $F(e^{it})$ of functions $F \in H^{*\varphi}$ (see II.2.1.2). According to this isomorphism we may define the norm in the space $H^{*\varphi}$ as the norm generated by of the respective elements of the space $L^{*\varphi}$ (see I.2.3.1). Namely, we take for $F \in H^{*\varphi}$

$$(*) \qquad \|F\|_\varphi = \|F(e^{i\cdot})\|_\varphi^*.$$

Then we get from Theorem I.2.3.2 and from Theorem II.1.3.2, immediately:

1.1.2. THEOREM. *$\|\cdot\|_\varphi$ possesses the following properties in $H^{*\varphi}$:*

1° $\|\cdot\|_\varphi$ is an F-norm,

2° if $|F_1(e^{it})| \leqslant |F_2(e^{it})|$ for almost all t, then $\|F_1\|_\varphi \leqslant \|F_2\|_\varphi$,

3° $\mu_\varphi(F) \leqslant \|F\|_\varphi$ if $\|F\|_\varphi \leqslant 1$; $\mu_\varphi(F) \leqslant 1$ implies $\|F\|_\varphi \leqslant 1$,

4° $\|F_n\|_\varphi \to 0$ if and only if $\mu_\varphi(aF_n) \to 0$ for every $a > 0$.

1.1.3. THEOREM. *If $F \in H^{*\varphi}$, then*

$$(*) \qquad \|F\|_\varphi = \inf\{k > 0 \mid \mu_\varphi(F/k) \leqslant k\}.$$

Proof. According to the definition of the norm generated by φ in the space $L^{*\varphi}$ (see I.2.3.1), we conclude from 1.1.1 (*) for $F \in H^{*\varphi}$ that

$$\|F\|_\varphi = \inf\left\{k > 0 \,\middle|\, \mathscr{I}_\varphi\left(\frac{F(e^{i\cdot})}{k}\right) \leqslant k\right\}.$$

Since $H^{*\varphi} \subset N'$, according to Theorem II.1.3.2 for every $k > 0$ there holds $\mathscr{I}_\varphi(F(e^{i\cdot})/k) = \mu_\varphi(F/k)$. Hence we conclude (*).

Obviously, formula (*) of 1.1.3 may serwe as a definition of the norm generated by φ in $H^{*\varphi}$; then equality (*) of 1.1.1 becomes a theorem for $F \in H^{*\varphi}$.

1.2.1. THEOREM. *Let F be an analytic function in the disc D. Then $T_r F \in K \subset H^{*\varphi}$ for each $0 \leqslant r < 1$, and $\|T_r F\|_\varphi$ is a non-decreasing function for $0 \leqslant r < 1$. Hence*

$$\sup\{\|T_r F\|_\varphi \mid 0 \leqslant r < 1\} = \lim_{r \to 1-} \|T_r F\|_\varphi.$$

Proof. Obviously, $T_r F \in K \subset H^{*\varphi}$ for each $0 \leqslant r < 1$. Now, by Theorems 1.2.1 and II.1.4.2 we have for $0 \leqslant r_1 < r_2 < 1$ and every $k > 0$, $\mu_\varphi(T_{r_1} F/k) \leqslant \mu_\varphi(T_{r_2} F/k)$. Hence we obtain

$$\begin{aligned}\|T_{r_1} F\|_\varphi &= \inf\{k > 0 \mid \mu_\varphi(T_{r_1} F/k) \leqslant k\} \leqslant \inf\{k > 0 \mid \mu_\varphi(T_{r_2} F/k) \leqslant k\} \\ &= \|T_{r_2} F\|_\varphi.\end{aligned}$$

1.2.2. THEOREM. *An analytic function F in the disc D belongs to $H^{*\varphi}$ if and only if $\sup\{\|T_r F\|_\varphi \mid 0 \leqslant r < 1\} < \infty$.*

1.2.3. THEOREM. *If $F \in H^{*\varphi}$, then*

$$\|F\|_\varphi = \sup\{\|T_r F\|_\varphi \mid 0 \leqslant r < 1\} = \lim_{r \to 1-} \|T_r F\|_\varphi.$$

Proof of 1.2.2 and 1.2.3. Let $F \epsilon H^{*\varphi}$. We take an arbitrary number $k > \|F\|_\varphi$. By Theorem 1.1.3 we have $\mu_\varphi(F/k) \leqslant k$. Hence, by Theorems 1.2.1 and II.1.4.2, we get $\mu_\varphi(T_r F/k) \leqslant k$ for each $0 \leqslant r < 1$. According to Theorem 1.1.3, this implies $\|T_r F\|_\varphi \leqslant k$ for each $0 \leqslant r < 1$. Hence, $\sup\{\|T_r F\|_\varphi | 0 \leqslant r < 1\} \leqslant k$. Thus, there holds the inequality:

$$(*) \qquad \sup\{\|T_r F\|_\varphi | 0 \leqslant r < 1\} \leqslant \|F\|_\varphi .$$

Now, let $\sup\{\|T_r F\|_\varphi | 0 \leqslant r < 1\} < \infty$. We take an arbitrary number $k > \sup\{\|T_r F\|_\varphi | 0 \leqslant r < 1\}$. Hence, by Theorem 1.1.3, we have $\mu_\varphi(T_r F/k) \leqslant k$ for each $0 \leqslant r < 1$. According to Theorem II.1.4.2, we get hence

$$\mu_\varphi(F/k) = \sup\{\mu_\varphi(T_r F/k) | 0 \leqslant r < 1\} \leqslant k.$$

Consequently, we obtain $F \epsilon H^{*\varphi}$. Moreover, by Theorem 1.1.3, $\|F\|_\varphi \leqslant k$. Thus, there holds the inequality

$$(**) \qquad \|F\|_\varphi \leqslant \sup\{\|T_r F\|_\varphi | 0 \leqslant r < 1\}.$$

If $F \epsilon H^{*\varphi}$, then inequalities $(*)$ and $(**)$ and Theorem 1.2.1 yield the equality given in Theorem 1.2.3. Thus, we have finished the proof of Theorems 1.2.2 and 1.2.3.

1.2.4. One may define a norm $\|\cdot\|_\varphi$ in $H^{*\varphi}$ also by means of the formula

$$\|F\|_\varphi = \sup\{\|F(re^{i\cdot})\|_\varphi^* | 0 \leqslant r < 1\}.$$

This definition of the norm $\|\cdot\|_\varphi$ does not require the knowledge of the space $H^{*\varphi}$ itself. Namely, from Theorems 1.2.1, 1.2.2, 1.2.3 and 1.1.1 follows that $\|F\|_\varphi$ may be defined by means of this formula for every functions F analytic in the disc D. The space $H^{*\varphi}$ is obtained then as the set of functions F analytic in the disc D for which $\|F\|_\varphi < \infty$. This property is not possessed by the definition of this norm by means of formula 1.1.1 $(*)$. By Remark II.1.3.3, we obtain even that there exist functions F analytic in the disc D, possessing a limit function $F(e^{i\cdot}) \epsilon L^{*\varphi}$, which do not belong to $H^{*\varphi}$. An example of such a function is given by $F(z) = \exp\big((1+z)/(1-z)\big)$.

1.3.1. THEOREM. *If* $F \epsilon H^{*\varphi}$, *then*

$$|F(z)| \leqslant \varphi_{-1}\left(\frac{\|F\|_\varphi}{\pi(1-|z|)}\right)\|F\|_\varphi \quad \textit{for } z \epsilon D.$$

Proof. We take an arbitrary $k > \|F\|_\varphi$. By Theorem 1.1.3, $\mu_\varphi(F/k) \leqslant k$. According to Theorem II.1.2.2, we have

$$\left|\frac{1}{k} F(z)\right| \leqslant \varphi_{-1}\left(\frac{\mu_\varphi(F/k)}{\pi(1-|z|)}\right) \quad \text{for } z \epsilon D.$$

Since φ_{-1} is an increasing function, we get hence

$$|F(z)| \leqslant \varphi_{-1}\left(\frac{k}{\pi(1-|z|)}\right)k \quad \text{for } z \in D.$$

Passing with $k \to \|F\|_\varphi$ we obtain the inequality given in the theorem, because φ_{-1} is a continuous function.

1.3.2. THEOREM. *The space $H^{*\varphi}$ is complete with respect to the norm $\|\cdot\|_\varphi$.*

Proof. Let (F_n) be an arbitrary sequence of functions from $H^{*\varphi}$ such that $\|F_n - F_m\|_\varphi \to 0$ as $n, m \to \infty$. Then, by Teorem 1.3.1, $F_n - F_m \to 0$ almost uniformly in the disc D as $n, m \to \infty$. Hence there exists a function F analytic in the disc D such that $F_m \to F$ almost uniformly in the disc D as $m \to \infty$. Now, we take an arbitrary number $\varepsilon > 0$. According to the assumption, there exists a positive integer n_0 such that $\|F_n - F_m\|_\varphi < \varepsilon$ for all $n, m \geqslant n_0$. Applying Theorems 1.1.3 and II.1.2.1, we obtain hence for all $0 \leqslant r < 1$, and $n, m \geqslant n_0$

$$\int_0^{2\pi} \varphi\left(\frac{|F_n(re^{it}) - F_m(re^{it})|}{\varepsilon}\right) dt = \mu_\varphi\left(r, \frac{F_n - F_m}{\varepsilon}\right) \leqslant \mu_\varphi\left(\frac{F_n - F_m}{\varepsilon}\right) \leqslant \varepsilon.$$

Keeping $0 \leqslant r < 1$ fixed and passing to the limit with $m \to \infty$, we get

$$\int_0^{2\pi} \varphi\left(\frac{|F_n(re^{it}) - F(re^{it})|}{\varepsilon}\right) dt = \mu_\varphi\left(r, \frac{F_n - F}{\varepsilon}\right) \leqslant \varepsilon$$

for all $0 \leqslant r < 1$ and $n \geqslant n_0$. Hence we obtain

$$\mu_\varphi\left(\frac{F_n - F}{\varepsilon}\right) \leqslant \varepsilon \quad \text{for } n \geqslant n_0.$$

From this inequality we deduce $F_{n_0} - F \in H^{*\varphi}$, and so $F \in H^{*\varphi}$. Moreover, by Theorem 1.1.3 we obtain from this inequality $\|F_n - F\|_\varphi \leqslant \varepsilon$ for $n \geqslant n_0$. But this proves $\|F_n - F\|_\varphi \to 0$ as $n \to \infty$.

In the sequel we shall use sometimes the symbol $[H^{*\varphi}, \|\cdot\|_\varphi]$ to denote the Fréchet space $H^{*\varphi}$ with the norm $\|\cdot\|_\varphi$.

1.4.1. THEOREM. *If $F \in H^{*\varphi}$, then $\|kF\|_\varphi / k$ is a non-increasing function for $k > 0$.*

Proof. Applying Theorem 1.1.3 we get for $k > 0$

$$\frac{1}{k}\|kF\|_\varphi = \frac{1}{k}\inf\left\{\varepsilon > 0 \,|\, \mu_\varphi\left(\frac{kF}{\varepsilon}\right) \leqslant \varepsilon\right\} = \inf\left\{\eta > 0 \,|\, \mu_\varphi\left(\frac{F}{\eta}\right) \leqslant k\eta\right\}.$$

Now, we take $0 < k_1 < k_2$. Since the inequality $\mu_\varphi(F/\eta) \leqslant k_1\eta$ implies $\mu_\varphi(F/\eta) \leqslant k_2\eta$ for every $\eta > 0$, we have

$$\frac{1}{k_1}\|k_1F\|_\varphi = \inf\left\{\eta > 0 \,|\, \mu_\varphi\left(\frac{F}{\eta}\right) \leqslant k_1\eta\right\} \geqslant \inf\left\{\eta > 0 \,|\, \mu_\varphi\left(\frac{F}{\eta}\right) \leqslant k_2\eta\right\}$$

$$= \frac{1}{k_2}\|k_2F\|_\varphi.$$

1.4.2. THEOREM. *Let $F \in H^{*\varphi}$. If $0 < \|F\|_\varphi \leqslant \delta$, then* $\left\|\dfrac{\delta F}{\|F\|_\varphi}\right\|_\varphi \leqslant \delta$.

Proof. Since $\delta/\|F\|_\varphi \geqslant 1$, the last theorem gives

$$\frac{\|F\|_\varphi}{\delta}\left\|\frac{\delta F}{\|F\|_\varphi}\right\|_\varphi \leqslant \|F\|_\varphi,$$

and we conclude the theorem.

1.5.1. We define for $F \in H^{*\varphi}$

$$[F]_\varphi = \inf\{k > 0 \,|\, \mu_\varphi(F/k) < \infty\}.$$

1.5.2. THEOREM. *The functional $[\cdot]_\varphi$ possesses the following properties in $H^{*\varphi}$:*

1° $[F]_\varphi = 0$ *if and only if* $F \in K^\varphi$,

2° $[aF]_\varphi = |a|\,[F]_\varphi$ *for an arbitrary complex number* a,

3° $[F_1+F_2]_\varphi \leqslant [F_1]_\varphi + [F_2]_\varphi$,

4° $[F]_\varphi \leqslant \|F\|_\varphi$,

5° $[F]_\varphi = \lim\limits_{k\to\infty}\|kF\|_\varphi/k$.

