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Boundaries, Martin's Axiom, and (P)-properties in dual Banach spaces

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Summary. Let X be a Banach space and $\mathcal{S}eq(X^{**})$ (resp., $X_{\aleph_0}$) the subset of elements $\psi \in X^{**}$ such that there exists a sequence $(x_n)_{n\geqslant 1} \subset X$ such that $x_n \to \psi$ in the w^*-topology of X^{**} (resp., there exists a separable subspace $Y \subset X$ such that $\psi \in \overline{Y}^{w^*}$). Then: (i) if $\mathrm{Dens}(X) \geqslant \aleph_1$, the property $X^{**} = X_{\aleph_0}$ (resp., $X^{**} = \mathcal{S}eq(X^{**})$) is $\aleph_1$-determined, i.e., X has this property iff Y has, for every subspace $Y \subset X$ with $\mathrm{Dens}(Y) = \aleph_1$; (ii) if $X^{**} = X_{\aleph_0}$, $(B(X^{**}), w^*)$ has countable tightness; (iii) under the Martin's axiom $MA(\omega_1)$ we have $X^{**} = \mathcal{S}eq(X^{**})$ iff $(B(X^*), w^*)$ has countable tightness and $\overline{\mathrm{co}}(B) = \overline{\mathrm{co}}^{w^*}(K)$ for every subspace $Y \subset X$, every w^*-compact subset K of Y^*, and every boundary $B \subset K$.

Keywords
Boundaries;
Martin's axiom;
equality $Seq(X^{**}) = X^{**}$;
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Dedicated to Professor Henryk Hudzik on the occasion of his 70th birthday.

1. Introduction and notation

If $(X, \|\cdot\|)$ is a Banach space, let $B(X)$ and $S(X)$ be the closed unit ball and unit sphere of X, respectively, and X^* its topological dual. By w we will denote the weak-topology of X and by w^* the weak*-topology of X^*. Let $\mathcal{S}eq(X^{**})$ be the subset of elements $\psi \in X^{**}$ such that there exists a sequence $(x_n)_{n\geqslant 1} \subset X$ such that $x_n \to \psi$ in the w^*-topology of X^{**}.

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The subspace $\mathcal{S}eq(X^{**})$ depends upon X but this will cause no confusion because the space X will always be fixed previously and clearly. $\mathcal{S}eq(X^{**})$ is a closed subspace of X^{**} (see [10]). Let $X_{\aleph_0}$ be the subspace of X^{**} defined by $X_{\aleph_0} := \cup\left\{\overline{D}^{w^*} : D \subset X \text{ countable}\right\}$. $X_{\aleph_0}$ is norm-closed. We study the properties "$X^{**} = X_{\aleph_0}$" and "$X^{**} = \mathcal{S}eq(X^{**})$" in Section 3. We prove that both properties are $\aleph_1$-determined, i.e., a Banach space X satisfies $X^{**} = X_{\aleph_0}$ (resp., $X^{**} = \mathcal{S}eq(X^{**})$) iff $Y^{**} = Y_{\aleph_0}$ (resp., $Y^{**} = \mathcal{S}eq(Y^{**})$ for every subspace $Y \subset X$ with $\mathrm{Dens}(Y) = \aleph_1$.

The countable tightness of the unit ball $(B(X^*), w^*)$ is related to the property "$X^{**} = X_{\aleph_0}$". Actually, in Section 4 we prove that the property "$X^{**} = X_{\aleph_0}$" implies that the unit ball $(B(X^*), w^*)$ has countable tightness.

If K is a w^*-compact subset of a dual Banach space X^*, a subset $B \subset K$ is said to be a (*James*) *boundary* of K if every $x \in X$ attains on B its maximum on K. For instance, K itself and the set of extreme points $Ext(K)$ of K are boundaries of K. If B is a boundary of K, then $\overline{\text{co}}^{w^*}(B) = \overline{\text{co}}^{w^*}(K)$ and also $\overline{\text{co}}(B) = \overline{\text{co}}^{w^*}(K)$ in some cases. But, in general, $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$. When X is separable the following facts are equivalent (see [4, 5] and Proposition 2.3 below):

(i) $X^{**} = \mathcal{S}eq(X^{**})$.

(ii) $\overline{\text{co}}(B) = \overline{\text{co}}^{w^*}(K)$ for every w^*-compact subset K of the dual Banach space X^* and every boundary B of K.

So, it is natural to ask for the relation between the previous conditions (i) and (ii) in the non-separable case. In Section 5 we examine this relation under the Martin's axiom $MA(\omega_1)$ in the non-separable case.

Let us introduce some definitions. A closed convex subset M of X is said to have *property* (C) of Corson if for every family $\mathcal{A}$ of closed convex subsets of M with empty intersection there is a countable subfamily $\mathcal{B}$ of $\mathcal{A}$ with empty intersection.

A w^*-compact subset $K \subset X^*$ is said to be *angelic* if for every subset S of K and for every point s in the w^*-closure of S there exists a sequence in S that w^*-converges to s.

Denote by ω_0 and ω_1 the first infinite ordinal and the first uncountable ordinal, respectively. If A is a set, $|A|$ will denote the cardinality of A and $\mathfrak{c} = |\mathbb{R}|$.

If θ is an ordinal, a basic sequence $\{x_\alpha : \alpha < \theta\} \subset X$ is of type ℓ_1^+ if there exist $u \in X^*$ and $\epsilon_0 > 0$ such that $\langle u, x_\alpha \rangle > \epsilon_0 > 0$ for every $\alpha < \theta$.

We define the (P)-properties of the dual X^* as follows:

(i) X^* has the (P)-*property* iff every w^*-compact subset $H \subset X^*$ satisfies $\overline{\text{co}}^{w^*}(H) = \overline{\text{co}}(H)$. Actually, by a result of Haydon [8], X^* has the (P)-property iff X fails to have a copy of ℓ_1.

(ii) X^* has the *super*-(P) *property* iff every w^*-compact subset $H \subset X^*$ and every boundary B of H satisfy $\overline{\text{co}}^{w^*}(B) = \overline{\text{co}}(H)$.

(iii) X^* has the *ultra-(P) property* iff Y^* is super-(P) for every subspace $Y \subset X$. X^* has the $\aleph_1$*-super-(P) property* iff Y^* is super-(P) for every subspace $Y \subset X$ such that $\mathrm{Dens}(Y) = \aleph_1$.

We shall consider only Banach spaces $(X, \|\cdot\|)$ over the real field $\mathbb{R}$. If $x_0 \in X$ and $r \geqslant 0$, let $B(x_0; r) := \{x \in X : \|x - x_0\| \leqslant r\}$. If $A \subset X$, $x \in X$, then $[A]$ and $\overline{[A]}$ denote the linear hull and the closed linear hull of A, respectively, and $\mathrm{dist}(x, A) = \inf\{\|x - a\| : a \in A\}$ the distance from x to A. $\mathrm{co}(A)$ denotes the convex hull of A, $\overline{\mathrm{co}}(A)$ is the $\|\cdot\|$-closure of $\mathrm{co}(A)$ and, if $A \subset X^*$, $\overline{\mathrm{co}}^{w^*}(A)$ the w^*-closure of $\mathrm{co}(A)$. If $A \subset X$ and $B \subset X^*$ are subspaces, we say that A and B 1-norm each other if

$$\forall a \in A \ \ \|a\| = \sup\{\langle x^*, a\rangle \ : \ x^* \in B(B)\}$$

and

$$\forall b \in B \ \ \|b\| = \sup\{\langle b, x\rangle \ : \ x \in B(A)\}.$$

2. Preliminaries

Let us state and prove the following lemma that will be used in the sequel.

