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Compactness and upper semicontinuity of solution set of functional-differential equations of hyperbolic type

1. Introduction. Let $\boldsymbol{R}^n$ denote n-dimensional Euclidean space with the norm $|\cdot|$. Denote by $\mathrm{Conv}(\boldsymbol{R}^n)$ the family of all non-empty compact and convex subsets of $\boldsymbol{R}^n$. $\mathrm{Conv}(\boldsymbol{R}^n)$ is a metric space with the Hausdorff metric h defined in the following way, $h(A, B) = \max\{\bar{h}(A, B), \bar{h}(B, A)\}$, for $A, B \in \mathrm{Conv}(\boldsymbol{R}^n)$ where $\bar{h}(A, B) = \max\{d(z, B)\colon z \in A\}$ and $d(z, B) = \min\{|z-b|\colon b \in B\}$. Let $a > 0$ and $b > 0$ be given, $P = [0, a] \times [0, b]$ and let us denote by $C(P)$ the Banach space of all continuous functions of P into $\boldsymbol{R}^n$ with the usual norm $\|\cdot\|$. Furthermore, we use the space $C_x(P)$ of all functions $u\colon P \to \boldsymbol{R}^n$ such that $u(\cdot, y)\colon [0, a] \to \boldsymbol{R}^n$ is measurable for fixed $y \in [0, b]$, $u(x, \cdot)\colon [0, b] \to \boldsymbol{R}^n$ is continuous for fixed $x \in [0, a]$ and such that

$$|u|_x = \int_0^a \max_{y \in [0, b]} |u(x, y)|\, dx < \infty .$$

Similarly we can define a space $C_y(P)$ with the norm $|\cdot|_y$. It was proved ([2]) that $(C_x(P), |\cdot|_x)$ and $(C_y(P), |\cdot|_y)$ are Banach spaces.

Let $F\colon P \times C(P) \times C_x(P) \times C_y(P) \to \mathrm{Conv}(\boldsymbol{R}^n)$ be a multivalued mapping satisfying the following Carathéodory type conditions:

(i) $F(\cdot, \cdot, z, u, v)$ is measurable for fixed $(z, u, v) \in C(P) \times C_x(P) \times C_y(P)$,

(ii) $F(x, y, \cdot, \cdot, \cdot)$ is continuous for fixed $(x, y) \in P$,

(iii) there exists a square Lebesgue integrable function $m\colon P \to R$ such that $h(F(x, y, z, u, v), \{0\}) \leqslant m(x, y)$ for $(z, u, v) \in C(P) \times C_x(P) \times C_y(P)$ and almost all $(x, y) \in P$.

Furthermore, it will be assumed that $F(x, y, z, \cdot, \cdot)$ satisfies the following strong Lipschitz condition:

(iv) there exists $k > 0$ such that

$$h(F(x, y, z, u, v), F(x, y, z, \bar{u}, \bar{v}))$$
$$\leqslant k\Big(\Big|\int_0^x [u(s, y) - u(s, y)]\, ds\Big| + \Big|\int_0^y [v(x, t) - v(x, t)]\, dt\Big|$$

for $z \in C(P)$; $u, \bar{u} \in C_x(P)$; $v, \bar{v} \in C_y(P)$ and a.a. $(x, y) \in P$.

Now, we will consider a generalized functional-differential equation of the form

(1) $z''_{xy}(x, y) \in F(x, y, z, z'_x, z'_y)$ for almost all $(x, y) \in P$

with the initial Darboux conditions:

(2) $z(x, 0) = \sigma(x)$, $z(0, y) = \tau(y)$, where $\sigma\colon [0, a] \to \boldsymbol{R}^n$, $\tau\colon [0, b] \to \boldsymbol{R}^n$ are given absolutely continuous functions such that $\sigma' \in L(0, a)$ and $\tau' = L(0, b)$.