Proof. 1° If $F \in K^\varphi$, then $\mu_\varphi(F/k) < \infty$ for every $k > 0$, and so $[F]_\varphi = 0$. Conversely, if $[F]_\varphi = 0$, then $\mu_\varphi(F/k) < \infty$ for every $k > 0$, whence $F \in K^\varphi$.

2° If $a = 0$, then 2° is obvious. In other case, Theorem II.1.5 2° gives

$$[aF]_\varphi = \inf\left\{k > 0 \,|\, \mu_\varphi\left(\frac{aF}{k}\right) < \infty\right\} = |a|\inf\left\{\varepsilon > 0 \,|\, \mu_\varphi\left(\frac{F}{\varepsilon}\right) < \infty\right\} = |a|\,[F]_\varphi.$$

3° We take arbitrary numbers $k_1 > [F_1]_\varphi$ and $k_2 > [F_2]_\varphi$. Then $\mu_\varphi(F_1/k_1) < \infty$ and $\mu_\varphi(F_2/k_2) < \infty$. By Theorem II.1.5 3° we get

$$\mu_\varphi\left(\frac{F_1+F_2}{k_1+k_2}\right) = \mu_\varphi\left(\frac{k_1}{k_1+k_2}\cdot\frac{F_1}{k_1} + \frac{k_2}{k_1+k_2}\cdot\frac{F_2}{k_2}\right) \leqslant \mu_\varphi\left(\frac{F_1}{k_1}\right) + \mu_\varphi\left(\frac{F_2}{k_2}\right) < \infty.$$

This means that $[F_1+F_2]_\varphi \leqslant k_1+k_2$. Hence we obtain the triangle inequality 3°.

4° We take an arbitrary number $k > \|F\|_\varphi$. By Theorem 1.1.3 we have then $\mu_\varphi(F/k) \leqslant k$. Hence $[F]_\varphi \leqslant k$, and inequality 4° follows.

5° We deduce from properties 2° and 4° of functional $[\cdot]_\varphi$ that $[F]_\varphi \leqslant \|kF\|_\varphi/k$ for every $k > 0$. Hence, taking into account Theorem 1.4.1 we get the inequality

$$[F]_\varphi \leqslant \lim_{k\to\infty} \frac{1}{k} \|kF\|_\varphi.$$

Now, we take an arbitrary number $\varepsilon > [F]_\varphi$. Then $\mu_\varphi(F/\varepsilon) < \infty$. Thus, there exists a number $k > 0$ such that $\mu_\varphi(F/\varepsilon) \leqslant k\varepsilon$. Since

$$\frac{1}{k} \|kF\|_\varphi = \inf\{\varepsilon > 0 \,|\, \mu_\varphi(F/\varepsilon) \leqslant k\varepsilon\},$$

we obtain $\|kF\|_\varphi/k \leqslant \varepsilon$. Hence we conclude from Theorem 1.4.1 that $\lim\limits_{k\to\infty}\|kF\|_\varphi/k \leqslant \varepsilon$. Thus, we have the inequality

$$\lim_{k\to\infty} \frac{1}{k} \|kF\|_\varphi \leqslant [F]_\varphi,$$

and the proof of 5° is finished.

In the following, the functional $[\cdot]_\varphi$ will be called the *pseudonorm* generated by φ.

1.5.3. THEOREM. *K^φ is a closed subspace of the space $[H^{*\varphi}, \|\cdot\|_\varphi]$.*

Proof. Let (F_n) be a sequence of functions from K^φ convergent in norm to the function $F \in H^{*\varphi}$. By the preceding theorem, we have $[F_n]_\varphi = 0$ for each n, and $\lim\limits_{n\to\infty} [F_n - F]_\varphi = 0$. Hence, the triangle inequality $0 \leqslant [F]_\varphi \leqslant [F_n - F]_\varphi + [F_n]_\varphi$ gives $[F]_\varphi = 0$. Thus, $F \in K^\varphi$.

In the sequel we shall use sometimes the symbol $[K^\varphi, \|\cdot\|_\varphi]$ to denote the Fréchet space K^φ with the norm $\|\cdot\|_\varphi$.

2. Structural properties of the space $H^{*\varphi}$

2.1.1. LEMMA. *Let (f_n) be a sequence of real-valued, non-negative functions integrable in $\langle 0, 2\pi)$ and convergent in measure to a function f integrable in $\langle 0, 2\pi)$. If*

$$\lim_{n\to\infty} \int_0^{2\pi} f_n(t)\,dt = \int_0^{2\pi} f(t)\,dt,$$

then for every measurable set $E \subset \langle 0, 2\pi)$ we have also

$$\lim_{n\to\infty} \int_E f_n(t)\,dt = \int_E f(t)\,dt.$$

Proof. Let us suppose the thesis of lemma does not hold for a measurable set $U \subset \langle 0, 2\pi)$. Then one may extract from the sequence (f_n) a subsequence $(f_{n'})$ convergent to almost everywhere in $\langle 0, 2\pi)$ and such that the limits

$$\lim_{n'\to\infty} \int_E f_{n'}(t)\,dt \quad \text{and} \quad \lim_{n'\to\infty} \int_{\langle 0,2\pi)\setminus E} f_{n'}(t)\,dt$$

exist and

$$\lim_{n'\to\infty} \int_E f_{n'}(t)\,dt \neq \int_E f(t)\,dt.$$

Then we have, by Fatou's lemma

$$\lim_{n'\to\infty} \int_E f_{n'}(t)\,dt > \int_E f(t)\,dt$$

and

$$\lim_{n'\to\infty} \int_{\langle 0,2\pi)\setminus E} f_{n'}(t)\,dt \geqslant \int_{\langle 0,2\pi)\setminus E} f(t)\,dt.$$

Adding these inequalities we obtain

$$\lim_{n'\to\infty} \int_0^{2\pi} f_{n'}(t)\,dt > \int_0^{2\pi} f(t)\,dt,$$

a contradiction to $(*)$.

2.1.2. THEOREM. *Let F_n $(n = 1, 2, \ldots)$ and F be functions of the class H^φ. If $F_n(e^{i\cdot}) \to F(e^{i\cdot})$ in measure on the interval $\langle 0, 2\pi)$ and $\mu_\varphi(F_n) \to \mu_\varphi(F)$, then $\mu_\varphi\big(\frac{1}{2}(F_n - F)\big) \to 0$.*

Proof. First, we prove the theorem under additional assumption that $F_n(e^{i\cdot}) \to F(e^{i\cdot})$ almost everywhere. Since $\big(\varphi(|F_n(e^{i\cdot})|)\big)$ is a sequence of non-negative and integrable functions on $\langle 0, 2\pi)$ convergent to the integrable function $\varphi(|F(e^{i\cdot})|)$ almost everywhere in $\langle 0, 2\pi)$, and since according to Theorem II.1.3.2

$$\lim_{n\to\infty} \int_0^{2\pi} \varphi(|F_n(e^{it})|)\,dt = \int_0^{2\pi} \varphi(|F(e^{it})|)\,dt,$$

the above lemma gives for every measurable set $E \subset \langle 0, 2\pi)$

$$\lim_{n\to\infty} \int_E \varphi(|F_n(e^{it})|)\,dt = \int_F \varphi(|F(e^{it})|)\,dt.$$

Since

$$\int_E \varphi\big(\tfrac{1}{2}|F_n(e^{it}) - F(e^{it})|\big)\,dt \leqslant \int_E \varphi\big(\tfrac{1}{2}|F_n(e^{it})| + \tfrac{1}{2}|F(e^{it})|\big)\,dt$$

$$\leqslant \int_E \varphi\big(\sup\{|F_n(e^{it})|, |F(e^{it})|\}\big)\,dt = \int_E \sup\{\varphi(|F_n(e^{it})|), \varphi(|F(e^{it})|)\}\,dt$$

$$\leqslant \int_E \varphi(|F_n(e^{it})|)\,dt + \int_E \varphi(|F(e^{it})|)\,dt,$$

we get hence

$$\overline{\lim_{n\to\infty}} \int_E \varphi\big(\tfrac{1}{2}|F_n(e^{it})-F(e^{it})|\big)\,dt \leqslant 2 \int_E \varphi\big(|F(e^{it})|\big)\,dt.$$

Let $\varepsilon > 0$ be arbitrary. Since $\mu_\varphi(F) = \int_0^{2\pi} \varphi\big(|F(e^{it})|\big)\,dt < \infty$, there exist $\delta > 0$ such that

$$\int_E \varphi\big(|F(e^{it})|\big)\,dt < \varepsilon/2.$$

for an arbitrary set $E \subset \langle 0, 2\pi)$ of measure mes $E < \delta$. But $F_n(e^{i\cdot}) \to F(e^{i\cdot})$ almost everywhere. Hence, by Egorov's theorem, there exists a set $E_0 \subset \langle 0, 2\pi)$ of measure mes$E_0 < \delta$ such that $F_n(e^{i\cdot}) \to F(e^{i\cdot})$ uniformly on $\langle 0, 2\pi)\setminus E_0$. Thus,

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \int_0^{2\pi} \varphi\big(\tfrac{1}{2}|F_n(e^{it})-F(e^{it})|\big)\,dt \\
&\quad\leqslant \overline{\lim_{n\to\infty}} \int_{E_0} \varphi\big(\tfrac{1}{2}|F_n(e^{it})-F(e^{it})|\big)\,dt + \overline{\lim_{n\to\infty}} \int_{\langle 0,2\pi)\setminus E_0} \varphi\big(\tfrac{1}{2}|F_n(e^{it})-F(e^{it})|\big)\,dt \\
&\quad\leqslant 2 \int_{E_0} \varphi\big(|F(e^{it})|\big)\,dt < \varepsilon.
\end{aligned}$$

Hence we conclude

$$\lim_{n\to\infty} \int_0^{2\pi} \varphi\big(\tfrac{1}{2}|F_n(e^{it})-F(e^{it})|\big)\,dt = 0.$$

By Theorem II.1.3.2 we deduce $\mu_\varphi\big(\tfrac{1}{2}(F_n - F)\big) \to 0$.

Now, let $F_n(e^{i\cdot}) \to F(e^{i\cdot})$ in measure on $\langle 0, 2\pi)$.

Let us suppose the thesis of the theorem does not hold. Then a subsequence $(F_{n'})$ may be extracted from the sequence (F_n) such that there exists the limit

$$(*) \qquad \lim_{n'\to\infty} \mu_\varphi\big(\tfrac{1}{2}(F_{n'} - F)\big) > 0$$

and $F_{n'}(e^{i\cdot}) \to F(e^{i\cdot})$ almost everywhere. Since $F_{n'}(e^{i\cdot}) \to F(e^{i\cdot})$ almost everywhere and $\mu_\varphi(F_{n'}) \to \mu_\varphi(F)$, we obtain from the above proved part of the theorem that $\mu_\varphi\big(\tfrac{1}{2}(F_{n'} - F)\big) \to 0$, a contradiction to $(*)$.

2.1.3. THEOREM. *If* $F \in H^\varphi$, *then* $\mu_\varphi\big(\tfrac{1}{2}(T_rF - F)\big) \to 0$ *as* $r \to 1-$.

Proof. Obviously, the functions T_rF belong to H^φ for $0 \leqslant r < 1$. But according to Theorem I.3.1.3, $T_rF(e^{i\cdot}) = F(re^{i\cdot}) \to F(e^{i\cdot})$ almost everywhere, and according to Theorems 1.2.1 and II.1.4.2,

$$\mu_\varphi(T_rF) = \mu_\varphi(r, F) \to \mu_\varphi(F) \qquad \text{as } r \to 1-.$$

Hence, by the previous theorem we obtain

$$\mu_\varphi\big(\tfrac{1}{2}(T_rF-F)\big) \to 0 \quad \text{as } r \to 1-.$$

2.1.4. THEOREM. *If $F \in H^\varphi$, then $\mu_\varphi\big(\frac{1}{2}(S_hF-F)\big) \to 0$ as $h \to 0$.*

Proof. Applying Theorem II.1.4.3 we find that $\mu_\varphi(S_hF) = \mu_\varphi(F)$ for any real h. Hence $S_hF \in H^\varphi$ for each h, and $\mu_\varphi(S_hF) \to \mu_\varphi(F)$ as $h \to 0$. Now, we may conclude Theorem 2.1.4 from Theorem 2.1.2 if we prove that $S_hF(e^{i\cdot}) \to F(e^{i\cdot})$ as $h \to 0$ in measure on $\langle 0, 2\pi)$.