2.1. Lemma. *Let X be a Banach space with* $\mathrm{Dens}(X) \geqslant \aleph_1$ *and* $X^{**} \neq X_{\aleph_0}$. *Let* $u \in S(X^{**}) \setminus X_{\aleph_0}$. *Then* $\mathrm{dist}(u, X_{\aleph_0}) > \epsilon_0 > 0$ *for some* $\epsilon_0 > 0$ *and there exist*

(A) *two sequences of separable subspaces* $\{A_\alpha : \alpha < \omega_1\}$ *and* $\{B_\alpha : \alpha < \omega_1\}$ *of X and X^*, resp., such that:* (i) A_α *and* B_α *1-norm each other;* (ii) $A_\alpha \subset A_\beta$ *and* $B_\alpha \subset B_\beta$ *for* $1 \leqslant \alpha < \beta < \omega_1$;

(B) *a monotone basic sequence* $\{x^*_\alpha : \alpha < \omega_1\} \subset S(X^*)$ *such that* $\langle u, x^*_\alpha\rangle > \epsilon_0 > 0$ *(so it is of type* ℓ_1^+*),* $x^*_\alpha \perp A_\alpha$ *(i.e.* $\langle x^*_\alpha, x\rangle = 0 \ \forall x \in A$*) and* $x^*_\alpha \in B_{\alpha+1}$ *for every* $1 \leqslant \alpha < \omega_1$.

Moreover, if $\mathrm{Dens}(X) = \aleph_1$, *the construction can be carried out so that*

$$X = \bigcup_{\alpha<\omega_1} \overline{A_\alpha}, \quad X_{\aleph_0} = \bigcup_{\alpha<\omega_1} \overline{A_\alpha}^{w^*} \quad \text{and} \quad x^*_\alpha \overset{w^*}{\to} 0 \qquad \text{for } \alpha \to \omega_1.$$

Proof. As $\mathrm{Dens}(X) \geqslant \aleph_1$ then there exists a family $\{x_\alpha : \alpha < \omega_1\}$ in $B(X)$ such that: (i) $\mathrm{Dens}([\{x_\alpha : \alpha < \omega_1\}]) = \aleph_1$; (ii) if $\mathrm{Dens}(X) = \aleph_1$, then $\{x_\alpha : \alpha < \omega_1\}$ is $\|\cdot\|$-dense in $B(X)$. Since $X_{\aleph_0}$ is $\|\cdot\|$-closed and $u \notin X_{\aleph_0}$ clearly, $\mathrm{dist}(u, X_{\aleph_0}) > \epsilon_0 > 0$ for some $1 > \epsilon_0 > 0$. We proceed by transfinite induction.

Step 1. Make $A_1 = \{0\} = B_1$. Choose $x^*_1 \in S(X^*)$ such that $\langle u, x^*_1\rangle > \epsilon_0$, and two separable subspaces $A_2 \subset X$ and $B_2 \subset X^*$ such that they 1-norm each other and $x_1 \in A_2, x^*_1 \in B_2$. This is done as follows. Consider $B_{21} = [x^*_1]$ and choose a separable subspace $A_{21} \subset X$ such that A_{21} 1-norms B_{21} and $x_1 \in A_{21}$. Next let $B_{22} \subset X^*$ be a separable subspace that

1-norms A_{21} and $B_{21} \subset B_{22}$. In the next step, we choose a separable subspace $A_{22} \subset X$ that 1-norms B_{22} and $A_{21} \subset A_{22}$. And so on. We put $A_2 := \overline{\cup_{n\geqslant 1} A_{2n}}$ and $B_2 := \overline{\cup_{n\geqslant 1} B_{2n}}$. Obviously $x_1^* \perp A_1$. Step 1 ends here. Note that we have built A_i, B_i for $i \leqslant 2$, and x_j^* for $j \leqslant 1$ fulfilling the above requirements (A) and (B).

Step 2. As $\mathrm{dist}(u, \overline{A_2}^{w^*}) > \epsilon_0$ (because $\overline{A_2}^{w^*} \subset X_{\aleph_0}$), there exist $x_2^* \in S(X^*)$, $x_2^* \perp A_2$, such that $\langle u, x_2^* \rangle > \epsilon_0$. Now we have

$$\|\lambda_1 x_1^*\| \leqslant \|\lambda_1 x_1^* + \lambda_2 x_2^*\| \quad \forall\, \lambda_1, \lambda_2 \in \mathbb{R}$$

because A_2 1-norms the subspace $[x_1^*]$ and $x_2^* \perp A_2$.

Let $A_2 \subset A_3 \subset X$ and $B_2 \subset B_3 \subset X^*$ be separable subspaces such that $x_2 \in A_3, x_2^* \in B_3$ and A_3, B_3 1-norm each other. This ends Step 2. Note that we have built A_i and B_i for $i \leqslant 3$, and x_j^* for $j \leqslant 2$ fulfilling the above requirements (A) and (B).

Step $\alpha < \omega_1$. Assume steps β for all $\beta < \alpha$ constructed. We have subspaces $A_{\beta+1}$ and $B_{\beta+1}$ of X and X^*, resp., such that $x_\beta \in A_{\beta+1}$ and $x_\beta^* \in S(B_{\beta+1})$, $\beta < \alpha$, verifying the requirements (A) and (B). We put

$$A_\alpha := \bigcup_{\beta<\alpha} A_{\beta+1} \quad \text{and} \quad B_\alpha := \bigcup_{\beta<\alpha} B_{\beta+1}.$$

Clearly, A_α and B_α are separable subspaces of X and X^*, resp., that 1-norm each other and $x_\beta \in A_\alpha$ for every $\beta < \alpha$.

As $\overline{A_\alpha}^{w^*} \subset X_{\aleph_0}$ we have $\mathrm{dist}(u, \overline{A_\alpha}^{w^*}) > \epsilon_0$ and so there exist $x_\alpha^* \in S(X^*) \cap A_\alpha^\perp$ such that $\langle u, x_\alpha^* \rangle > \epsilon_0$. Note that for $x^* \in \overline{[\{x_\beta^* : \beta < \alpha\}]}$ and $\lambda \in \mathbb{R}$, we have $\|x^*\| \leqslant \|x^* + \lambda x_\alpha^*\|$ because A_α 1-norms the subspace $\overline{[\{x_\beta^* : \beta < \alpha\}]} \subset \overline{B_\alpha}$ and $x_\alpha^* \perp A_\alpha$.

Let $A_\alpha \subset A_{\alpha+1} \subset X$ and $B_\alpha \subset B_{\alpha+1} \subset X^*$ be separable subspaces such that $x_\alpha \in A_{\alpha+1}$, $x_\alpha^* \in B_{\alpha+1}$, and $A_{\alpha+1}, B_{\alpha+1}$ 1-norm each other.

Transfinite induction ensures that all steps can be constructed for $\alpha < \omega_1$.

Finally, if $\mathrm{Dens}(X) = \aleph_1$, as $\{x_\alpha : \alpha < \omega_1\}$ is $\|\cdot\|$-dense in $B(X)$, clearly

$$X = \bigcup_{\alpha<\omega_1} \overline{A_\alpha}, X_{\aleph_0} = \bigcup_{\alpha<\omega_1} \overline{A_\alpha}^{w^*} \quad \text{and} \quad x_\alpha^* \overset{w^*}{\to} 0 \qquad \text{for } \alpha \to \omega_1.$$

□

2.2. Proposition. *Let X be a Banach space and consider the following statements:*

(i) $(B(X^{**}), w^*)$ *is angelic.*

(ii) X^* *has the property (C) of Corson.*

(iii) X^* *fails to have an uncountable basic sequence of type ℓ_1^+.*

(iv) $X^{**} = Seq(X^{**})$.