We will say that $F\colon P \times C(P) \times C_x(P) \times C_y(P) \to \mathrm{Conv}(\boldsymbol{R}^n)$ has the *Volterra's property* if $F(x, y, z, u, v) = F(x, y, \bar{z}, \bar{u}, \bar{v})$ for every $(x, y) \in P$ and (z, u, v), $(\bar{z}, \bar{u}, \bar{v}) \in C(P) \times C_x(P) \times C_y(P)$ such that $z|_{[0,x]\times[0,y]} = \bar{z}|_{[0,x]\times[0,y]}$, $u|_{[0,x]\times[0,y]} = \bar{u}|_{[0,x]\times[0,y]}$, $v|_{[0,x]\times[0,y]} = \bar{v}|_{[0,x]\times[0,y]}$, where for given $w\colon P \to \boldsymbol{R}^n$ and $(x, y) \in P$, $w|_{[0,x]\times[0,y]}$ denotes the restriction of w to the rectangle $[0, x] \times [0, y] \subset P$. Let us denote by $C(0, a)$ and $C(0, b)$ spaces of all continuous functions of $[0, a]$ and $[0, b]$ respectively into $\boldsymbol{R}^n$ with the usual norms, denoted by $\|\cdot\|_a$ and $\|\cdot\|_b$, respectively. By A we will denote the subset of space $C(0, a) \times C(0, b)$ containing all pairs of absolutely continuous functions (σ, τ) such that $\sigma(0) = \tau(0)$.

2. Compactness and upper semicontinuity. Suppose $F\colon P \times C(P) \times C_x(P) \times C_y(P) \to \mathrm{Conv}(\boldsymbol{R}^n)$ satisfies the Carathéodory conditions (i)–(iii), the Lipschitz condition (iv) and let F have Volterra's property. Denote by $\mathfrak{X}(\sigma, \tau)$ the solution set of problem (1)–(2), that is the set of all solutions of (1)–(2). In virtue of remark given in ([3]) $\mathfrak{X}(\sigma, \tau)$ is non-empty for every $(\sigma, \tau) \in A$. Now we can prove the following theorem.

THEOREM 1. *If M is a compact subset of A, then $\mathfrak{X}(M) = \bigcup \{ \mathfrak{X}(\sigma, \tau)\colon (\sigma, \tau) \in M\}$ is a compact subset of $C(P)$.*

Proof. Let M' and M'' be projections of M into $C(0, a)$ and $C(0, b)$ respectively. Then there exist $d_1, d_2 \in R$ such that $\|\sigma\|_a < d_1$ and $\|\tau\|_b < d_2$ for every $(\sigma, \tau) \in M$.

If $z \in \mathfrak{X}(M)$, then there exists a point $(\sigma, \tau) \in M$ such that

$$z(x, y) = \sigma(x) + \tau(y) - \sigma(0) + \int_0^x \int_0^y z''_{xy}(s, t)\, ds\, dt.$$

Hence,

$$\|z\| \leqslant 2\|\sigma\|_a + \|\tau\|_b + \iint_P m(x, y)\, dx\, dy.$$

Thus $\mathfrak{X}(M)$ is bounded. Let (x_1, y_1), $(x_2, y_2) \in P$. Then

$$\|z(x_2, y_2) - z(x_1, y_1)\|$$
$$\leqslant |\sigma(x_2) - \sigma(x_1)| + |\tau(y_2) - \tau(y_1)| + \int_0^{x_1} \int_{y_1}^{y_2} m(s, t)\, ds\, dt + \int_{x_1}^{x_2} \int_0^{y_2} m(s, t)\, ds\, dt.$$

Therefore $\mathfrak{X}(M)$ is equicontinuous. Thus by Ascoli's Theorem $\overline{\mathfrak{X}(M)}$ is compact in $C(P)$.

Now, we shall show that $\mathfrak{X}(M) = \overline{\mathfrak{X}(M)}$. Let (z_k) be a sequence of $\mathfrak{X}(M)$ convergent to $z \in C(P)$. In virtue of Lemmas 1.2 and 1.3 in ([4]) and Theorem 2.8 given in ([1]), z is absolutely continuous and

$$z''_{xy}(x, y) \in \bigcap_{i=1}^{\infty} \overline{\mathrm{co}} \bigcup_{k=i}^{\infty} (z_k)''_{xy}(x, y)$$

$$\subset \bigcap_{i=1}^{\infty} \overline{\mathrm{co}} \bigcup_{k=i}^{\infty} F\big(x, y, z_k, (z_k)'_x, (z_k)'_y\big) \subset F(x, y, z, z'_x, z'_y)$$

for almost all $(x, y) \in P$.