We take numbers $\varepsilon, \eta > 0$ arbitrarily. Since $F(re^{i\cdot}) \to F(e^{i\cdot})$ almost everywhere as $r \to 1-$, we have

$$\operatorname{mes}\{t \in \langle 0, 2\pi) \mid |F(re^{it})-F(e^{it})| \geqslant \eta/3\} < \varepsilon/2$$

for some r, $0 \leqslant r < 1$. Since the function $F(re^{i\cdot})-F(^{i\cdot})$ is 2π-periodic, we have also

$$\operatorname{mes}\{t \in \langle 0, 2\pi) \mid |F(re^{i(t+h)})-F(e^{i(t+h)})| \geqslant \eta/3\} < \varepsilon/2$$

for every h. Next, the function $F(re^{i\cdot})$ is 2π-periodic and continuous, and thus it is uniformly continuous. Hence there exists a $\delta > 0$ such that

$$|F(re^{i(t+h)})-F(re^{it})| < \eta/3 \quad \text{for } |h| < \delta$$

for all t. Thus we get for $|h| < \delta$

$$\begin{aligned}
\operatorname{mes}&\{t \in \langle 0, 2\pi) \mid |F(e^{i(t+h)})-F(e^{it})| \geqslant \eta\} \\
&\leqslant \operatorname{mes}\{t \in \langle 0, 2\pi) \mid |F(e^{i(t+h)})-F(re^{i(t+h)})| \geqslant \eta/3\}+ \\
&\quad + \operatorname{mes}\{t \in \langle 0, 2\pi) \mid |F(re^{i(t+h)})-F(re^{it})| \geqslant \eta/3\}+ \\
&\quad + \operatorname{mes}\{t \in \langle 0, 2\pi) \mid |F(re^{it})-F(e^{it})| \geqslant \eta/3\} < \varepsilon/2+\varepsilon/2 = \varepsilon.
\end{aligned}$$

This proves $S_hF(e^{i\cdot}) \to F(e^{i\cdot})$ as $h \to 0$ in measure on $\langle 0, 2\pi)$.

2.1.5. THEOREM. *If $F \in K^\varphi$, then $\|T_rF-F\|_\varphi \to 0$ as $r \to 1-$.*

Proof. If $F \in K^\varphi$, then $aF \in H^\varphi$ for each $a > 0$. Hence, by Theorem 2.1.3, $\mu_\varphi\big(\frac{1}{2}a(T_rF-F)\big) \to 0$ as $r \to 1-$ for each $a > 0$. Applying Theorem 1.1.2 (4°) we thus obtain $\|T_rF-F\|_\varphi \to 0$ as $r \to 1-$.

2.1.6. THEOREM. *If $F \in K^\varphi$, then $\|S_hF-F\|_\varphi \to 0$ as $h \to 0$.*

This theorem is deduced from Theorem 2.1.4 in a similar manner as Theorem 2.1.5 from Theorem 2.1.3.

2.2.1. THEOREM. *$[K^\varphi, \|\cdot\|_\varphi]$ is a separable space, and set of polynomials with complex rational coefficients is dense in $[K^\varphi, \|\cdot\|_\varphi]$.*

Proof. Let F be an arbitrary function from K^φ. We take an arbitrary number $\varepsilon > 0$. Taking into account Theorem 2.1.5 we choose r such that $\|T_rF-F\|_\varphi < \frac{1}{2}\varepsilon$, $0 \leqslant r < 1$. Now, let us develop F in a power series

$$F(z) = a_0+a_1z+\ldots+a_nz^n+\ldots \quad \text{for } z \in D.$$

Then the function $T_r F$ may be developed in the power series

$$T_r F(z) = F(rz) = a_0 + a_1 rz + \ldots + a_n r^n z^n + \ldots$$

convergent in the disc $\{z \mid |z| < 1/r\}$, whence uniformly convergent in the closed disc $\bar{D} = \{z \mid |z| \leqslant 1\}$. We take n so large that the polynomial

$$P(z) = a_0 + a_1 rz + \ldots + a_n r^n z^n$$

satisfies the inequality $|T_r F(z) - P(z)| \leqslant \frac{1}{4}\varepsilon\varphi_{-1}(\varepsilon/4\pi)$ for all $z \in \bar{D}$. Now, we choose complex rational numbers b_k such that

$$|a_k r^k - b_k| \leqslant \frac{\varepsilon}{2^{k+3}}\varphi_{-1}\left(\frac{\varepsilon}{4\pi}\right) \quad \text{for } k = 0, 1, \ldots, n.$$

The polynomial $Q(z) = b_0 + b_1 z + \ldots + b_n z^n$ has complex rational coefficients and

$$\begin{aligned}
|T_r F(z) - Q(z)| &\leqslant |T_r F(z) - P(z)| + |P(z) - Q(z)| \\
&\leqslant |T_r F(z) - P(z)| + \sum_{k=0}^{n} |a_k r^k - b_k| \\
&\leqslant \frac{\varepsilon}{4}\varphi_{-1}\left(\frac{\varepsilon}{4\pi}\right) + \sum_{k=0}^{n} \frac{\varepsilon}{2^{k+3}}\varphi_{-1}\left(\frac{\varepsilon}{4\pi}\right) \leqslant \frac{\varepsilon}{2}\varphi_{-1}\left(\frac{\varepsilon}{4\pi}\right)
\end{aligned}$$

for all $z \in \bar{D}$. Hence we get, by Theorem II.1.3.2,

$$\mu_\varphi\left(\frac{2}{\varepsilon}(T_r F - Q)\right) = \int_0^{2\pi} \varphi\left(\frac{2}{\varepsilon}|T_r F(e^{it}) - Q(e^{it})|\right) dt \leqslant \frac{\varepsilon}{2}.$$

Thus, according to Theorem 1.1.3 we have $\|T_r F - Q\|_\varphi \leqslant \frac{1}{2}\varepsilon$. Consequently,

$$\|F - Q\|_\varphi \leqslant \|F - T_r F\|_\varphi + \|T_r F - Q\|_\varphi < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves the set of polynomials with complex rational coefficients is dense in $[K^\varphi, \|\cdot\|_\varphi]$.

2.2.2. THEOREM. *The space $[H^{*\varphi}, \|\cdot\|_\varphi]$ is separable if and only if φ satisfies condition (Δ_2).*

Proof. If φ safisfies condition (Δ_2), then according to Theorem II.3.3.2 we have $H^{*\varphi} = K^\varphi$ and we conclude from the preceding theorem that $[H^{*\varphi}, \|\cdot\|_\varphi]$ is separable.

Now, let us suppose φ does not satisfy condition (Δ_2). By Lemma I.1.3.3 the function φ does not satisfy condition (Δ_a) for any $a > 1$. Hence there exists a sequence of numbers $u_n > (n+1)^2$ such that

$$\varphi\left(\left(1 + \frac{1}{n+1}\right)u_n\right) > 2^n \varphi(u_n), \quad \varphi(u_n) \geqslant 1.$$

Since

$$\left(1+\frac{1}{n+1}\right)\frac{u_n}{u_n-1} = \frac{n+2}{n+1}\left(1+\frac{1}{u_n-1}\right) < \frac{n+2}{n+1}\left(1+\frac{1}{n^2+2n}\right) = 1+\frac{1}{n},$$

we have also

$$\varphi\left(\left(1+\frac{1}{n}\right)(u_n-1)\right) > 2^n\varphi(u_n).$$

We take a sequence of pairwise disjoint sets E_n in $\langle 0, 2\pi)$ such that

$$\operatorname{mes} E_n = \frac{2\pi}{2^n\varphi(u_n)}$$

and we define real functions

$$f_n(t) = \begin{cases} u_n-1 & \text{for } t\in E_n, \\ 0 & \text{for } t\in\langle 0, 2\pi)\setminus E_n. \end{cases}$$

Next, we define a family of real functions

$$f_\eta(t) = 1+\eta_1 f_1(t)+\eta_2 f_2(t)+\ldots+\eta_n f_n(t)+\ldots,$$

where $\eta = (\eta_n)$ is an arbitrary sequence of terms 0 and 1. We have for functions of this family

$$\begin{aligned}\mathscr{I}_\varphi(f_\eta) &\leqslant \mathscr{I}_\varphi(1+f_1+f_2+\ldots+f_n+\ldots) \\ &\leqslant 2\pi\varphi(1)+\sum_{n=1}^{\infty}\varphi(u_n)\operatorname{mes} E_n \leqslant 2\pi\varphi(1)+2\pi.\end{aligned}$$

Hence and from fact that $f_\eta(t) \geqslant 1$ for all $t\in\langle 0, 2\pi)$ we get, by Lemma II.3.1.1, $\log f_\eta(\cdot)\in L^1$. Applying Theorem I.3.2.5 we obtain function $F_\eta\in N'$ such that $|F_\eta(e^{it})| = f_\eta(t)$ for almost all $t\in\langle 0, 2\pi)$. By Theorem II.1.3.2 we have $\mu_\varphi(F_\eta) \leqslant 2\pi\varphi(1)+2\pi$ and this means that $F_\eta\in H^\varphi \subset H^{*\varphi}$ for every η. Now, we take two different sequences $\eta' = (\eta_n')$ and $\eta'' = (\eta_n'')$, and let $\eta_m' \neq \eta_m''$. Then we get, taking into account Theorem II.1.3.2

$$\begin{aligned}&\mu_\varphi\big(2(F_{\eta'}-F_{\eta''})\big) \\ &\geqslant \mu_\varphi\left(\left(1+\frac{1}{m}\right)(F_{\eta'}-F_{\eta''})\right) = \int_0^{2\pi}\varphi\left(\left(1+\frac{1}{m}\right)|F_{\eta'}(e^{it})-F_{\eta''}(e^{it})|\right)dt \\ &\geqslant \int_0^{2\pi}\varphi\left(\left(1+\frac{1}{m}\right)\big||F_{\eta'}(e^{it})|-|F_{\eta''}(e^{it})|\big|\right)dt = \int_0^{2\pi}\varphi\left(\left(1+\frac{1}{m}\right)|f_{\eta'}(t)-f_{\eta''}(t)|\right)dt \\ &\geqslant \varphi\left(\left(1+\frac{1}{m}\right)(u_m-1)\right)\cdot\operatorname{mes} E_m > 2\pi > 2.\end{aligned}$$

This proves that $\|F_{\eta'}-F_{\eta''}\|_\varphi \geqslant \frac{1}{2}$. Thus, there is a set of power of continuum of functions in $H^{*\varphi}$ whose distances are $\geqslant \frac{1}{2}$. Hence the space $[H^{*\varphi}, \|\cdot\|_\varphi]$ is not separable.

2.3. THEOREM. *If* $F \in H^{*\varphi}$, *then*

$$[F]_\varphi \leqslant \inf\{\|F-G\|_\varphi | G \in K^\varphi\} \leqslant \overline{\lim_{r\to 1-}} \|T_r F - F\|_\varphi \leqslant 2[F]_\varphi.$$

Proof. We deduce from Theorem 1.4.1 that $\inf\{\|k(F-G)\|_\varphi/k | G \in K^\varphi\}$ is a non-increasing function for $k > 0$. Now, we take an arbitrary number $\varepsilon > \inf\{\frac{1}{2}\|k(F-G)\|_\varphi | G \in K^\varphi\}$. There exist a number $k_0 > 0$ and a function $G_0 \in K^\varphi$ such that $\varepsilon \geqslant \|k_0(F-G_0)\|_\varphi/k_0$. Hence we get, by Theorems 1.4.1 and 1.5.2 (5°)

$$\varepsilon \geqslant \frac{1}{k_0}\|k_0(F-G_0)\|_\varphi \geqslant \lim_{k\to\infty} \frac{1}{k}\|k(F-G_0)\|_\varphi = [F-G_0]_\varphi.$$

But $[F-G_0]_\varphi = [F]_\varphi$, because $G_0 \in K^\varphi$. Thus $\varepsilon \geqslant [F]_\varphi$. Hence, there holds the inequality

$$(*) \qquad [F]_\varphi \leqslant \liminf_{k\to\infty} \left\{\frac{1}{k}\|k(F-G)\|_\varphi | G \in K^\varphi\right\} \leqslant \inf\{\|F-G\|_\varphi | G \in K^\varphi\}.$$

Now, we take an arbitrary number $\varepsilon > 2[F]_\varphi$. Then $\mu_\varphi(2F/\varepsilon) < \infty$, and this means that $2F/\varepsilon \in H^\varphi$. By Theorem 2.1.3 we get $\mu_\varphi\big((T_r F - F)/\varepsilon\big) \to \to 0$ as $r \to 1-$, Hence we conclude, applying Theorem 1.1.2 (4°) that $\overline{\lim_{r\to 1-}} \|T_r F - F\|_\varphi \leqslant \varepsilon$. Thus there holds the inequality

$$(**) \qquad \overline{\lim_{r\to 1-}} \|T_r F - F\|_\varphi \leqslant 2[F]_\varphi.$$

Since $T_r F \in K^\varphi$ for every $0 \leqslant r < 1$, inequalities (*) and (**) imply the inequality given in the theorem.

3. Comparison of convergence of sequences

3.1.1. THEOREM. *Let* (F_n) *be a sequence of functions from* H^φ. *If* $\mu_\varphi(F_n) \to 0$, *then* $F_n \to 0$ *almost uniformly in the disc* D.