(v) X^* *is super-(P).*

(vi) *X has the property (C) and fails to have a copy of ℓ_1.*

Then always (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi).

Proof. (i) $\Rightarrow$ (ii) follows from the Pol characterization of property (C) (see [12, Theorem 3.4])

(ii) $\Rightarrow$ (iii). Suppose that in X^* there exists an uncountable basic sequence $\mathcal{B} := \{u_\alpha : \alpha < \omega_1\}$ of type ℓ_1^+. We define $C_\beta := \overline{\text{co}}(\{u_\alpha : \beta \leqslant \alpha < \omega_1\})$ $\forall \beta < \omega_1$.

Claim 1. The family of convex closed sets $\{C_\beta : \beta < \omega_1\}$ has the countable intersection property.

Indeed, given a countable subset $\mathcal{F} \subset [1, \omega_1)$, if $\alpha = \sup \mathcal{F}$, then $\alpha < \omega_1$ and $\varnothing \neq C_\alpha \subset \cap_{\beta \in \mathcal{F}} C_\beta$.

Claim 2. $\cap_{\beta < \omega_1} C_\beta = \varnothing$.

Indeed, since $\{u_\alpha : \alpha < \omega_1\}$ is a basic sequence, then

$$\cap_{\beta<\omega_1} \overline{[\{u_\alpha : \beta \leqslant \alpha < \omega_1\}]} = \{0\}.$$

As $C_\beta \subset \overline{[\{u_\alpha : \beta \leqslant \alpha < \omega_1\}]}$, we have $\cap_{\beta<\omega_1} C_\beta \subset \{0\}$. On the other hand, since $\{u_\alpha : \alpha < \omega_1\}$ is of type ℓ_1^+, there exist $z \in X^{**}$ and $\epsilon_0 > 0$ such that $\langle z, u_\beta \rangle \geqslant \epsilon_0$ $\forall \beta < \omega_1$. Thus $\langle z, w \rangle \geqslant \epsilon_0$ $\forall w \in C_\beta$, and so $0 \notin C_\beta$ $\forall \beta < \omega_1$. Therefore $\cap_{\beta<\omega_1} C_\beta = \varnothing$.

Taking into account (ii), Claim 1, and Claim 2 we get a contradiction that proves the implication (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv). Let $z \in X^{**}$. By Lemma 2.1 there exist a closed separable subspace $Y \subset X$ such that $z \in \overline{Y}^{w^*}$. On the other hand, X fails to have a copy of ℓ_1 (otherwise we would find in X^* a copy of $\ell_1(\mathfrak{c})$, which contradicts (iii)). So Y fails to have a copy of ℓ_1. Identifying Y^{**} with $\overline{Y}^{w^*}$, by Odell–Rosenthal Theorem [11] we obtain that there is a sequence $\{y_n : n \geqslant 1\} \subset Y$ such that $y_n \to z$ in (X^{**}, w^*). Hence $z \in Seq(X^{**})$.

(iv) $\Rightarrow$ (v). See, for instance, Cor. 2.10 of [6].

(v) $\Rightarrow$ (vi). First, "X^* is super-(P)" implies "X^* is (P)" and this fact implies, by a result of Haydon [8], that X fails to have a copy of ℓ_1. Let's see that $X \in (C)$. Assume that X does not have the property (C). By the characterization of property (C) (see [12]) there exist a convex subset $A \subset B(X^*)$ such that $0 \in \overline{A}^{w^*}$, but $0 \notin \overline{\text{co}}^{w^*}(D)$, for all countable subsets $D \subset A$. Let

$$B_0 := \bigcup \left\{ \overline{\text{co}}^{w^*}(D) \,:\, D \subset A \text{ countable} \right\}.$$

Obviously, $0 \notin B_0$. Moreover, it is clear that B_0 is a convex $\|\cdot\|$-closed boundary of $\overline{A}^{w^*}$. Thus X^* is not super-(P), a contradiction which proves that X is (C). $\square$

Recall (see [3]) that a Hausdorff compact space K is said to be a Rosenthal compact when K is homeomorphic to a compact subset of the space $(\mathcal{B}_1(S), \tau_p)$ where (i) S is a Polish space and $\mathcal{B}_1(S)$ is the space of 1-Baire functions $f: S \to \mathbb{R}$; (ii) τ_p is the pointwise convergence topology on S. Godefroy proved in [3, Theorem 13] that $C(K) \in (C)$ whenever K is a Rosenthal compact.

2.3. Proposition. *For every separable Banach space X the following are equivalent:*

(i) $(B(X^{**}), w^*)$ *is angelic.*

(i') $(B(X^{**}), w^*)$ *is a Rosenthal compact.*

(ii) X^* *has the property (C) of Corson.*

(ii') $C(B(X^{**}), w^*)$ *has the property (C) of Corson.*

(iii) X^* *fails to have an uncountable basic sequence of type ℓ_1^+.*

(iv) $X^{**} = Seq(X^{**})$.

(v) X^* *is super-(P).*

(vi) X *fails to have a copy of ℓ_1.*

(vii) X^* *fails to have a copy of $\ell_1(\mathfrak{c})$.*

Proof. (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (vi) follow from Proposition 2.2.

(i') ⇒ (i) follows from a result of Bourgain, Fremlin and Talagrand [1, 3F. Theorem].

(i') ⇒ (ii') follows from [3, Theoreme 13].

(ii') ⇒ (ii) is obvious.

(vi) ⇒ (i'). X being separable, $(B(X^*), w^*)$ is a metrizable compact topological space, i.e., a Polish space. Since $X^{**} = \mathcal{S}eq(X^{**})$ (by Odell–Rosenthal results [11]), all elements of $B(X^{**})$ are 1-Baire on $(B(X^*), w^*)$. So $(B(X^{**}), w^*)$ is a Rosenthal compact since it is a compact subspace of the space of 1-Baire functions $\mathcal{B}_1(B(X^*), w^*)$ endowed with the pointwise convergence topology.

(iii) ⇒ (vii) is obvious.

(vii) ⇒ (vi). Assume that there exists an isomorphic embedding $T: \ell_1 \to X$. Then the conjugate operator $T^*: \ell_\infty \to X^*$ is a quotient mapping. It is well known that $\ell_1(\mathfrak{c}) \subset \ell_\infty$. Let $(e_i)_{i<\mathfrak{c}}$ be the canonical basis of $\ell_1(\mathfrak{c})$ and $(u_i)_{i<\mathfrak{c}} \subset X^*$ a bounded sequence such that $T^*(u_i) = e_i$. Then $(u_i)_{i<\mathfrak{c}}$ is a basic sequence equivalent to $(e_i)_{i<\mathfrak{c}}$ and so X^* has a copy of $\ell_1(\mathfrak{c})$, but this fact contradicts (vii). □

3. The properties "$X^{**} = X_{\aleph_0}$" and "$X^{**} = \mathcal{S}eq(X^{**})$" are $\aleph_1$-determined

First, two auxiliary lemmas.