Furthermore, for every $z_k \in \mathfrak{X}(M)$ there exists $(\sigma_k, \tau_k) \in M$ such that $\sigma_k(x) = z_k(x, 0)$, $\tau_k(y) = z_k(0, y)$. Since M is compact subset of A there exists a subsequence, also denoted by (σ_k, τ_k) convergent to a point $(\sigma, \tau) \in M$. Thus,

$$|z(x, 0) - \sigma(x)| \leqslant |z(x, 0) - z_k(x, 0)| + |z_k(x, 0) - \sigma_k(x)| + |\sigma_k(x) - \sigma(x)|$$

for every $k = 1, 2, 3, \ldots,$

then we obtain $z(x, 0) = \sigma(x)$ for $x \in [0, a]$ and similarly, $z(0, y) = \tau(y)$ for $y \in [0, b]$. The above shows that $z \in \mathfrak{X}(M)$ and the proof is complete.

Now we can prove that a mapping $\mathfrak{X}$: $M \ni (\sigma, \tau) \to \mathfrak{X}(\sigma, \tau) \in \mathrm{Comp}\big(C(P)\big)$ is upper semicontinuous.

THEOREM 2. *The mapping* $\mathfrak{X}$ *is upper semicontinuous.*

Proof. Assume that $\mathfrak{X}$ is not upper semicontinuous at $(\sigma_0, \tau_0) \in M$, that is there exists $\varepsilon_0 > 0$ such that for all $\delta > 0$, $\mathfrak{X}[B_\delta(\sigma_0, \tau_0)] \not\subset \mathfrak{X}^{\varepsilon_0}(\sigma_0, \tau_0)$, where $B_\delta(\sigma_0, \tau_0)$ denotes a neighbourhood of (σ_0, τ_0) at the radius $\delta > 0$. Choose z_k such that $z_k \in \mathfrak{X}[M \cap B_{1/k}(\sigma_0, \tau_0)]$ and $z_k \notin \mathfrak{X}^{\varepsilon_0}(\sigma_0, \tau_0)$. Now $z_k \in \overline{\mathfrak{X}[M \cap B_1(\sigma_0, \tau_0)]}$ which is compact by Theorem 1. There exists a subsequence, also denoted by (z_k) covering to $z \in \mathfrak{X}[M \cap B_1(\sigma_0, \tau_0)]$. Let $\varepsilon > 0$ and $(\sigma_k, \tau_k) \in B_{1/k}(\sigma_0, \tau_0)$ be given. Choose k_0 and k_1 such that $k > k_0$ implies $\|z - z_k\| < \frac{1}{2}\varepsilon$ and $k > k_1$ implies $\|\sigma_k - \sigma_0\|_a < \frac{1}{2}\varepsilon$.

Then for $x \in [0, a]$ and $k > \max(k_0, k_1)$, we have

$$|z(x, 0) - \sigma_0(x)| \leqslant |z(x, 0) - z_k(x, 0)| +$$
$$+ |z_k(x, 0) - \sigma_k(x)| + \sigma_k(x) - \sigma_0(x)| \leqslant \tfrac{1}{2}\varepsilon + 0 + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Since ε is any positive number, we have $z(x, 0) = \sigma_0(x)$ and similarly $z(0, y) = \tau_0(y)$. Thus, $z \in \mathfrak{X}(\sigma_0, \tau_0)$. But $z_k \notin \mathfrak{X}^{\varepsilon_0}(\sigma_0, \tau_0)$. Therefore $z \notin \mathfrak{X}(\sigma_0, \tau_0)$. From this contradiction we conclude that $\mathfrak{X}$ is upper semicontinuous. The proof is complete.

References

[1] J. L. Davy, *Properties of the solution set of a generalized differential equation*, Bull. Austral. Math. Soc. 6 (1972), 379–398.

[2] K. Deimling, *A Carathéodory theory for systems of integral equations*, Ann. Math. (1969), 217–260.

[3] W. Sosulski, *Existence theorem for generalized functional-differential equations of hyperbolic type*, Comment. Math. this volume, 149–152.

[4] –, *Properties of the solution of generalized differential equation of hyperbolic type*, W.S.Inż. Zielona Góra (1982) (in Polish).