This follows from Theorem II.1.2.2 immediately

3.1.2. THEOREM. *Let* (F_n) *be a sequence of functions from* H^φ. *Then* $\mu_\varphi(F_n) \to 0$ *if and anly if* $\mathscr{I}_\varphi\big(F_n(e^{i\cdot})\big) \to 0$.

This follows from Theorem II.1.3.2.

3.2.1. THEOREM. *Let the inequality*

$$\varphi_2(u) \leqslant d\varphi_1(u) \quad \textit{for } u \geqslant u_0$$

be satisfied for some constants $d > 0$ and $u_0 \geqslant 0$. Then $\mu_{\varphi_1}(F_n) \to 0$ implies $\mu_{\varphi_2}(F_n) \to 0$ for an arbitrary sequence (F_n) of the class H^{φ_1}.

This theorem is obtained from the previous one applying Theorems I.2.1.6 and I.2.2.1.

3.2.2. In order to abbreviate the formulation of some theorems we introduce the following notion:

We denote by $(H^\infty)_0^\varphi$ the set of sequences (F_n) such that $F_n \in H^\infty$ and $\mu_\varphi(F_n) \to 0$.

3.2.3. THEOREM. *The inclusion*

$$(*) \qquad \bigcap_{\nu=1}^{\infty} (H^\infty)_0^{\varphi_\nu} \subset \bigcup_{\nu=1}^{\infty} (H^\infty)_0^{\tilde{\varphi}_\nu}$$

holds if and only if for some positive integers m and n and for some constants $d > 0$ and $u_0 \geqslant 0$ there is satisfied the inequality

$$(**) \qquad \tilde{\varphi}_n(u) \leqslant d \cdot \sup\{\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u)\} \quad \text{for } u \geqslant u_0.$$

Proof. We write $\hat{\varphi}_m(u) = \sup\{\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u)\}$. Inequality $(**)$ may be written in the form

$$(***) \qquad \tilde{\varphi}_n(u) \leqslant d\tilde{\varphi}_m(u) \quad \text{for } u \geqslant u_0.$$

If inequality $(**)$ holds, then of course $(***)$ holds, too, and so by Theorem 3.2.1 $(H^\infty)_0^{\hat{\varphi}_m} \subset (H^\infty)_0^{\tilde{\varphi}_n}$. Since

$$\varphi_\nu(u) \leqslant \hat{\varphi}_m(u) \leqslant \varphi_1(u)+\varphi_2(u)+\ldots+\varphi_m(u) \quad \text{for } \nu = 1, 2, \ldots, m,$$

we have

$$\bigcap_{\nu=1}^{\infty} (H^\infty)_0^{\varphi_\nu} \subset \bigcap_{\nu=1}^{m} (H^\infty)_0^{\varphi_\nu} = (H^\infty)_0^{\hat{\varphi}_m} \subset (H^\infty)_0^{\tilde{\varphi}_n} \subset \bigcup_{\nu=1}^{\infty} (H^\infty)_0^{\tilde{\varphi}_\nu}.$$

If $(**)$ does not hold, then of course $(***)$ does not hold, too. Then for each m and n there exists $u_{m,n} > 0$ such that

$$\tilde{\varphi}_n(u_{m,n}) > 2^{m+n}\hat{\varphi}_m(u_{m,n}) \quad \text{and} \quad \varphi_1(u_{m,n}) > m.$$

We take pairwise disjoint sets $E_{m,n}$ in the interval $\langle 0, 2\pi)$ of measures

$$\operatorname{mes} E_{m,n} = \frac{2\pi m}{2^{m+n}\hat{\varphi}_m(u_{m,n})}$$

and we define real functions

$$f_m(t) = \begin{cases} u_{m,n} & \text{for } t \in E_{m,n},\ n = 1, 2, \ldots, m, \\ (\hat{\varphi}_m)_{-1}(1/m) & \text{elsewhere in } \langle 0, 2\pi). \end{cases}$$

We have for each $\nu \leqslant m$

$$\mathscr{I}_{\varphi_\nu}(f_m) \leqslant \mathscr{I}_{\hat{\varphi}_m}(f_m) = \int_0^{2\pi} \hat{\varphi}_m\big(f_m(t)\big)\,dt \leqslant 2\pi\frac{1}{m} + \sum_{n=1}^{m} \hat{\varphi}_m(u_{m,n})\operatorname{mes}E_{m,n}$$

$$\leqslant 2\pi\frac{1}{m}2\pi + \frac{m}{2^m}$$

and

$$\mathscr{I}_{\tilde{\varphi}_\nu}(f_m) = \int_0^{2\pi} \tilde{\varphi}_\nu\big(f_m(t)\big)\,dt \geqslant \tilde{\varphi}_\nu(u_{m,\nu})\cdot\operatorname{mes}E_{m,\nu} > 2\pi m.$$

Now, let us note that every function f_m is bounded on the interval $\langle 0, 2\pi)$, and $\log f_m(\cdot)\in L^1$. By Theorem I.3.2.5 there exist functions $F_m \in N'$ such that $|F_m(e^{it})| = f_m(t)$ for almost all $t \in \langle 0, 2\pi)$. We show that every function F_m is bounded in the disc D. Namely, applying Theorem II.1.3.2 to the function $\varphi(u) = u^p$, $p > 0$, we get

$$\mu_\varphi(F_m) = \mathscr{I}_\varphi(f_m) \leqslant 2\pi\big(\sup\{f_m(t)\,|\,0 \leqslant t < 2\pi\}\big)^p$$

and so, by Theorem II.1.2.2

$$|F_m(z)| \leqslant \left(\frac{2}{1-|z|}\right)^{1/p} \sup\{f_m(t)\,|\,0 \leqslant t < 2\pi\} \quad \text{for } z \in D.$$

Passing to the limit as $p \to \infty$ we obtain

$$|F_m(z)| \leqslant \sup\{f_m(t)\,|\,0 \leqslant t < 2\pi\} \quad \text{for } z \in D.$$

Hence we deduce that every function F_m is bounded in the disc D. Applying Theorem II.1.3.2 once more for $\nu \leqslant m$ we get now

$$\mu_{\varphi_\nu}(F_m) = \mathscr{I}_{\varphi_\nu}(f_m) \leqslant 2\pi\frac{1}{m} + 2\pi\frac{m}{2^m} \quad \text{and} \quad \mu_{\tilde{\varphi}_\nu}(F_m) = \mathscr{I}_{\tilde{\varphi}_\nu}(f_m) > 2\pi m.$$

Hence we conclude that the sequence (F_n) belongs to $(H^\infty)_0^{\varphi_\nu}$ for each ν, and does not belong to $(H^\infty)_0^{\tilde{\varphi}_\nu}$ for any ν.

3.2.4. Theorem. *If $\mu_{\varphi_1}(F_n) \to 0$ implies $\mu_{\varphi_2}(F_n) \to 0$ for every sequence (F_n) of functions from H^∞, then there are constants $d > 0$ and $u_0 \geqslant 0$ such that*

$$\varphi_2(u) \leqslant d\varphi_1(u) \quad \textit{for } u \geqslant u_0.$$

This follows from the previous theorem, immediately. It is a converse to Theorem 3.2.1.

3.2.5. Theorem. *The following theorems of Chapter II remain valid, if we replace class H^φ by class of sequences $(H^\infty)_0^\varphi$, respectively:* 3.1.2, 3.1.3, 3.1.5, 3.1.6, 3.1.7, 3.1.8 *and* 3.1.9.

Indeed, let us remark that Theorems II.3.1.2 and II.3.1.3 formulated for classes of sequences $(H^\infty)_0^\varphi$, follow from Theorem 3.2.3, immediately. Moreover, Theorem II.3.1.4 also in case of classes of sequences $(H^\infty)_0^\varphi$ is contained in the above quoted Theorem 3.2.1 and 3.2.4. Next theorems of Chapter II mentioned above are obtained for classes of sequences $(H^\infty)_0^\varphi$ analogously us for classes of functions H^φ.

3.3.1. Convergence of sequences in the space $H^{*\varphi}$ meant in the sense $\mu_\varphi(F_n - F) \to 0$ is not in general of linear character. Therefore we introduce also in $H^{*\varphi}$ the following notion of convergence:

A sequence (F_n) of functions from $H^{*\varphi}$ is called *φ-convergent or modular convergent* to a function $F \epsilon H^{*\varphi}$, in writing $F_n \xrightarrow{\varphi} F$, if $\mu_\varphi(\lambda(F_n - F)) \to 0$ for a constant $\lambda > 0$ (depending in general on the sequence (F_n)).

This convergence is of a linear character. Obviously, the norm convergence in $H^{*\varphi}$ is also of a linear character. Let us recall that a sequence (F_n) of functions from $H^{*\varphi}$ is norm convergent to a function $F \epsilon H^{*\varphi}$, i.e. $\|F_n - F\|_\varphi \to 0$ if and only if $\mu_\varphi(\lambda(F_n - F)) \to 0$ for every $\lambda > 0$ (see Theorem 1.1.2.4°).

It is obvious that $\|F_n\|_\varphi \to 0$ implies $\mu_\varphi(F_n) \to 0$ and $\mu_\varphi(F_n) \to 0$ implies $F_n \xrightarrow{\varphi} 0$ for an arbitrary sequence (F_n) of functions from $H^{*\varphi}$.

3.3.2. THEOREM. (α) *If $\varphi_2 \dashv \varphi_1$, then $F_n \xrightarrow{\varphi_1} 0$ implies $F_n \xrightarrow{\varphi_2} 0$ for an arbitrary sequence (F_n) of functions from $H^{*\varphi_1}$.*

(β) *If $F_n \xrightarrow{\varphi_1} 0$ implies $F_n \xrightarrow{\varphi_2} 0$ for an arbitrary sequence (F_n) of functions from H^∞, then $\varphi_2 \dashv \varphi_1$.*

3.3.3. THEOREM. (α) *If $\varphi_2 \dashv \varphi_1$, then $\|F_n\|_{\varphi_1} \to 0$ implies $\|F_n\|_{\varphi_2} \to 0$ for an arbitrary sequence (F_n) of functions from $H^{*\varphi_1}$.*

(β) *If $\|F_n\|_{\varphi_1} \to 0$ implies $\|F_n\|_{\varphi_2} \to 0$ for an arbitrary sequence (F_n) of functions from H^∞, then $\varphi_2 \dashv \varphi_1$.*

Proof of Theorems 3.3.2 and 3.3.3. Let $\varphi_2 \dashv \varphi_1$. By Theorem II.3.2.1 we have then $H^{*\varphi_1} \subset H^{*\varphi_2}$, and so a sequence (F_n) of functions from $H^{*\varphi_1}$ is simultaneously a sequence of functions from $H^{*\varphi_2}$. Since the inequality

$$\varphi_2(u) \leqslant a \cdot \varphi_1(bu) \qquad \text{for } u \geqslant u_0$$

is satisfied for some constants $a, b > 0$ and $u_0 \geqslant 0$, we deduce from Theorem 3.2.1 that for a given $\lambda > 0$, $\mu_{\varphi_1}(\lambda F_n) \to 0$ implies $\mu_{\varphi_2}\left(\frac{\lambda}{b} F_n\right) \to 0$. Hence $F_n \xrightarrow{\varphi_1} 0$ implies $F_n \xrightarrow{\varphi_2} 0$, and $\|F_n\|_{\varphi_1} \to 0$ implies $\|F_n\|_{\varphi_2} \to 0$. Thus, part ($\alpha$) of Theorems 3.3.2 and 3.3.3 is proved. Part (β) of these Theorems follows from part (β) of the following theorem, immediately:

3.3.4. THEOREM. (α) *If $\varphi_2 \dashv \varphi_1$, then $\|F_n\|_{\varphi_1} \to 0$ implies $F_n \xrightarrow{\varphi_2} 0$ for an arbitrary sequence (F_n) of functions from $H^{*\varphi_1}$.*

(β) *If* $\|F_n\|_{\varphi_1} \to 0$ *implies* $F_n \xrightarrow{\varphi_2} 0$ *for every sequence* (F_n) *of functions from* H^∞, *then* $\varphi_2 \dashv \varphi_1$.

Proof. Part (α) of this theorem follows from part (α) of Theorems 3.3.2 and 3.3.3.

To prove part (β), we write $\hat{\varphi}_m(u) = \varphi_1(mu)$ and $\tilde{\varphi}_m(u) = \varphi_2(u/m)$. Let us remark that the assumption of part (β) of our theorem may be written in the form

$$\bigcap_{\nu=1}^{\infty} (H^\infty)_0^{\hat{\varphi}_\nu} \subset \bigcup_{\nu=1}^{\infty} (H^\infty)_0^{\tilde{\varphi}_\nu}.$$

Hence it follows, by Theorem 3.2.3 that there are positive integers m, n and constants $d > 0$ and $u_0 \geqslant 0$ such that

$$\varphi_2(u/n) = \tilde{\varphi}_n(u) \leqslant d\sup\{\hat{\varphi}_1(u), \hat{\varphi}_2(u), \ldots, \hat{\varphi}_m(u)\} = d\varphi_1(mu) \quad \text{for } u \geqslant u_0.$$

Hence we conclude that $\varphi_2 \dashv \varphi_1$, thus finishing the proof of part (β) of Theorem 3.3.4.