3.1. Lemma (Odell–Rosenthal [11]). *Let X be a Banach space. The following are equivalent:*

(i) *X does not have a copy of ℓ_1;*

(ii) *$\mathcal{S}eq(X^{**}) = X_{\aleph_0}$.*

Proof. (i) ⇒ (ii). First, always $\mathcal{S}eq(X^{**}) \subset X_{\aleph_0}$. Let $z \in X_{\aleph_0}$. Then there exists a separable closed subspace $Y \subset X$ such that $z \in \overline{Y}^{w^*}$. As Y is separable and fails to have a copy of ℓ_1, by a result of Odell–Rosenthal [11] there exists a sequence $\{y_n : n \geqslant 1\} \subset Y$ such that $y_n \overset{w^*}{\to} z$. Thus $z \in \mathcal{S}eq(X^{**})$ and so $X_{\aleph_0} = \mathcal{S}eq(X^{**})$.

(ii) ⇒ (i). First observe that $(\ell_1)_{\aleph_0} = \ell_1^{**} = \ell_\infty^*$ (trivial) and also $\mathcal{S}eq(\ell_1^{**}) = \ell_1$ because ℓ_1 is weakly sequentially complete. Suppose that X contains a subspace Y isomorphic to ℓ_1. So there exists $u \in Y_{\aleph_0} \setminus \mathcal{S}eq(Y^{**}) = Y^{**} \setminus \mathcal{S}eq(Y^{**})$. If we consider Y^{**} as a subspace of X^{**} (in fact, $Y^{**} = \overline{Y}^{w^*} \subset X^{**}$), then $u \in X_{\aleph_0}$ (since $u \in Y_{\aleph_0} \subset X_{\aleph_0}$) but $u \notin \mathcal{S}eq(X^{**})$ because, if $u \in \mathcal{S}eq(X^{**})$, by [11, SubLemma, p. 378], we get $u \in \mathcal{S}eq(Y^{**})$, a contradiction. Thus X fails to have a copy of ℓ_1. □

3.2. Lemma. *Let X be a Banach space, $A \subset X$ a countable subset, $C \subset X$ a closed subspace, and $u \in X^{**}$ such that $u \in \overline{A}^{w^*} \cap \overline{C}^{w^*}$. Then there exists a separable subspace $D \subset C$ such that $u \in \overline{D}^{w^*}$.*

Proof. Let $co_{\mathbb{Q}}(A)$ denote the family of finite convex combinations of elements of A with rational coefficients. Clearly, $|co_{\mathbb{Q}}(A)| \leqslant \aleph_0$. For each $a \in co_{\mathbb{Q}}(A)$ we choose $c_a \in C$ such that $\|c_a - a\| \leqslant 2\,\mathrm{dist}(a, C)$. It is enough to prove the following

Claim. $u \in \overline{\{c_a : a \in co_{\mathbb{Q}}(A)\}}^{w^*}$.

Indeed, if $\epsilon > 0$ and $x_1^*, \ldots, x_p^* \in S(X^*)$, we consider the following convex w^*-neighborhood of u

$$W(u; x_1^*, \ldots, x_p^*; \epsilon) := \left\{z \in X^{**} \;:\; |\langle z - u, x_i^*\rangle| \leqslant \epsilon \;:\; i = 1, \ldots, p\right\}.$$

Let $A_0 := A \cap W(u; x_1^*, \ldots, x_p^*; \epsilon/2)$. Clearly, $co(A_0) \subset W(u; x_1^*, \ldots, x_p^*; \epsilon/2)$.

SubClaim. $\inf\{\|c - d\| : c \in C, d \in \overline{\mathrm{co}}(A_0)\} = \inf\{\|c - d\| : c \in C, d \in co_{\mathbb{Q}}(A_0)\} = 0$.

First, clearly $\inf\{\|c - d\| : c \in C, d \in \overline{\mathrm{co}}(A_0)\} = \inf\{\|c - d\| : c \in C, d \in co_{\mathbb{Q}}(A_0)\}$ because $\overline{\mathrm{co}}(A_0) = \overline{\mathrm{co}}_{\mathbb{Q}}(A_0)$. Suppose that $\inf\{\|c - d\| : c \in C, d \in \overline{\mathrm{co}}(A_0)\} > 0$. Then by the Hahn–Banach separation theorem there exists $x^* \in X^*$ fulfilling

$$\inf\langle x^*, C\rangle > \sup\big\langle x^*, co(A_0)\big\rangle. \tag{1}$$

Thus $\langle u, x^*\rangle \geqslant \inf\langle x^*, C\rangle$ (because $u \in \overline{C}^{w^*}$) and also $\sup\langle x^*, co(A_0)\rangle \geqslant \langle u, x^*\rangle$ since $u \in \overline{A_0}^{w^*} \subset \overline{\text{co}}^{w^*}(A_0)$. By (1), we get $\langle u, x^*\rangle > \langle u, x^*\rangle$, a contradiction which proves the SubClaim.

Therefore, there exists $a \in co_{\mathbb{Q}}(A_0)$ such that $\|c_a - a\| \leqslant \epsilon/2$. Hence for $i = 1, \dots, p$ we have

$$\left|\langle c_a - u, x_i^*\rangle\right| \leqslant \left|\langle c_a - a, x_i^*\rangle\right| + \left|\langle a - u, x_i^*\rangle\right| \leqslant \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

In consequence, $c_a \in W(u; x_1^*, \dots, x_p^*; \epsilon)$. Since $W(u; x_1^*, \dots, x_p^*; \epsilon)$ is arbitrary, this fact proves Claim and Lemma. □

3.3. Proposition. *Let X be a Banach space. Then:*

(i) *The property $X^{**} = X_{\aleph_0}$ is $\aleph_1$-determined, that is, $X^{**} = X_{\aleph_0}$ iff every subspace $Y \subset X$ with* $\text{Dens}(Y) = \aleph_1$ *satisfies $Y^{**} = Y_{\aleph_0}$.*

(ii) *If* $\text{Dens}(X) \geqslant \aleph_1$, *the following are equivalent:*
   (ii.1) $X^{**} = \mathcal{S}eq(X^{**})$;
   (ii.2) *every subspace $Y \subset X$ satisfies $Y^{**} = \mathcal{S}eq(Y^{**})$;*
   (ii.3) *every subspace $Y \subset X$ with* $\text{Dens}(Y) = \aleph_1$ *satisfies $Y^{**} = \mathcal{S}eq(Y^{**})$.*

Proof. (i). Assume that $X^{**} = X_{\aleph_0}$ and let $Y \subset X$ be a closed subspace of X. Identifying Y^{**} with $\overline{Y}^{w^*} \subset X^{**}$ and applying Lemma 3.2, we get $Y^{**} = Y_{\aleph_0}$.