3.3.5. Theorem. (α) *If for each positive integer* m *there exist constants* $d_m > 0$ *and* $u_m \geqslant 0$ *such that*

$$(*) \qquad \varphi_2(mu) \leqslant d_m\varphi_1(u) \quad \text{for } u \geqslant u_m,\ m = 1, 2, \ldots,$$

then $F_n \xrightarrow{\varphi_1} 0$ *implies* $\|F_n\|_{\varphi_2} \to 0$ *for an arbitrary sequence* (F_n) *of functions from* $H^{*\varphi_1}$.

(β) *If* $F_n \xrightarrow{\varphi_1} 0$ *implies* $\|F_n\|_{\varphi_2} \to 0$ *for every sequence* (F_n) *of functions from* H^∞, *then for each positive integer* m *there exist constants* $d_m > 0$ *and* $u_m \geqslant 0$ *such that* $(*)$ *holds.*

Proof. If $(*)$ is satisfied, then according to Theorem II.3.2.4 we have the inclusion $H^{*\varphi_1} \subset K^{\varphi_2}$. Hence the sequence (F_n) of functions from $H^{*\varphi_1}$ is simultaneously a sequence of functions from K^{φ_2}. By Theorem 3.2.1 we deduce from $(*)$ that for a given $\lambda > 0$, $\mu_{\varphi_1}(\lambda F_n) \to 0$ implies $\mu_{\varphi_2}(m\lambda F_n) \to 0$ for each positive integer m. Hence $F_n \xrightarrow{\varphi_1} 0$ implies $\|F_n\|_{\varphi_2} \to 0$, and part (α) of the theorem is proved.

Let $\hat{\varphi}_m(u) = \varphi_1(u/m)$ and $\tilde{\varphi}_m(u) = \varphi_2(mu)$. If $F_n \xrightarrow{\varphi_1} 0$ implies $\|F_n\|_{\varphi_2} \to 0$ for every sequence (F_n) of functions from H^∞, then

$$(H^\infty)_0^{\hat{\varphi}_1} \subset \bigcup_{\nu=1}^{\infty} (H^\infty)_0^{\hat{\varphi}_\nu} \subset \bigcap_{\nu=1}^{\infty} (H^\infty)_0^{\tilde{\varphi}_\nu} \subset (H^\infty)_0^{\tilde{\varphi}_m}$$

for $m = 1, 2, \ldots$ Hence we get, applying Theorem 3.2.4 that for each positive integer m there exist constants $d_m > 0$ and $u_m \geqslant 0$ such that $(*)$ holds.

3.4.1. Theorem. (α) *If* $\varphi_1 \sim \varphi_2$ *and* φ_2 *satisfies condition* (Δ_2), *then for an arbitrary sequence* (F_n) *of functions from* $H^{*\varphi_2}$, $F_n \xrightarrow{\varphi_2} 0$ *if and only if* $\|F_n\|_{\varphi_1} \to 0$.

(β) *If for every sequence* (F_n) *of functions from* H^∞, $F_n \xrightarrow{\varphi_2} 0$ *if and only if* $\|F_n\|_{\varphi_1} \to 0$, *then* $\varphi_1 \sim \varphi_2$ *and* φ_2 *satisfies condition* (Δ_2).

Proof. If $\varphi_1 \sim \varphi_2$ and φ_2 satisfies condition (Δ_2), then applying Theorem II.3.3.1 we have $K^{\varphi_1} = H^{*\varphi_2}$. Hence the sequence (F_n) of functions from $H^{*\varphi_2}$ is a sequence of functions from K^{φ_1}, simultaneously. Since φ_2 satisfies condition (Δ_2), we deduce by Lemma I.1.3.3 that for each positive integer m there are constants $d_m > 0$ and $u_m \geqslant 0$ such that

$$\varphi_2(mu) \leqslant d_m \varphi_2(u) \quad \text{for } u \geqslant u_m \ (m = 1, 2, \ldots).$$

Hence we obtain, according to part (α) of the previous theorem, that $F_n \xrightarrow{\varphi_2} 0$ implies $\|F_n\|_{\varphi_2} \to 0$. Thus $F_n \xrightarrow{\varphi_2} 0$ if and only if $\|F_n\|_{\varphi_2} \to 0$. Next, by Theorem 3.3.3 (α), from the assumption $\varphi_1 \sim \varphi_2$ we deduce that $\|F_n\|_{\varphi_2} \to 0$ if and only if $\|F_n\|_{\varphi_1} \to 0$. So we obtain that $F_n \xrightarrow{\varphi_2} 0$ if and only if $\|F_n\|_{\varphi_1} \to 0$, thus finishing the proof of part (α) of the theorem.

Now, let us suppose that $F_n \xrightarrow{\varphi_2} 0$ if and only if $\|F_n\|_{\varphi_1} \to 0$ for every sequence (F_n) of functions from H^∞. Then, by Theorem 3.3.4 (β), $\varphi_2 \dashv \varphi_1$, i.e.

$$(*) \qquad \varphi_2(u) \leqslant a\varphi_1(bu) \quad \text{for } u \geqslant u_0 \ (a, b > 0, u_0 \geqslant 0),$$

and by Theorem 3.3.5 (β),

$$(**) \qquad \varphi_1(mu) \leqslant d_m \varphi_2(u) \quad \text{for } u \geqslant u_m \ (d_m > 0, u_m \geqslant 0)$$

for $m = 1, 2, \ldots$ We may suppose the constant b in $(*)$ to be a positive integer, for in other case we could take the smallest positive integer greater than b in place of b. From $(*)$ and $(**)$ we get

$$\varphi_2(2u) \leqslant a \cdot \varphi_1(b \cdot 2u) \leqslant a d_{2b} \varphi_2(u) \quad \text{for } u \geqslant \sup\{\tfrac{1}{2}u_0, u_{2b}\}.$$

Hence $\varphi_1 \sim \varphi_2$ and φ_2 satisfies condition (Δ_2).

3.4.2. Theorem. *The following four conditions are mutually equivalent*:

1° φ *satisfies condition* (Δ_2),

2° $F_n \xrightarrow{\varphi} 0$ *if and only if* $\mu_\varphi(F_n) \to 0$,

3° $\mu_\varphi(F_n) \to 0$ *if and only if* $\|F_n\|_\varphi \to 0$,

4° $F_n \xrightarrow{\varphi} 0$ *if and only if* $\|F_n\|_\varphi \to 0$.

Here we take in 2°, 3° *and* 4° *sequences* (F_n) *of functions from* $H^{*\varphi}$.

Proof. The equivalency of conditions 1° and 4° follows from the previous theorem, immediately. If condition 4° holds, then of course conditions 2° and 3° hold, too. Now, if 2° holds, then $\mu_\varphi(\frac{1}{2}F_n) \to 0$ implies $\mu_\varphi(F_n) \to 0$ for every sequence (F_n) of functions from H^∞. Hence, by Theorem 3.2.4 the inequality

$$\varphi(u) \leqslant d\varphi(\tfrac{1}{2}u) \quad \text{for } u \geqslant u_0$$

is satisfied for some constants $d > 0$ and $u_0 \geqslant 0$. We conclude that φ satisfies condition (Δ_2). Similarly, if 3° holds, then $\mu_\varphi(F_n) \to 0$ implies $\mu_\varphi(2F_n) \to 0$ for every sequence (F_n) of functions from H^∞. Hence, by Theorem 3.2.4, φ satisfies condition (Δ_2). Consequently 2° as well as 3° implies 1°.

3.4.3. THEOREM. *If φ does not satisfy condition (Δ_2), then there exist sequences (F_n) and (G_n) of functions from H^∞ such that*

(α) $\mu_\varphi(F_n) \to 0$ *and* $\mu_\varphi(\lambda F_n) \to \infty$ *for* $\lambda > 1$,

(β) $\mu_\varphi(\lambda G_n) \to 0$ *for* $0 \leqslant \lambda < 1$ *and* $\mu_\varphi(G_n) \to \infty$.

Proof. Let $\tilde{\varphi}_\nu(u) = \varphi\big((1+1/\nu)u\big)$ and $\varphi_\nu(u) = \varphi(u)$. By Lemma I.1.3.3 inequality (**) from Theorem 3.2.3 is equivalent to the fact that φ satisfies condition (Δ_2). Thus, if φ does not satisfy condition (Δ_2), then the proof of Theorem 3.2.3 follows existence of a sequence (F_n) of functions from H^∞ such that

$$\mu_\varphi(F_m) = \mu_{\varphi_\nu}(F_m) \leqslant 2\pi\frac{1}{m} + 2\pi\frac{m}{2^m}$$

and

$$\mu_\varphi\left(\left(1+\frac{1}{\nu}\right)F_m\right) = \mu_{\tilde{\varphi}_\nu}(F_m > 2\pi m.$$

for each $\nu \leqslant m$. Hence we get (α). Now, let

$$\varphi_\nu(u) = \varphi\left(\left(1-\frac{1}{\nu+1}\right)u\right) \quad \text{and} \quad \tilde{\varphi}_\nu(u) = \varphi(u).$$

It is easily verified that inequality (**) of Theorem 3.2.3 is equivalent to the fact that φ satisfies condition (Δ_2). Hence, if φ does not satisfy condition (Δ_2), then there exists a sequence (G_n) of functions from H^∞ such that

$$\mu_\varphi\left(\left(1-\frac{1}{\nu+1}\right)G_m\right) = \mu_{\varphi_\nu}(G_m) \leqslant 2\pi\frac{1}{m} + 2\pi\frac{m}{2^m}$$

and

$$\mu_\varphi(G_m) = \mu_{\tilde{\varphi}_\nu}(G_m) > 2\pi m$$

for each $\nu \leqslant m$. Hence we get (β).

IV. SPACES $H^{*\varphi}$ WITH AN s-HOMOGENEOUS NORM $(0 < s \leqslant 1)$

1. Spaces $H^{*\varphi}$ with an s-homogeneous norm $(0 < s \leqslant 1)$

1.1.1. In case when $\varphi(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and ψ is convex φ-function, we may define an s-homogeneous norm on the space $H^{*\varphi}$ by means of that defined in $L^{*\varphi}$ (see I.2.4.1) in a similar manner as in III.1.1.1. Namely, we take for $F \in H^{*\varphi}$

$$(*) \qquad \|F\|_{s\varphi} = \|F(e^{i\cdot})\|^*_{s\varphi}.$$

Since the norms $\|\cdot\|_\varphi^*$ and $\|\cdot\|_{s\varphi}^*$ are then equivalent in $L^{*\varphi}$, we deduce that also the norms $\|\cdot\|_\varphi$ and $\|\cdot\|_{s\varphi}$ are equivalent in $H^{*\varphi}$.

1.1.2. THEOREM. *Let $\varphi(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and ψ is a convex φ-function. If $F \in H^{*\varphi}$, then*

$$(*)\qquad \|F\|_{s\varphi} = \inf\left\{k > 0 \,|\, \mu_\varphi\left(\frac{F}{k^{1/s}}\right) \leqslant 1\right\}.$$

Proof. According to the definition of the s-homogeneous norm $\|\cdot\|_{s\varphi}^*$ in $L^{*\varphi}$ (see I.2.4.1) and to 1.1.1 (*) for $F \in H^{*\varphi}$ we have

$$\|F\|_{s\varphi} = \inf\left\{k > 0 \,|\, \mathscr{I}_\varphi\left(\frac{F(e^{i\cdot})}{k^{1/s}}\right) \leqslant 1\right\}.$$

Since $H^{*\varphi} \subset N'$, we deduce from Theorem II.1.3.2 that for every $k > 0$ there holds $\mathscr{I}_\varphi\big(F(e^{i\cdot})/k^{1/s}\big). = \mu_\varphi\big(F/k^{1/s}\big)$ Hence we obtain formula (*) of the theorem.

It is obtain that in case of $\varphi(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and ψ is a convex φ-function, formula (*) of 1.1.2 may be applied as the definition of the s-homogeneous norm $\|\cdot\|_{s\varphi}$ in $H^{*\varphi}$; then formula (*) of 1.1.1 becomes a theorem for $F \in H^{*\varphi}$.

1.1.3. THEOREM. *Let $\varphi(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and ψ is a convex φ-function, and let F be an analytic function in the disc D. Then $\|T_rF\|_{s\varphi}$ is a non-decreasing function of the variable r, $0 \leqslant r < 1$. Hence*

$$\sup\{\|T_rF\|_{s\varphi} | 0 \leqslant r < 1\} = \lim_{r\to 1-} \|T_rF\|_{s\varphi}.$$

Proof. By Theorems II.1.2.1 and II.1.4.2 we have $0 \leqslant r_1 < r_2 < 1$ and for every $k > 0$, $\mu_\varphi(T_{r_1}F/k^{1/s}) \leqslant \mu_\varphi(T_{r_2}F/k^{1/s})$. Hence we obtain, according to the last theorem

$$\begin{aligned}\|T_{r_1}F\|_{s\varphi} &= \inf\{k > 0 \,|\, \mu_\varphi(T_{r_1}F/k^{1/s}) \leqslant 1\}\\ &\leqslant \inf\{k > 0 \,|\, \mu_\varphi(T_{r_2}F/k^{1/s}) \leqslant 1\} = \|T_{r_2}F\|_{s\varphi}.\end{aligned}$$

1.1.4. THEOREM. *Let $\varphi(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and ψ is a convex φ-function. A function F analytic in the disc D belongs to $H^{*\varphi}$ if and only if $\sup\{\|T_rF\|_{s\varphi} | 0 \leqslant r < 1\} < \infty$. If, moerover, $F \in H^{*\varphi}$, then*

$$\|F\|_{s\varphi} = \sup\{\|T_rF\|_{s\varphi} | 0 \leqslant r < 1\} = \lim_{r\to 1-} \|T_rF_{s\varphi}\|.$$

Proof. Let $F \in H^{*\varphi}$. We take an arbitrary number $k > \|F\|_{s\varphi}$. By Theorem 1.1.2 we have then $\mu_\varphi(F/k^{1/s}) \leqslant 1$. Hence, applying Theorems II.1.2.1 and II.1.4.2 we get $\mu_\varphi(T_rF/k^{1/s}) \leqslant 1$ for every $0 \leqslant r < 1$. According to Theorem 1.1.2 we obtain $\|T_rF\|_{s\varphi} \leqslant k$ for every $0 \leqslant r < 1$. Thus, $\sup\{\|T_rF\|_{s\varphi} | 0 \leqslant r < 1\} \leqslant k$. Hence there holds the inequality

$$(*)\qquad \sup\{\|T_rF\|_{s\varphi} | 0 \leqslant r < 1\} \leqslant \|F\|_{s\varphi}.$$

Now, let $\sup\{\|T_rF\|_{s\varphi}|0 \leqslant r < 1\} < \infty$ hold for a function F analytic in the disc D. We take an arbitrary number k satisfying the inequality $k > \sup\{\|T_rF\|_{s\varphi}|0 \leqslant r < 1\}$. Hence $\mu_\varphi(T_rF/k^{1/s}) \leqslant 1$ for $0 \leqslant r < 1$. Then, by Theorem II.1.4.2 we get $\mu_\varphi(F/k^{1/s}) \leqslant 1$. Hence we obtain $F \epsilon H^{*\varphi}$. Moreover, by Theorem 1.1.2, $\|F\|_{s\varphi} \leqslant k$. Thus the following inequality holds:

$$(**) \qquad \|F\|_{s\varphi} \leqslant \sup\{\|T_rF\|_{s\varphi}|0 \leqslant r < 1\}.$$

If $F \epsilon H^{*\varphi}$, then inequalities (*) and (**) yield the equality given in the theorem.

1.1.5. If $\varphi(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and ψ is a convex φ-function, then the norm $\|\cdot\|_{s\varphi}$ in $H^{*\varphi}$ may be defined also by means of the formula

$$\|F\|_{s\varphi} = \sup\{\|F(re^{i\cdot})\|^*_{s\varphi}|0 \leqslant r < 1\}.$$

If we define the norm $\|\cdot\|_{s\varphi}$ in $H^{*\varphi}$ by means of this formula, then equality (*) of 1.1.1 for $F \epsilon H^{*\varphi}$ will follow from Theorem 1.1.4.

The definition of the norm $\|\cdot\|_{s\varphi}$ be means of this formula has the property, that it does not require the knowledge of the space $H^{*\varphi}$ itself. Namely, it follows from Theorems 1.1.3 and 1.1.4 that for every function F analytic in the disc D, $\|F\|_{s\varphi}$ may be defined by the above formula, and that the space $H^{*\varphi}$ is obtained then as the set of functions F analytic in the disc D for which $\|F\|_{s\varphi} < \infty$. The definition of this norm by means of formula (*) of 1.1.1 does not possess this property; for example, the function $F(z) = \exp\left(\frac{1+z}{1-z}\right)$ has a limit function $F(e^{i\cdot})$ belonging to $L^{*\varphi}$, but it does not belong to $H^{*\varphi}$.

1.1.6. Theorem. *Let $\varphi(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and ψ is a convex φ-function. If $F \epsilon H^{*\varphi}$, then*

$$|F(z)| \leqslant \varphi_{-1}\left(\frac{1}{\pi(1-|z|)}\right)\|F\|^{1/s}_{s\varphi} \quad \textit{for } z \epsilon D.$$

Proof. We take an arbitrary number $k > \|F\|_{s\varphi}$. By Theorem 1.1.2 we have $\mu_\varphi(F/k^{1/s}) \leqslant 1$. From Theorem II.1.2.2 we get

$$\left|\frac{1}{k^{1/s}}F(z)\right| \leqslant \varphi_{-1}\left(\frac{\mu_\varphi(F/k^{1/s})}{\pi(1-|z|)}\right) \quad \text{for } z \epsilon D.$$

Since φ_{-1} is increasing, we obtain hence

$$|F(z)| \leqslant \varphi_{-1}\left(\frac{1}{\pi(1-|z|)}\right)k^{1/s} \quad \text{for } z \epsilon D.$$

Passing with k to the limit, $k \to \|F\|_{s\varphi}$, we get the required inequality.

1.2.1. If ψ is a convex φ-function satisfying conditions (0_1) and (∞_1), then a homogeneous norm may be defined in $H^{*\psi}$ also by means of the formula

$$\|F\|_{(\psi)} = \|F(e^{i\cdot})\|^*_{(\psi)},$$

where $\|\cdot\|^*_{(\psi)}$ is the homogeneous norm in $L^{*\psi}$ defined in I.2.5.1. This norm is equivalent to any of the norms $\|\cdot\|_\psi$ and $\|\cdot\|_{1\psi}$ and the equivalency of $\|\cdot\|_{(\psi)}$ and $\|\cdot\|_{1\psi}$ is given by means of the inequality $\|F\|_{1\psi} \leqslant \|F\|_{(\psi)} \leqslant 2\|F\|_{1\psi}$, where $F \in H^{*\psi}$ (see I.2.5.1).

1.2.2. THEOREM. *If $F \in H^{*\psi}$, where ψ is a convex φ-function satisfying conditions (0_1) and (∞_1), then*

$$\|F\|_{(\psi)} = \inf\left\{\frac{1}{k}\big(1+\mu_\psi(kF)\big)\Big|\ k > 0\right\}.$$

This follows from Theorem I.2.5.2 and II.1.3.2 immediately.

1.2.3. THEOREM. *Let ψ be a convex function satisfying conditions (0_1) and (∞_1), and let F be an analytic function in the disc D. Then $\|T_rF\|_{(\psi)}$ is a non-decreasing function for $0 \leqslant r < 1$.*

This follows from Theorem 1.2.1 and from the previous theorem, immediately.

1.2.4. THEOREM. *Let ψ be a convex φ-function satisfying conditions (0_1) and (∞_1). A function F analytic in the disc D belongs to $H^{*\psi}$ if and only if $\sup\{\|T_rF\|_{(\psi)}|0 \leqslant r < 1\} < \infty$. If, moreover, $F \in H^{*\psi}$, then*

$$\|F\|_{(\psi)} = \sup\{\|T_rF\|_{(\psi)}|0 \leqslant r < 1\} = \lim_{r\to 1-}\|T_rF\|_{(\psi)}.$$

Proof. Let $F \in H^{*\psi}$. By Theorem II.1.2.1 and II.1.4.2 we have $\mu_\psi(kT_rF) \leqslant \mu_\psi(kF)$ for every $0 \leqslant r < 1$ and every $k > 0$. Hence we obtain, by Theorem 1.2.2

$$\sup\{\|T_rF\|_{(\psi)}|0 \leqslant r < 1\} \leqslant \|F\|_{(\psi)}.$$

Now, let us suppose that $\sup\{\|T_rF\|_{(\psi)}|0 \leqslant r < 1\} < \infty$ for a function F analytic in the disc D.

Since $\|G\|_{1\psi} \leqslant \|G\|_{(\psi)}$ for all $G \in H^{*\psi}$, we have then $\sup\{\|T_rF\|_{1\psi}|\ 0 \leqslant r < 1\} < \infty$. Applying Theorem 1.1.4 we get hence $F \in H^{*\psi}$. Now, we take an arbitrary number $k > \sup\{\|T_rF\|_{(\psi)}|\ 0 \leqslant r < 1\}$. According to the definition of the norm $\|\cdot\|^*_{(\psi)}$ in $L^{*\psi}$ (see I.2.5.1) we have then

$$\sup\left\{\sup\left\{\int_0^{2\pi}|F(re^{it})g(t)|\,dt\,\Big|\,\mathscr{I}_{\psi'}(g) \leqslant 1,\ g \in L^{\psi'}\right\}\Big|\ 0 \leqslant r < 1\right\} < k.$$

Thus

$$\int_0^{2\pi}|F(re^{it})g(t)|\,dt < k$$

for all $0 \leqslant r < 1$ and all functions $g \in L^{\psi'}$ such that $\mathscr{I}_{\psi'}(g) \leqslant 1$. Keeping the function g fixed and passing to the limit as $r \to 1-$ we get, by Fatou's lemma,

$$\int_0^{2\pi} |F(e^{it})g(t)|\,dt \leqslant k$$

for all functions $g \in L^{\psi'}$ such that $\mathscr{I}_{\psi'}(g) \leqslant 1$. Hence we obtain $\|F\|_{(\psi)} = \|F(e^{i\cdot})\|^*_{(\psi)} \leqslant k$, and for $F \in H^{*\psi}$ we conclude the inequality

$$\|F\|_{(\psi)} \leqslant \sup\{\|T_r F\|_{(\psi)} | 0 \leqslant r < 1\},$$

thus finishing the proof.

1.3. In Hardy spaces H^p, $0 < p \leqslant 1$, a p-homogeneous norm is defined by the formula

$$\|F\|_p = \frac{1}{2\pi}\int_0^{2\pi} |F(e^{it})|^p\,dt, \qquad F \in H^p$$

([17], Chapter VII). Spaces H^p, $0 < p \leqslant 1$, are a special case of spaces $H^{*\varphi}$, obtained by taking $\varphi(u) = u^p$, where $0 < p \leqslant 1$. Let us find in this case the connection between norms $\|\cdot\|_\varphi$ and $\|\cdot\|_{p\varphi}$, and the norm $\|\cdot\|_p$:

$$\|F\|_\varphi = \inf\left\{k > 0 \Big| \int_0^{2\pi} \left|\frac{F(e^{it})}{k}\right|^p dt \leqslant k\right\} = \inf\left\{k > 0 \Big| \int_0^{2\pi} |F(e^{it})|^p\,dt \leqslant k^{1+p}\right\}$$
$$= \inf\{k > 0 | 2\pi\|F\|_p \leqslant k^{1+p}\} = (2\pi)^{1/(1+p)}\|F\|_p^{1/(1+p)}$$

and

$$\|F\|_{p\varphi} = \inf\left\{k > 0 \Big| \int_0^{2\pi} \left|\frac{F(e^{it})}{k^{1/p}}\right|^p dt \leqslant 1\right\} = \inf\left\{k > 0 \Big| \int_0^{2\pi} F(e^{it})|^p dt \leqslant k\right\}$$
$$= \inf\{k > 0 | 2\pi\|F\|_p \leqslant k\} = 2\pi\|F\|_p.$$

In Hardy spaces H^p, $p > 1$, a homogeneous norm is defined by the formula

$$\|F\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |F(e^{it})|^p\,dt\right)^{1/p}, \qquad F \in H^p$$

([17], Chapter VII; 2). Spaces H^p, $p > 1$, are just spaces $H^{*\psi}$ for $\psi(u) = u^p$, where $p > 1$. Let us compare norms $\|\cdot\|_\psi$ and $\|\cdot\|_{1\psi}$ with the norm $\|\cdot\|_p$ also in this case:

$$\|F\|_\psi = \inf\left\{k > 0 \Big| \int_0^{2\pi} \left|\frac{F(e^{it})}{k}\right|^p dt \leqslant k\right\} = \inf\left\{k > 0 \Big| \int_0^{2\pi} |F(e^{it})|^p dt \leqslant k^{1+p}\right\}$$
$$= \inf\{k > 0 | 2\pi\|F\|_p^p \leqslant k^{1+p}\} = (2\pi)^{1/(1+p)}\|F\|_p^{p/(1+p)}$$

and

$$\|F\|_{1\psi} = \inf\left\{k>0 \,\Big|\, \int_0^{2\pi} \left|\frac{F(e^{it})}{k}\right|^p dt \leqslant 1\right\} = \inf\left\{k>0 \,\Big|\, \int_0^{2\pi} |F(e^{it})|^p dt < k^p\right\}$$

$$= \inf\{k>0 \,|\, 2\pi\|F\|_p^p \leqslant k^p\} = (2\pi)^{1/p}\|F\|_p.$$

Since $\psi(u) = u^p$, $p>1$, is a convex φ-function satisfying conditions (0_1) and (∞_1), in H^p is defined yet homogeneous norm $\|\cdot\|_{(\psi)}$. We find the connection between this norm and the norm $\|\cdot\|_p$. First, we have

$$\|F\|_{(\psi)} = \inf\left\{\frac{1}{k}\Big(1+\int_0^{2\pi} |kF(e^{it})|^p dt\Big) \,\Big|\, k>0\right\} = \inf\left\{\frac{1}{k}(1+k^p 2\pi\|F\|_p^p) \,\Big|\, k>0\right\}.$$

Since the real function $(1+k^p 2\pi\|F\|_p^p)/k$ for $k>0$, where $F\in H^p$ and $F\neq 0$, assumes its least value if $-1/k^2+(p-1)k^{p+2}2\pi\|F\|_p^p = 0$, i.e. for $k = \big((p-1)^{1/p}(2\pi)^{1/p}\|F\|_p\big)^{-1}$, we have

$$\|F\|_{(\psi)} = (p-1)^{1/p}(2\pi)^{1/p}\|F\|_p\left(1+\frac{1}{p-1}\right) = \frac{p^{1/p}}{(p')^{1/p'}}(2\pi)^{1/p}\|F\|_p,$$

where $1/p+1/p' = 1$.