Now suppose that $X^{**} \neq X_{\aleph_0}$. In the sequel we construct a closed subspace $Y \subset X$ with $\text{Dens}(Y) = \aleph_1$ such that $Y^{**} \neq Y_{\aleph_0}$. We repeat the construction of Lemma 2.1 and, with the notation of this Lemma, we put $Y = \cup_{\alpha<\omega_1}\overline{A_\alpha}$, which is a closed subspace of X such that $\text{Dens}(Y) = \aleph_1$ and $Y_{\aleph_0} = \cup_{\alpha<\omega_1}\overline{A_\alpha}^{w^*}$. Now we construct a sequence $\{w_\alpha : 1 \leqslant \alpha < \omega_1\} \subset B(X^{**})$ such that $w_\alpha \in B(\overline{A_\alpha}^{w^*})$ and $\langle w_\alpha, x_\beta^*\rangle = \langle u, x_\beta^*\rangle > \epsilon_0 > 0$ for $\beta < \alpha < \omega_1$. Put $w_1 = 0$, and for $2 \leqslant \alpha < \omega_1$ consider the operators

$$A_\alpha \overset{i}{\hookrightarrow} X$$
$$A_\alpha^* \overset{i^*}{\twoheadleftarrow} X^* \overset{j}{\hookleftarrow} B_\alpha$$
$$A_\alpha^{**} = \overline{A_\alpha}^{w^*} \overset{i^{**}}{\hookrightarrow} X^{**} \overset{j^*}{\twoheadrightarrow} B_\alpha^*,$$

where "$\hookrightarrow$" means isometric inclusion and "$\twoheadleftarrow$" canonical 1-quotient. The operator $i^* \circ j =: \psi : B_\alpha \hookrightarrow A_\alpha^*$ is an isometric inclusion because A_α 1-norms B_α. Whence $j^* \circ i^{**} = \psi^* : \overline{A_\alpha}^{w^*} \twoheadrightarrow B_\alpha^*$ is a $w^* - w^*$-continuous canonical 1-quotient. Thus

$$\psi^*\left(B(\overline{A_\alpha}^{w^*})\right) = B(B_\alpha^*).$$

Since $j^*(u) \in B(B_\alpha^*)$, there exists $w_\alpha \in B(\overline{A_\alpha}^{w^*})$ such that $\psi^*(w_\alpha) = j^*(u)$. Hence for all $\beta < \alpha$ we have

$$\begin{aligned}\langle w_\alpha, x_\beta^*\rangle &= \left\langle i^{**}(w_\alpha), j(x_\beta^*)\right\rangle = \left\langle j^* \circ i^{**}(w_\alpha), x_\beta^*\right\rangle = \left\langle \psi^*(w_\alpha), x_\beta^*\right\rangle \\ &= \left\langle j^*(u), x_\beta^*\right\rangle = \langle u, x_\beta^*\rangle > \epsilon_0.\end{aligned}$$

Choose $w_0 \in \bigcap_{\beta<\omega_1} \overline{\{w_\gamma : \beta \leqslant \gamma < \omega_1\}}^{w^*}$ arbitrarily. Then

(i) $w_0 \in \overline{B(Y)}^{w^*} = B(Y^{**})$;

(ii) $w_0 \notin Y_{\aleph_0}$, because $w_0 \notin \overline{A_\alpha}^{w^*}$ $\forall \alpha < \omega_1$, since $\langle w_0, x_\alpha^*\rangle \geqslant \epsilon_0$ but $x_\alpha^* \perp A_\alpha$ $\forall \alpha < \omega_1$.

Therefore $Y^{**} \neq Y_{\aleph_0}$.

(ii). (ii.1) $\Rightarrow$ (ii.2) follows from [11, SubLemma, p. 378]. (ii.2) $\Rightarrow$ (ii.3) is obvious.

(ii.3) $\Rightarrow$ (ii.1). First, X does not have a copy of ℓ_1. Indeed, otherwise there exists a subspace $Y \subset X$ with $\mathrm{Dens}(Y) = \aleph_1$ containing a copy of ℓ_1. So, by Lemma 3.1, $\mathcal{S}eq(Y^{**}) \neq Y_{\aleph_0} \subset Y^{**}$, which contradicts (ii.3). Thus, every subspace $Y \subset X$ fulfills $Y_{\aleph_0} = \mathcal{S}eq(Y^{**})$ by Lemma 3.1. Moreover by (B_3) every subspace $Y \subset X$ with $\mathrm{Dens}(Y) = \aleph_1$ satisfies $Y^{**} = Y_{\aleph_0}$. By (i) we get $X^{**} = X_{\aleph_0}$ and finally $X^{**} = \mathcal{S}eq(X^{**})$, since $X_{\aleph_0} = \mathcal{S}eq(X^{**})$ by Lemma 3.1. □

4. The property "$X^{**} = X_{\aleph_0}$" implies "$(B(X^*), w^*) \in (CT)$"

A topological space (X, τ) has *countable tightness* (in short, $X \in (CT)$) iff for every subset $A \subset X$ and every $u \in \overline{A}$ there exists a countable subset $A_0 \subset A$ such that $u \in \overline{A_0}$. If (X, τ) is a topological vector space and $C \subset X$ is a convex subset of X, we say that C has *convex countable tightness* (in short, $C \in (CCT)$) if for every subset $A \subset C$ and every $u \in \overline{A}$ there exists a countable subset $D \subset A$ such that $u \in \overline{\mathrm{co}}(D)$. Our aim in this Section is to prove that if X is a Banach space the fact "$X^{**} = X_{\aleph_0}$ " implies "$(B(X^*), w^*)$ is (CT)".

4.1. Lemma. *Let X be a Banach space and $A \subset B(X^*)$ such that $0 \in \overline{A}^{w^*}$ but $0 \notin \overline{D}^{w^*}$ for every countable subset $D \subset A$. Then*

(i) *$0 \notin \overline{A}^{w}$, where w is the weak topology of X^*.*

(ii) *If $F(A) = \bigcup\{\overline{D}^{w^*} : D \subset A, |D| \leqslant \aleph_0\}$, $F(A)$ is w-closed and $0 \notin F(A)$.*

(iii) *There exist $\eta > 0$ and $v \in X^{**}$ such that $|\{a \in A : \langle v, a\rangle > \eta\}| \geqslant \aleph_1$.*

Proof. (i). First, $0 \notin \overline{A}^{w}$, because every Banach space has countable tightness for the weak topology (see [14, p. 229] for instance) and so the fact $0 \in \overline{A}^{w}$ would imply that there exists a countable subset $D \subset A$ with $0 \in \overline{D}^{w}$. Since $\overline{D}^{w} \subset \overline{D}^{w^*}$, we would get that $0 \in \overline{D}^{w^*}$, which is not true.

(ii). Let $u \in \overline{F(A)}^{w}$. Since every Banach space has countable tightness for the weak topology, there exists $D \subset A$ with $|D| \leqslant \aleph_0$ such that $u \in \overline{D}^{w} \subset \overline{D}^{w^*}$, that is, $u \in F(A)$. Obviously, $0 \notin F(A)$, by hypothesis.

(iii). Since $0 \notin \overline{A}^{w}$, there exist $\eta > 0$ and vectors $v_1, \dots, v_n$ in X^{**} such that $A \subset \bigcup_{i=1}^{n} \{x^* \in X^* : \langle v_i, x^* \rangle > \eta\}$. A is uncountable by hypothesis, and so for some $j \in \{1, \dots, n\}$ we have necessarily that $|\{a \in A : \langle v_j, a\rangle > \eta\}| \geqslant \aleph_1$. Now pick $v := v_j$. □

4.2. Lemma. *Let X be a Banach space, $Y \subset X$ a separable subspace, $A \subset B(X^*)$ with $0 \in \overline{A}^{w^*}$ and $0 \notin F(A) := \bigcup\{\overline{D}^{w^*} : D \subset A, |D| \leqslant \aleph_0\}$, and $A_0 \subset A$ such that $|A_0| \leqslant \aleph_0$. Then there exist $D \subset A \setminus A_0$ with $|D| \leqslant \aleph_0$ and $z \in \overline{D}^{w^*}$ such that $z \upharpoonright Y = 0$.*

Proof. Let $\{y_n : n \geqslant 1\}$ be a dense family in $B(Y)$ and denote

$$F(A \setminus A_0) := \bigcup \{\overline{D}^{w^*} : D \subset A \setminus A_0, |D| \leqslant \aleph_0\}.$$

Consider the w^*-open neighborhoods of 0

$$V_n := \left\{x^* \in X^* : |\langle x^*, y_i\rangle| < \tfrac{1}{n},\ i = 1, \dots, n\right\} \quad \forall\ n \geqslant 1.$$