2. Problems of existence of an s-homogeneous norm $(0<s\leqslant 1)$ in $H^{*\varphi}$

2.1.1. THEOREM. *If an F-norm $\|\cdot\|_0$ is defined in $H^{*\varphi}$ such that*

1° $H^{*\varphi}$ is complete with respect to this norm,

2° $F_n\in H^{*\varphi}$, $\|F_n\|_0\to 0$ imply $F_n\to 0$ almost uniformly in the disc D, then this norm is equivalent to the norm generated by φ.

Proof. Let $I(\cdot)$ be the identity map of $H^{*\varphi}$ onto, itself. If $\|F_n-F\|_0\to 0$ and $\|I(F_n)-G\|_\varphi\to 0$, then $F_n\to F$ almost uniformly in the disc D, and also $I(F_n)\to G$ almost uniformly in the disc D, whence $I(F) = G$. From the closed graph theorem ([1], p. 41, Theorem 7) follows that $\|F_n\|_0\to 0$ implies $\|F_n\|_\varphi\to 0$ for every sequence (F_n) of functions from $H^{*\varphi}$. The fact that $\|F_n\|_\varphi\to 0$ implies $\|F_n\|_0\to 0$ is proved, analogously.

2.1.2. THEOREM. *An s-homogeneous norm $\|\cdot\|_0$ $(0<s\leqslant 1)$ satisfying the conditions*

1° $H^{*\varphi}$ is complete with respect to this norm,

2° $F_n\in H^{*\varphi}$, $\|F_n\|_0\to 0$ imply $F_n\to 0$ almost uniformly in the disc D, exist in $H^{*\varphi}$ if and only if $\varphi(u)\sim\psi(u^s)$, where ψ is a convex φ-function.

Proof. Let us suppose, there exists an s-homogeneous norm $\|\cdot\|_0$ in $H^{*\varphi}$ $(0<s\leqslant 1)$ satisfying conditions 1° and 2°. By the previous theorem, this norm is equivalent to the norm generated by φ. Hence there exists a constant δ, $0<\delta<1$, such that $\|F\|_0\leqslant 2\delta$ implies $\|F\|_\varphi\leqslant 1$, and

$\|F\|_\varphi \leqslant 2\delta$ implies $\|F\|_0 \leqslant 1$. We shall show that then the following inequality holds for $0 < a^s \leqslant \delta$ and $a^s\varphi(u\delta^{-1}) \geqslant 1$:

$$(*) \qquad \varphi(au) \leqslant 2\delta^{-2}a^s\varphi(u\delta^{-1}).$$

Let us suppose this inequality is not satisfied. Then there exist numbers $\hat{a}$ and $\hat{u}$ such that $0 < \hat{a}^s \leqslant \delta$, $\hat{a}^s\varphi(\hat{u}\delta^{-1}) \geqslant 1$ and $\varphi(\hat{a}\hat{u}) > 2\delta^{-2}\hat{a}^s\varphi(\hat{u}\delta^{-1})$. Let us denote by n a positive integer for which $\delta/2 \leqslant n\hat{a}^s \leqslant \delta$. We define, in $\langle 0, 2\pi)$, n closed and disjoint intervals $\langle t_\nu', t_\nu''\rangle$ of length $t_\nu'' - t_\nu' = \frac{5}{3}\delta\varphi(\hat{u}\delta^{-1})^{-1}$. Let $\tau_\nu' = t_\nu' + \frac{1}{6}\delta\varphi(\hat{u}\delta^{-1})^{-1}$ and $\tau_\nu'' = t_\nu'' - \frac{1}{6}\delta\varphi(\hat{u}\delta^{-1})^{-1}$. Next, we define n real functions continuous on the interval $\langle 0, 2\pi)$:

$$f_\nu(t) = \begin{cases} 0 & \text{for } 0 \leqslant t < t_\nu' \text{ and } t_\nu'' \leqslant t < 2\pi, \\ \dfrac{t - t_\nu'}{\tau_\nu' - t_\nu'} & \text{for } t_\nu' \leqslant t < \tau_\nu', \\ 1 & \text{for } \tau_\nu' \leqslant t < \tau_\nu'', \\ \dfrac{t_\nu'' - t}{t_\nu'' - \tau_\nu''} & \text{for } \tau_\nu'' \leqslant t < t_\nu''. \end{cases}$$

Since the functions f_ν are continuous on $\langle 0, 2\pi)$, we may choose for every $\varepsilon > 0$ trigonometric polynomials

$$T_\nu(t) = \sum_{k=-m_\nu}^{m_\nu} c_{k\nu}e^{ikt}$$

such that $|f_\nu(t) - T_\nu(t)| < \varepsilon/n$ for all $t \in \langle 0, 2\pi)$, $\nu = 1, 2, \ldots, n$. Let $m = \sup\{m_1, m_2, \ldots, m_n\}$. The complex polynomials

$$P_\nu(z) = \sum_{k=-m_\nu}^{m_\nu} c_{k\nu}z^{k+m}$$

possess a limit function $P_\nu(e^{it}) = e^{imt}T_\nu(t)$. Hence

$$\big|f_\nu(t) - |P_\nu(e^{it})|\big| = \big|f_\nu(t) - |T_\nu(t)|\big| \leqslant |f_\nu(t) - T_\nu(t)| < \varepsilon/n$$

and

$$\Big|\sum_{\nu=1}^n f_\nu(t) - \Big|\sum_{\nu=1}^n P_\nu(e^{it})\Big|\Big| = \Big|\sum_{\nu=1}^n f_\nu(t) - \Big|\sum_{\nu=1}^n T_\nu(t)\Big|\Big|$$
$$\leqslant \Big|\sum_{\nu=1}^n f_\nu(t) - \sum_{\nu=1}^n T_\nu(t)\Big| \leqslant \sum_{\nu=1}^n |f_\nu(t) - T_\nu(t)| < \varepsilon$$

for all $t \in \langle 0, 2\pi)$. Since the function φ is continuous for $u \geqslant 0$ and the functions f_ν are uniformly bounded, we may suppose that ε is chosen so small that the following inequalities hold:

$$\big|\varphi(\hat{u}\delta^{-1}f_\nu(t)) - \varphi(\hat{u}\delta^{-1}|P_\nu(e^{it})|)\big| < \frac{1}{6\pi}\delta \quad \text{for } \nu = 1, 2, \ldots, n$$

and

$$\left|\varphi\left(\hat{a}\hat{u}\sum_{\nu=1}^{n} f_\nu(t)\right) - \varphi\left(\hat{a}\hat{u}\left|\sum_{\nu=1}^{n} P_\nu(e^{it})\right|\right)\right| < \frac{1}{6\pi}$$

for all $t \in \langle 0, 2\pi)$. Hence we get for $\nu = 1, 2, \ldots, n$

$$\begin{aligned}\mu_\varphi\left(\frac{\hat{u}P_\nu}{2\delta}\right) &\leqslant \mu_\varphi(\hat{u}\delta^{-1}P_\nu) = \int_0^{2\pi} \varphi\left(\hat{u}\delta^{-1}|P_\nu(e^{it})|\right)dt \\ &\leqslant \int_0^{2\pi} \varphi\left(\hat{u}\delta^{-1}f_\nu(t)\right)dt + \tfrac{1}{3}\delta \leqslant \varphi(\hat{u}\delta^{-1})(t_\nu'' - t_\nu') + \tfrac{1}{3}\delta \\ &\leqslant \tfrac{5}{3}\delta + \tfrac{1}{3}\delta = 2\delta,\end{aligned}$$

and this means that $\|\hat{u}P_\nu\|_\varphi \leqslant 2\delta$. Thus we have $\|\hat{u}P_\nu\|_0 \leqslant 1$ for $\nu = 1, 2, \ldots, n$ and

$$\left\|\hat{a}\hat{u}\sum_{\nu=1}^{n} P_\nu\right\|_0 = \hat{a}^s \left\|\sum_{\nu=1}^{n} \hat{u}P_\nu\right\|_0 \leqslant \hat{a}^s \sum_{\nu=1}^{n} \|\hat{u}P_\nu\|_0 \leqslant \hat{a}^s n \leqslant \delta.$$

Hence $\|\hat{a}\hat{u}\sum_{\nu=1}^{n} P_\nu\|_\varphi \leqslant 1$, and so $\mu_\varphi\left(\hat{a}\hat{u}\sum_{\nu=1}^{n} P_\nu\right) \leqslant 1$. But, on the other hand,

$$\begin{aligned}\mu_\varphi\left(\hat{a}\hat{u}\sum_{\nu=1}^{n} P_\nu\right) &= \int_0^{2\pi} \varphi\left(\hat{a}\hat{u}\left|\sum_{\nu=1}^{n} P_\nu(e^{it})\right|\right)dt > \int_0^{2\pi} \varphi\left(\hat{a}\hat{u}\sum_{\nu=1}^{n} f_\nu(t)\right)dt - \tfrac{1}{3} \\ &> \sum_{\nu=1}^{n} \varphi(\hat{a}\hat{u})(\tau_\nu'' - \tau_\nu') - \tfrac{1}{3} = n\varphi(\hat{a}\hat{u}) \cdot \tfrac{4}{3}\delta\varphi(u\delta^{-1})^{-1} - \tfrac{1}{3} \\ &> \tfrac{4}{3} n\delta \frac{2}{\delta^2}\hat{a}^s - \tfrac{1}{3} \geqslant \tfrac{4}{3}\frac{2}{\delta}\frac{\delta}{2} - \tfrac{1}{3} = 1.\end{aligned}$$

Thus we have a contradiction. Consequently, inequality (*) is satisfied. Let us remark that inequality (*) holds also for $\delta < a^s \leqslant 1$ and $u \geqslant 0$, since then we have

$$\varphi(au) \leqslant \delta^2\delta^{-2}\varphi(u\delta^{-1}) \leqslant 2\delta^{-2}a^s\varphi(u\delta^{-1}).$$

Now, substituting $a^s = \varphi(u\delta^{-1})^{-1}$ into (*) we get $\varphi\left(\varphi(u\delta^{-1})^{-1/s}u\right) \leqslant 2\delta^{-2}$ for $u \geqslant \delta\varphi_{-1}(1)$. From this inequality follows the existence of a constant $v_0 > 0$ such that $\varphi(u\delta^{-1})^{-1/s}u \leqslant \delta v_0^{1/s}$ for $u \geqslant \delta v_0^{1/s}$. Hence, taking $u = \delta v^{1/s}$, we obtain the inequality $\varphi(v^{1/s})v^{-1} \geqslant v_0^{-1}$ for $v \geqslant v_0$. We take $v_2 \geqslant v_1 \geqslant v_0$. Then $0 < v_1 v_2^{-1} \leqslant 1$ and $v_1 v_2^{-1}\varphi(v_2^{1/s}) \geqslant v_1 v_0^{-1} \geqslant 1$. Now, substituting $a^s = v_1 v_2^{-1}$ and $u = \delta v_2^{1/s}$ into (*) we get the inequality

$$\tfrac{1}{2}\delta^2 v_1^{-1}\varphi(\delta v_1^{1/s}) \leqslant v_2^{-1}\varphi(v_2^{1/s}) \qquad \text{for } v_2 \geqslant v_1 \geqslant v_0.$$

Hence, by Lemma I.1.2.3, $\varphi(v^{1/s})$ is equivalent to a convex φ-function $\psi(v)$, i.e. $\varphi(u) \sim \psi(u^s)$.

Conversely, let $\varphi(u) \sim \varphi_1(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and ψ is a convex φ-function. By Theorem II.3.2.1 we then $H^{*\varphi} = H^{*\varphi_1}$. Applying 1.1.1 we conclude that $\|\cdot\|_{s\varphi_1}$ is an s-homogeneous norm in $H^{*\varphi}$ such that the space $H^{*\varphi}$ is complete with respect to this norm. Moreover, from The orem 1.1.6 we deduce that $F_n \in H^{*\varphi}$ and $\|F_n\|_{s\varphi_1} \to 0$ implies $F_n \to 0$-almost uniformly in the disc D.

Now, some corollaries will be deduced from Theorem 2.1.2.