As $0 \in \overline{A \setminus A_0}^{w^*} \subset \overline{F(A \setminus A_0)}^{w^*}$, then $V_n \cap F(A \setminus A_0) \neq \varnothing$ $\forall n \geqslant 1$. Choose $z_n \in V_n \cap F(A \setminus A_0)$ $\forall n \geqslant 1$. Clearly $\langle z_n, y_i\rangle \underset{n\to\infty}{\to} 0$ for $i \geqslant 1$, whence we get $\langle z_n, y\rangle \underset{n\to\infty}{\to} 0$ for every $y \in Y$. Let D_n be a countable subset of $A \setminus A_0$ such that $z_n \in \overline{D_n}^{w^*}$ $\forall n \geqslant 1$, and $D := \bigcup_{n \geqslant 1} D_n$. It is clear that $|D| \leqslant \aleph_0$ and $z_n \in \overline{D}^{w^*}$ $\forall n \geqslant 1$. Let z be a w^*-limit point of $\{z_n : n \geqslant 1\}$. Obviously, $z \in \overline{D}^{w^*}$ and $z \upharpoonright Y = 0$. □

4.3. Lemma. *Let X be a Banach space with* $\mathrm{Dens}(X) = \aleph_1$. *The following are equivalent:*

(i) $(B(X^*), w^*) \notin (CT)$.

(ii) *There exists in $B(X^*)$ a sequence $\{y^*_\alpha : \alpha < \omega_1\}$ such that*

  (ii.1) *$y^*_\alpha \to 0$ in the w^*-topology of X^* when $\alpha \to \omega_1$;*

  (ii.2) *for every $\beta < \omega_1$ we have $0 \notin \overline{\{y^*_\alpha : \alpha < \beta\}}^{w^*}$;*

  (ii.3) *there exist $\eta > 0$ and $v \in X^{**}$ such that $|\{\alpha < \omega_1 : \langle v, y^*_\alpha\rangle > \eta\}| = \aleph_1$.*

Proof. (ii) ⇒ (i) is immediate by (ii.1) and (ii.2).

(i) ⇒ (ii). As $(B(X^*), w^*)$ is not (CT), there exists $A \subset B(X^*)$ such that $0 \in \overline{A}^{w^*}$ but $0 \notin F(A) := \bigcup\{\overline{D}^{w^*} : D \subset A,\ |D| \leqslant \aleph_0\}$. Let $\{x_\alpha : \alpha < \omega_1\} \subset B(X)$ be a $\|\cdot\|$-dense family in $B(X)$. In the sequel we construct sequences $\{Y_\alpha : \alpha < \omega_1\}$, $\{D_\alpha : \alpha < \omega_1\}$ and $\{y^*_\alpha : \alpha < \omega_1\} \subset B(X^*)$ such that

(1) Y_α is a separable closed subspace of X and $x_\alpha \in Y_\alpha \subset Y_\beta$ for $\alpha \leqslant \beta < \omega_1$;

(2) $\{D_\alpha : \alpha < \omega_1\}$ is a family of countable pairwise disjoint subsets of A;

(3) $y^*_\alpha \in \overline{D_\alpha}^{w^*}$ and $y^*_\alpha \perp Y_\alpha$ for $\alpha < \omega_1$.

We use transfinite induction.

Step 1. Let $Y_1 = [x_1]$. By Lemma 4.2, there exist $D_1 \subset A$ with $|D_1| \leqslant \aleph_0$ and $y^*_1 \in \overline{D_1}^{w^*}$ such that $y^*_1 \perp Y_1$.

Step 2. Let $Y_2 = \overline{[Y_1 \cup \{x_2\}]}$, which is separable. By Lemma 4.2 there exist $D_2 \subset A \setminus D_1$ with $|D_2| \leqslant \aleph_0$ and $y^*_2 \in \overline{D_2}^{w^*}$ such that $y^*_2 \perp Y_2$.

Step $\alpha < \omega_1$. Assume that we have constructed the elements Y_β, D_β and y^*_β for $\beta < \alpha$ fulfilling (1), (2) and (3). Let $Y_\alpha = \overline{[\{x_\alpha\} \cup (\bigcup_{\beta<\alpha} Y_\beta)]}$, which is a separable subspace of X. By Lemma 4.2 there exist $D_\alpha \subset A \setminus \bigcup_{\beta<\alpha} D_\beta$ with $|D_\alpha| \leqslant \aleph_0$ and $y^*_\alpha \in \overline{D_\alpha}^{w^*}$ such that $y^*_\alpha \perp Y_\alpha$.

We can carry out the construction for every $\alpha < \omega_1$.

By construction, it is clear that $X = \bigcup_{\alpha<\omega_1} Y_\alpha$, $y^*_\alpha \to 0$ in the w^*-topology of X^* as $\alpha \to \omega_1$ and

$$0 \notin \bigcup_{\beta<\omega_1} \overline{\{y^*_\alpha : \alpha < \beta\}}^{w^*}.$$

Finally, by Lemma 4.1, there exist $\eta > 0$ and $\nu \in B(X^{**})$ such that $|\{\alpha < \omega_1 : \langle \nu, y^*_\alpha\rangle > \eta\}| = \aleph_1$. □

We say that a Banach space X is $\aleph_1$-(CT) if every $Y \subset X$ with $\mathrm{Dens}(Y) \leqslant \aleph_1$ satisfies $(B(Y^*), w^*) \in (CT)$.

4.4. Proposition. *Let X be a Banach space. The following are equivalent:*

(i) $(B(X^*), w^*) \in (CT)$.

(ii) *X is $\aleph_1$-(CT).*

Hence the property $(B(X^*), w^*) \in (CT)$ is $\aleph_1$-determined.

Proof. (i) ⇒ (ii) is obvious because the property (CT) passes over to compact quotient spaces.

(ii) ⇒ (i). Suppose that $(B(X^*), w^*)$ is not (CT). Then $\mathrm{Dens}(X) > \aleph_1$ and without loss of generality we may assume that there exists $A \subset B(X^*)$ such that $0 \in \overline{A}^{w^*}$ but $0 \notin F(A) := \bigcup\{\overline{D}^{w^*} : D \subset A,\ |D| \leqslant \aleph_0\}$. From this fact we deduce a contradiction. Since $F(A)$ is a w-closed subset of X^* (see Lemma 4.1) and $0 \notin F(A)$, $\mathrm{dist}(0, F(A)) > \epsilon$ for some $\epsilon > 0$. Let $\{x_\alpha : \alpha < \omega_1\} \subset B(X)$ be such that $\mathrm{Dens}(\overline{[\{x_\alpha : \alpha < \omega_1\}]}) = \aleph_1$. Now we construct sequences $\{Y_\alpha : \alpha < \omega_1\}$, $\{D_\alpha : \alpha < \omega_1\}$ and $\{x^*_\alpha : \alpha < \omega_1\} \subset B(X^*)$ such that

(1) Y_α is a separable closed subspace of X and $x_\alpha \in Y_\alpha \subset Y_\beta$ for $\alpha \leqslant \beta < \omega_1$;

(2) $\{D_\alpha : \alpha < \omega_1\}$ is a family of countable pairwise disjoint subsets of A;

(3) $x^*_\alpha \in \overline{D_\alpha}^{w^*}$ and $x^*_\alpha \perp Y_\alpha$ for $\alpha < \omega_1$;

(4) if $i_\alpha \colon Y_\alpha \to X$ is the canonical inclusion, then

$$0 \notin \overline{i^*_\alpha(\cup_{\beta<\alpha} D_\beta)}^{w^*} = i^*_\alpha(\overline{\cup_{\beta<\alpha} D_\beta}^{w^*}) \quad \forall\ \alpha < \omega_1.$$

We use transfinite induction as in Lemma 4.3.