2.1.3. THEOREM. *If* $\lim\limits_{u\to\infty} u^{-s}\varphi(u) = 0$ *for a given* $0 < s \leqslant 1$, *then no s-homogeneous norm may be defined in $H^{*\varphi}$ such that $H^{*\varphi}$ is complete with respect to this norm and convergence to* 0 *with respect to this norm implies almost uniform convergence to* 0 *in the disc D.*

Proof. Let us suppose that conversely, an s-homogeneous norm possessing the above mentioned properties may be defined in $H^{*\varphi}$. From the proof of Theorem 2.1.2 follows that there exists a constant $v_0 > 0$ such that $\varphi(v^{1/s})v^{-1} \geqslant v_0^{-1}$ for $v \geqslant v_0$. Hence we get $\lim\limits_{u\to\infty} u^{-s}\varphi(u) \geqslant v_0^{-1} > 0$, a contradiction with our assumption.

2.1.4. THEOREM. *In Hardy spaces H^p, $0 < p < 1$, no s-homogeneous norm may be defined, $p < s \leqslant 1$, such that H^p is complete with respect to this norm and convergence to* 0 *with respect to this norm implies almost uniform convergence to* 0 *in the disc D.*

This theorem follows from Theorem 2.1.3 immediately, since $\lim\limits_{u\to\infty} u^{-s}u^p = 0$ for $0 < p < s \leqslant 1$.

In the special case $s = 1$, Theorem 2.1.4 gives the known result of Livingstone [5] (compare also [4]).

2.1.5. Remark. There exist spaces $H^{*\varphi}$ such that for no s, $0 < s \leqslant 1$, an s-homogeneous norm may be defined in $H^{*\varphi}$ such that $H^{*\varphi}$ is complete with respect to this norm and convergence to 0 with respect to this norm implies almost uniform convergence to 0 in the disc D.

We show this by the example of the space $H^{*\varphi}$, where $\varphi(u) = \log^p(1+u)$, $p > 1$.

First of all let us note that function $\varphi(u) = \log^p(1+u)$, $p > 1$, is a log-convex φ-function. This follows from Theorem I.1.6.2 immediately, because the function

$$p(t) = t\frac{d\varphi}{du}(t) = p\log^{p-1}(1+t)\frac{t}{1+t}$$

is positive and non-decreasing for $t > 0$ and $p(t) \to \infty$ as $t \to \infty$. But

$$\lim_{u\to\infty} u^{-s}\log^p(1+u) = \left(\lim_{u\to\infty}\frac{\log(1+u)}{u^{s/p}}\right)^p = 0$$

for every $0 < s \leqslant 1$; hence, we conclude from Theorem 2.1.3 that for no s, $0 < s \leqslant 1$, an s-homogeneous norm satisfying the above formulated conditions may defined in the space $H^{*\varphi}$, where $\varphi(u) = \log^p(1+u)$, $p > 1$.

2.2.1. A set $X \subset H^{*\varphi}$ is called *bounded* in the space $[H^{*\varphi}, \|\cdot\|_\varphi]$, if for every sequence (F_n) of elements $F_n \in X$ and every sequence of numbers (a_n) convergent to 0, $\|a_n F_n\|_\varphi \to 0$.

If X is a bounded set in $[H^{*\varphi}, \|\ \|_\varphi]$, then there exists a positive integer n such that $\|F/n\|_\varphi \leqslant 1$ for all $F \in X$, whence applying the triangle inequality to the norm we get $\|F\|_\varphi \leqslant n$. This means that a set X bounded in the space $[H^{*\varphi}, \|\cdot\|_\varphi]$ is in this space bounded in the norm.

We prove now that

2.2.2. THEOREM. *If φ satisfies condition (V_2), then every set $X \in H^{*\varphi}$ bounded in the norm of the space $[H^{*\varphi}, \|\cdot\|_\varphi]$ is bounded in this space.*

Proof. If φ satisfies condition (V_2), then the inequality $2\varphi(u) \leqslant \varphi(du)$ for $u \geqslant u_0$ is satisfied for some constants $d > 1$ and $u_0 \geqslant 0$. Hence we get for an arbitrary positive integer n

$$2^n\varphi(u) \leqslant 2^{n-1}\varphi(du) \leqslant \ldots \leqslant \varphi(d^n u) \qquad \text{for } u \geqslant u_0.$$

Substituting $u = d^{-n}v$ to this inequality we obtain

$$\varphi(d^{-n}v) \leqslant 2^{-n}\varphi(v) \qquad \text{for } v \geqslant d^n u_0.$$

Now, let δ be a fixed number such that $\|F\|_\varphi < \delta$ for all $F \in X$. We take an arbitrary number $\varepsilon > 0$ and we choose n so large that there hold the inequalities

$$\frac{\delta}{2^{n-1}} < \varepsilon \quad \text{and} \quad \frac{1}{u_0}\varphi_{-1}\left(\frac{\delta}{2^{n+1}\pi}\right) \leqslant 1.$$

We set

$$\eta = \frac{1}{(2d)^n u_0}\varphi_{-1}\left(\frac{\delta}{2^{n+1}\pi}\right).$$

We denote $E(F) = \{t \in \langle 0, 2\pi) \mid |F(e^{it})| \geqslant \delta d^n u_0\}$ for $F \in X$. Now, applying Theorem II.1.3.2 we get for an arbitrary complex number a such that $|a| < \eta$ and for an arbitrary $F \in X$

$$\mu_\varphi\left(\frac{2^{n-1}}{\delta}aF\right) = \int_0^{2\pi}\varphi\left(2^{n-1}|a|\frac{|F(e^{it})|}{\delta}\right)dt = \int_{E(F)}\ldots + \int_{\langle 0,2\pi)\setminus E(F)}\ldots$$

$$\leqslant \int_{E(F)}\varphi\left(\frac{1}{d^n}\frac{|F(e^{it})|}{\delta}\right)dt + 2\pi\varphi\left(\eta(2d)^n u_0\right)$$

$$\leqslant \frac{1}{2^n}\int_{E(F)}\varphi\left(\frac{|F(e^{it})|}{\delta}\right)dt + \frac{\delta}{2^n} \leqslant \frac{1}{2^n}\mu_\varphi\left(\frac{F}{\delta}\right) + \frac{\delta}{2^n}$$

$$\leqslant \frac{\delta}{2^n} + \frac{\delta}{2^n} = \frac{\delta}{2^{n-1}}.$$

But this proves that $\|aF\|_\varphi \leqslant \delta/2^{n-1} < \varepsilon$ for all complex numbers a with moduli $|a| < \eta$ and for all $F \in X$. Hence it follows that X is a bounded set in the space $[H^{*\varphi}, \|\cdot\|_\varphi]$.

2.2.3. Theorem. *If φ does not satisfy condition (V_2), then none of the balls $\{F \in H^{*\varphi} | \ \|F\|_\varphi \leqslant \delta\}$, $\delta > 0$, is a bounded set in the space $[H^{*\varphi}, \|\cdot\|_\varphi]$.*

Proof. If φ does not satisfy condition (V_2), then there exists a sequence of numbers (u_n) such that

$$2\varphi(2^{-n}u_n) > \varphi(u_n) > 2\delta.$$

We define sets E_n in the interval $\langle 0, 2\pi)$ of measures $\operatorname{mes} E_n = \frac{1}{2}\delta\varphi(u_n)^{-1}$, and real functions

$$f_n(t) = \begin{cases} u_n & \text{for } t \in E_n, \\ \delta\varphi_{-1}\left(\dfrac{\delta}{4\pi}\right) & \text{for } t \in \langle 0, 2\pi)\setminus E_n. \end{cases}$$

Obviously, $\log f_n(\cdot) \in L^1$ for $n = 1, 2, \ldots$ Hence, by Theorem I.3.2.5 there exist functions $F_n \in N'$ such that $|F_n(e^{it})| = f_n(t)$ for almost all t from $\langle 0, 2\pi)$. Applying Theorem II.1.3.2 to there functions we get

$$\mu_\varphi\left(\frac{F_n}{\delta}\right) = \mathscr{I}_\varphi\left(\frac{f_n}{\delta}\right) \leqslant \varphi(u_n)\cdot\operatorname{mes} E_n + \frac{\delta}{4\pi}\cdot 2\pi = \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

and

$$\mu_\varphi\left(\frac{4}{\delta}\,\frac{F_n}{2^{n+2}}\right) = \mu_\varphi\left(\frac{F_n}{2^n\delta}\right) = \mathscr{I}_\varphi\left(\frac{f_n}{2^n\delta}\right) > \varphi(2^{-n}u_n)\cdot\operatorname{mes} E_n > \frac{\delta}{4}.$$

Hence

$$\|F_n\|_\varphi \leqslant \delta \quad \text{and} \quad \left\|\frac{1}{2^{n+2}}F_n\right\|_\varphi > \frac{\delta}{4}$$

for all n. But this proves that the ball $\{F \in H^{*\varphi} | \ \|F\|_\varphi \leqslant \delta\}$ is not a bounded set in the space $[H^{*\varphi}, \|\cdot\|_\varphi]$.

2.2.4. Theorem. *An s-homogeneous norm $\|\cdot\|_0$, $0 < s \leqslant 1$, such that $H^{*\varphi}$ is complete with respect to this norm and convergence to 0 with respect to this norm implies almost uniform convergence to 0 in the disc D exists in $H^{*\varphi}$ if and only if φ satisfies condition (V_2).*

Proof. If an s-homogeneous norm $\|\cdot\|_0$, $0 < s \leqslant 1$, possessing the above properties exists in $H^{*\varphi}$, then this norm is equivalent to the norm generated by φ, according to Theorem 2.1.1. Hence there exists a $\delta > 0$ such that $\|F\|_\varphi \leqslant \delta$ implies $\|F\|_0 \leqslant 1$. Since the norm $\|\cdot\|_0$ is s-homogeneous, the set $\{F \in H^{*\varphi} | \ \|F\|_0 \leqslant 1\}$ is bounded in the space $[H^{*\varphi}, \|\cdot\|_0]$. But both norms $\|\cdot\|_0$ and $\|\cdot\|_\varphi$ are equivalent; hence the set $\{F \in H^{*\varphi} | \ \|F\|_0 \leqslant 1\}$ is bounded also in the space $[H^{*\varphi}, \|\cdot\|_\varphi]$. Thus the set $\{F \in H^{*\varphi} \mid \|F\|_\varphi \leqslant \delta\}$ is bounded in this space. Hence it follows, by Theorem 2.2.3 that φ satisfies condition (V_2).

Conversely, if φ satisfies condition (V_2), then we may conclude from Lemma I.1.4.3 that there exist a number $s' > 0$ and a convex φ-function ψ such that $\varphi \sim \varphi_1$, where $\varphi_1(u) = \psi(u^{s'})$. Let $s = \inf\{s', 1\}$. Since $0 < s \leqslant 1$ and $\varphi_1(u^{1/s}) = \psi(u^{s'/s})$ is a convex φ-function, according to 1.1.1 one may define an s-homogeneous norm $\|\cdot\|_{s\varphi_1}$ in $H^{*\varphi_1}$. Obviously, the space $H^{*\varphi_1}$ is complete with respect to the norm $\|\cdot\|_{s\varphi_1}$. Moreover, it follows from Theorem 1.1.6 that if $F_n \in H^{*\varphi_1}$ and $\|F_n\|_{s\varphi_1} \to 0$, then $F_n \to 0$ almost uniformly in the disc D. But $\varphi \sim \varphi_1$. From Theorem II.3.2.1 we get $H^{*\varphi} = H^{*\varphi_1}$ and this concludes the proof.

Taking into accound Lemma I.1.4.3 it is easily seen that Theorem 2.2.4 is weaker than Theorem 2.1.2. Nevertheless, we proved it independently of Theorem 2.1.2.

2.3. THEOREM. *All theorems of this Section remain valid if we replace $H^{*\varphi}$ by K^φ in the formulations of these theorems.*

Theorem 2.1.1 for K^φ is proved analogously as for $H^{*\varphi}$. Theorem 2.1.2 for K^φ holds, since the polynomials used in the proof of this theorem are obviously elements of K^φ, and on the other hand, K^φ is a subspace of the space $H^{*\varphi}$. Theorem 2.1.3 for K^φ is a corollary to Theorem 2.1.2 formulated for K^φ. Remark 2.1.5 remains valid for K^φ, because the function $\varphi(u) = \log^p(1+u)$, $p > 1$, satisfies condition (Δ_2) and so $H^{*\varphi} = K^\varphi$, by Theorem II.3.3.2. Next, Theorem 2.2.2 remains valid for K^φ, since K^φ is a subspace of the space $H^{*\varphi}$. Theorem 2.2.3 in the formulation for K^φ holds, because the functions f_n used in the proof of this theorem are bounded in $\langle 0, 2\pi)$, and thus the analytic functions F_n obtained applying Theorem I.3.2.5 belong to $H^\infty \subset K^\varphi$ (compare the proof of Theorem III.3.2.2). Finally, Theorem 2.2.4 for K^φ is proved analogously as for $H^{*\varphi}$.

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