Step 1. Let $Y_1 = [x_1]$. By Lemma 4.2, there exist $D_1 \subset A$ with $|D_1| \leqslant \aleph_0$ and $x^*_1 \in \overline{D_1}^{w^*}$ such that $x^*_1 \perp Y_1$.

Step 2. Since $\mathrm{dist}(0, \overline{D_1}^{w^*}) > \epsilon$, there exists a finite family $F_2 \subset B(X)$ such that

$$\overline{D_1}^{w^*} \subset \left\{x^* \in X^* : \sup\langle x^*, F_2\rangle > \epsilon\right\}.$$

Let $Y_2 = \overline{[Y_1 \cup \{x_2\} \cup F_2]}$, which is separable. Clearly, $0 \notin \overline{D_1}^{w^*}$ and also $0 \notin i^*_2(\overline{D_1}^{w^*})$ because $F_2 \subset Y_2$. By Lemma 4.2 there exist $D_2 \subset A \setminus D_1$ with $|D_2| \leqslant \aleph_0$ and $x^*_2 \in \overline{D_2}^{w^*}$ such that $x^*_2 \perp Y_2$.

Step $\alpha < \omega_1$**.** Assume that we have constructed the elements Y_β, D_β and x^*_β for $\beta < \alpha$ fulfilling (1), (2), (3) and (4). Since $\mathrm{dist}(0, \overline{\cup_{\beta<\alpha} D_\beta}^{w^*}) > \epsilon$, there exists a finite family $F_\alpha \subset B(X)$ such that

$$\overline{\cup_{\beta<\alpha} D_\beta}^{w^*} \subset \left\{x^* \in X^* : \sup\langle x^*, F_\alpha\rangle > \epsilon\right\}.$$

Let $Y_\alpha = \overline{[\{x_\alpha\} \cup (\bigcup_{\beta<\alpha} Y_\beta) \cup F_\alpha]}$, which is a separable subspace of X. Note that $0 \notin \overline{i^*_\alpha(\cup_{\beta<\alpha} D_\beta)}^{w^*}$ because $F_\alpha \subset Y_\alpha$. By Lemma 4.2, there exist $D_\alpha \subset A \setminus \bigcup_{\beta<\alpha} D_\beta$ with $|D_\alpha| \leqslant \aleph_0$ and $x^*_\alpha \in \overline{D_\alpha}^{w^*}$ such that $x^*_\alpha \perp Y_\alpha$.

We can carry out the induction for every $\alpha < \omega_1$.

Let $Y := \cup_{\alpha<\omega_1} Y_\alpha$ (which is a closed subspace of X with $\mathrm{Dens}(Y) = \aleph_1$), $i \colon Y \to X$ the canonical inclusion and $y^*_\alpha = i^*(x^*_\alpha)$, $\alpha < \omega_1$. We have

(ii.1) $y^*_\alpha \xrightarrow{w^*} 0$ as $\alpha \to \omega_1$ in $(B(Y^*), w^*)$ because $y^*_\alpha \perp Y_\alpha$, $Y := \cup_{\alpha<\omega_1} Y_\alpha$, and $Y_\beta \subset Y_\alpha$ for $\beta < \alpha < \omega_1$;

(ii.2) $0 \notin \overline{\{y^*_\beta : \beta < \alpha\}}^{w^*}$ for every $\alpha < \omega_1$ because $0 \notin \overline{i^*_\alpha(\cup_{\beta<\alpha} D_\beta)}^{w^*}$;

(ii.3) by Lemma 4.1, (ii.1) and (ii.2), there exist $\eta > 0$ and $v \in Y^{**}$ such that $|\{\alpha < \omega_1 : \langle v, y^*_\alpha\rangle > \eta\}| = \aleph_1$.

By Lemma 4.3 we get $(B(Y^*), w^*) \notin (CT)$, which is the contradiction we are looking for. $\square$

4.5. Proposition. *Let X be a Banach space such that $X^{**} = X_{\aleph_0}$. Then $(B(X^*), w^*)$ is (CT).*

Proof. Suppose that $(B(X^*), w^*)$ is not (CT). We will deduce a contradiction. By Proposition 4.4 we may assume that $\mathrm{Dens}(X) = \aleph_1$. By Lemma 4.3, passing to a subsequence if necessary, there exist a sequence $\{x^*_\alpha : \alpha < \omega_1\}$ in $B(X^*)$, $u \in B(X^{**})$ and $\epsilon_0 > 0$ such that

(i.1) $x^*_\alpha \to 0$ in the w^*-topology of X^* as $\alpha \to \omega_1$;

(i.2) $\langle u, x^*_\alpha\rangle > \epsilon_0$ $\forall \alpha < \omega_1$.

Let $D := \{x_n : n \geqslant 1\} \subset X$ be a sequence such that $u \in \overline{D}^{w^*}$. Then

$$[1, \omega_1) = \bigcup_{n\geqslant 1}\{\alpha < \omega_1 \,:\, \langle x^*_\alpha,\, x_n\rangle > \epsilon_0\}.$$

In consequence, there exists $m \in \mathbb{N}$ such that $|\{\alpha < \omega_1 : \langle x^*_\alpha, x_m\rangle > \epsilon_0\}| = \aleph_1$, which contradicts (i.1). Thus $(B(X^*), w^*)$ is (CT). □

5. The equality $X^{**} = \mathcal{S}eq(X^{**})$ and Martin's axiom $MA(\omega_1)$

In this Section we see that under Martin's axiom $MA(\omega_1)$, for a Banach space X the property "$X^{**} = \mathcal{S}eq(X^{**})$" is equivalent to the property $(B(X^*), w^*) \in (CT)$ and the property "X^* is super-(P)" (or similar properties). We begin by introducing some notions (see [2]).

Let $(\mathcal{P}, \leqslant)$ be a partially ordered set (a "poset"). Two elements $p, q \in \mathcal{P}$ are said to be *compatible* if there exists $r \in \mathcal{P}$ such that $p \leqslant r$ and $q \leqslant r$. Otherwise, we say that p, q are *incompatible*. We say that $\mathcal{P}$ satisfies the CCC (countable chain condition) property if for every uncountable subset $\mathcal{P}_1$ of $\mathcal{P}$ there exist at least two compatible elements $p, q \in \mathcal{P}_1$. A subset $\mathcal{Q} \subset \mathcal{P}$ is said *cofinal* in $\mathcal{P}$ if for every $p \in \mathcal{P}$ there exists $q \in \mathcal{Q}$ such that $p \leqslant q$. A subset $\mathcal{R} \subset \mathcal{P}$ is said $\uparrow$*-directed* (*or up-directed*) if for every pair of elements $p, q \in \mathcal{R}$ there exists $r \in \mathcal{R}$ such that $p \leqslant r$ and $q \leqslant r$.

For each cardinal κ let $MA(\kappa)$ be the following statement:

> "If $(\mathcal{P}, \leqslant)$ is a CCC poset and $\mathcal{F}$ a family of cofinal subsets of $\mathcal{P}$ with $|\mathcal{F}| \leqslant \kappa$, there exists a $\uparrow$-directed subset $\mathcal{R} \subset \mathcal{P}$ such that $\mathcal{R}$ intersects every element of $\mathcal{F}$."

It is well known that $MA(\omega_0)$ is true but $MA(\mathfrak{c})$ is false (see [2]).

5.1. Definition. $\mathfrak{m}$ is the minimum cardinal such that $MA(\mathfrak{m})$ is false.

Of course $\omega_1 \leqslant \mathfrak{m} \leqslant \mathfrak{c}$.

5.2. Definition. Martin's axiom MA is the statement that $\mathfrak{m} = \mathfrak{c}$. In other words, if $(\mathcal{P}, \leqslant)$ is a CCC poset and $\mathcal{F}$ a family of cofinal subsets in $\mathcal{P}$ with $|\mathcal{F}| < \mathfrak{c}$, there exists a $\uparrow$-directed subset $\mathcal{R} \subset \mathcal{P}$ such that $\mathcal{R}$ intersects every element of $\mathcal{F}$.

Clearly, $(CH) \Rightarrow MA$ ((CH) = continuum hypothesis, that is, $\omega_1 = \mathfrak{c}$) and $MA + \neg(CH) \Leftrightarrow \omega_1 < \mathfrak{m} = \mathfrak{c}$.

5.3. Definition. Martin's axiom $MA(\omega_1)$ is the statement that $\omega_1 < \mathfrak{m}$. In other words, $MA(\omega_1)$ is the following statement: If $(\mathcal{P}, \leqslant)$ is a poset fulfilling CCC and $\{\mathcal{D}_\alpha : \alpha < \omega_1\}$ is a family of cofinal subsets in $\mathcal{P}$, there exists a $\uparrow$-directed subset $\mathcal{J}$ in $\mathcal{P}$ such that $\mathcal{J} \cap \mathcal{D}_\alpha \neq \varnothing \ \forall \alpha < \omega_1$.

The following Definition is a topological version of Martin's axiom $MA(\omega_1)$ (see [9, 3.4.Theorem]). Recall that a topological space X is CCC if every family of pairwise disjoint open subsets of X is countable.

5.4. Definition. Martin's axiom $MA(\omega_1)$ is the following statement: If K is a compact Hausdorff CCC space and $\{\mathcal{D}_\alpha : \alpha < \omega_1\}$ is a family of dense open subsets of K, then $\cap\{\mathcal{D}_\alpha : \alpha < \omega_1\}$ is dense in K.

5.5. Proposition ($MA(\omega_1)$). *Let X be a Banach space such that* $\mathrm{Dens}(X) = \aleph_1$. *The following are equivalent:*

(i) $X^{**} = \mathcal{S}eq(X^{**})$.

(ii) X^* *is super-*(P) *and* $(B(X^*), w^*) \in (CT)$.

Proof. The implication (i) $\Rightarrow$ (ii) always holds, by Proposition 2.2 and Proposition 4.5.

(ii) $\Rightarrow$ (i). We suppose that

$$X^* \text{ is super-}(P) \text{ and } (B(X^*), w^*) \in (CT), \text{ but } X^{**} \neq \mathcal{S}eq(X^{**}) \qquad (*)$$

and deduce a contradiction. Since X^* is super-(P), X fails to have a copy ℓ_1 (see Proposition 2.2). In consequence, $\mathcal{S}eq(X^{**}) = X_{\aleph_0}$ by Lemma 3.1 and $X_{\aleph_0} \neq X^{**}$ by $(*)$. Thus, we can apply Lemma 2.1. So there exist $\epsilon_0 > 0$, $u \in S(X^{**})$, a monotone basic sequence $\{x_\alpha^* : \alpha < \omega_1\} \subset S(X^*)$ and separable closed subspaces $\{A_\alpha : \alpha < \omega_1\}$ of X such that

(a) $A_\alpha \subset A_\beta$ if $\alpha < \beta < \omega_1$, and $X = \bigcup_{\alpha<\omega_1} A_\alpha$;

(b) $x_\alpha^* \perp A_\alpha \ \forall \alpha < \omega_1$, whence $x_\alpha^* \overset{w^*}{\to} 0$ as $\alpha \to \omega_1$;

(c) $\langle u, x_\alpha^* \rangle \geqslant \epsilon_0 \ \forall \alpha < \omega_1$.

Let $\tilde{T}: X \to \ell_\infty(\omega_1)$ be a continuous operator defined by $\tilde{T}(x) := (\langle x_\alpha^*, x\rangle)_{\alpha<\omega_1}$. By (b), $\tilde{T}(X) \subset \ell_\infty^c(\omega_1)$(=elements of $\ell_\infty(\omega_1)$ with countable support). Moreover,

$$\mathrm{supp}(\tilde{T}(X)) := \{\alpha \in \omega_1 : \exists x \in X \langle x_\alpha^*, \ x\rangle \neq 0\} = \omega_1.$$

Indeed, since $\|x_\alpha^*\| = 1 \ \forall \alpha < \omega_1$, there exists $x \in X$ such that $\langle x_\alpha^*, x\rangle \neq 0$. By the proof of [7, Th. 4.46] (here the axiom $MA(\omega_1)$ is necessary), there exists an uncountable subset

$\Gamma \subset \omega_1$ such that if we define $T(x) := (\langle x_\gamma^*, x\rangle)_{\gamma\in\Gamma}$ $\forall x \in X$, then $T(X)$ is a non-separable subspace of $c_0(\Gamma)$. Note that $\langle u, x_\gamma^*\rangle \geqslant \epsilon_0 > 0$ $\forall \gamma \in \Gamma$.

Claim. X^* is not super-(P).

Indeed, consider the operator $T\colon X \to c_0(\Gamma)$ such that $T(x) = (\langle x_\gamma^*, x\rangle)_{\gamma\in\Gamma}$. Let $B := \{e_\gamma : \gamma \in \Gamma\}$ be the canonical basis of $\ell_1(\Gamma) = c_0^*(\Gamma)$ and $K := \overline{B}^{w^*}$. Observe that K is the w^*-compact subset $K = \{e_\gamma : \gamma \in \Gamma\} \cup \{0\}$ of $(\ell_1(\Gamma), w^*)$ such that $0 \notin \overline{\mathrm{co}}(B)$. Moreover, B is a boundary of K. Let $H := T^*(K)$ and $B_0 := T^*(B)$. Clearly, B_0 is a boundary of the w^*-compact subset H and $B_0 = \{x_\gamma^* : \gamma \in \Gamma\}$. Since $\langle u, x_\gamma^*\rangle \geqslant \epsilon_0$ $\forall \gamma \in \Gamma$, we have $\langle u, x^*\rangle \geqslant \epsilon_0$ $\forall x^* \in \overline{\mathrm{co}}(B_0)$. Thus $0 \notin \overline{\mathrm{co}}(B_0)$ but $0 \in H$, and this implies that X^* is not super-(P).

Therefore we get a contradiction, which proves the implication (ii) ⇒ (i). □

5.6. Proposition $(MA(\omega_1))$. *For every Banach space X the following statements are equivalent:*

(i) $X^{**} = \mathcal{S}eq(X^{**})$.

(ii) X^* *is ultra-*(P) *and* $(B(X^*), w^*) \in (CT)$.

(iii) X^* *is* $\aleph_1$*-super-*(P) *and* $(B(X^*), w^*) \in (CT)$.

Proof. (i) ⇒ (ii) follows from Proposition 3.3, Proposition 2.2 and Proposition 4.5. (ii) ⇒ (iii) is obvious. Finally (iii) ⇒ (i) follows from Proposition 5.5 and the fact that the property $X^{**} = \mathcal{S}eq(X^{**})$ is $\aleph_1$-determined, by Proposition 3.3. □

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