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Summing multinorms defined by Orlicz spaces and symmetric sequence spaces

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Summary. We develop the notion of the (X_1, X_2)-summing power-norm based on a Banach space E, where X_1 and X_2 are symmetric sequence spaces. We study the particular case when X_1 and X_2 are Orlicz spaces ℓ_Φ and ℓ_Ψ respectively and analyze under which conditions the (Φ, Ψ)-summing power-norm becomes a multinorm. In the case when E is also a symmetric sequence space L, we compute the precise value of $\|(\delta_1, \cdots, \delta_n)\|_n^{(X_1,X_2)}$ where (δ_k) stands for the canonical basis of L, extending known results for the (p,q)-summing power-norm based on the space ℓ_r which corresponds to $X_1 = \ell_p$, $X_2 = \ell_q$, and $E = \ell_r$.

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1. Introduction

In the recent decade the use of multinormed spaces and their variations has been shown to be very fruitful for several purposes. The theory of multinorms was introduced by H. G. Dales and M. E. Polyakov in [7], and it is strongly connected to the theory of absolutely summing operators and tensor norms, among other things (see [3–5, 18]). Recently a new link to the theory of Banach lattices has also been found via the notion of p-multinormed spaces (see [6]), extending the results previously considered by G. Pisier and his student (see [7, Theorem 4.56] or [16]) in the case of multinorms. The concepts

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such as "multinorm", "dual multinorm" and "p-multinorm" are particular cases of a more general notion called either a "special norm" (see [7,18]) or a "power-norm" (see [2,6]). We shall use the latter terminology, and recall here the definition. A sequence $(\|\cdot\|_n : n \in \mathbb{N})$ of norms defined on E^n, where $(E, \|\cdot\|)$ is a normed space over the complex field $\mathbb{C}$, is called a *power-norm* based on E if $\|x\|_1 = \|x\|$ for each $x \in E$, and the axioms:

(P1) $\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\|_n = \|\boldsymbol{x}\|_n$;

(P2) $\|(x_1, \ldots, x_n, 0)\|_{n+1} = \|\boldsymbol{x}\|_n$;

(P3) $\|\boldsymbol{\alpha x}\|_n \leqslant (\max_{1\leqslant i\leqslant n}|\alpha_i|)\|\boldsymbol{x}\|_n$

are satisfied for each permutation σ of the set $\{1, \ldots, n\}$, $n \in \mathbb{N}$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, where $\boldsymbol{\alpha x} = (\alpha_1 x_1, \ldots, \alpha_n x_n)$.

A power-norm based on E is said to be a *multinorm* whenever the extra axiom:

(M) $\|(x_1, \ldots, x_{n-1}, x_n, x_n)\|_{n+1} = \|\boldsymbol{x}\|_n$

is satisfied for all $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$.

In the category of multinormed spaces, the *maximum* and the *minimum multinorm* based on a normed space E, denoted by $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$ and $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$, respectively, and defined by the property

$$\|\boldsymbol{x}\|_n^{\min} \leqslant \|\boldsymbol{x}\|_n \leqslant \|\boldsymbol{x}\|_n^{\max}, \quad \boldsymbol{x} \in E^n,\ n \in \mathbb{N}$$

for any multinorm $(\|\cdot\|_n : n \in \mathbb{N})$ based on E, were considered and studied in [4,7]. It is easy to see that

$$\|\boldsymbol{x}\|_n^{\min} = \max_{1\leqslant i\leqslant n} \|x_i\|$$

for any $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$, and a description of $\|\boldsymbol{x}\|_n^{\max}$ was found in [7, Theorem 3.33]).

We recall that for each multinorm $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ based on E, there exists a sequence measuring the "rate of growth" of the multinorm (see [7, Definition 3.1]), defined by

$$\varphi_{\|\cdot\|_n}(E) = \sup\Big\{\|\boldsymbol{x}\|_n \ :\ \|x_1\| = \cdots = \|x_n\| = 1\Big\};$$

in other words, $\varphi_{\|\cdot\|_n}(E)$ is the norm of the identity operator from $(E^n, \|\cdot\|_n^{min})$ into $(E^n, \|\cdot\|_n)$.

The two fundamental questions in the theory of multinormed spaces based on a concrete normed space E is the determination under which conditions $\|\boldsymbol{x}\|_n \approx \|\boldsymbol{x}\|_n^{\min}$ for all $\boldsymbol{x} \in E^n$ and $n \in \mathbb{N}$, and the calculation, or at least an estimation of $\varphi_{\|\cdot\|_n}(E)$ for particular multinorms based on E.

In this paper we address these questions for the sequence Orlicz space $E = \ell_\Phi$ and multinorms defined with $(1,\psi)$-summing operators, which are motivated by the corresponding results already known for ℓ_p-spaces and $(1,p)$-summing multinorms.

We recall two basic power-norms to be considered and generalized in the sequel: the so-called weak p-summing power-norm, denoted by $\mu_{p,n}(\cdot)$ in [5, 7, 12], and the (p,q)--summing power-norm, first introduced in [7] for $p \leqslant q$ and used later in [2] for $1 \leqslant p, q \leqslant \infty$, and denoted by $\|\cdot\|_n^{(p,q)}$.

Let E be a normed space and let $1 \leqslant p, q < \infty$. For each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$, we define the *weak p-summing power-norm* by

$$\mu_{p,n}(\boldsymbol{x}) = \sup\left\{\left(\sum_{i=1}^{n} |\langle x_i, \lambda\rangle|^p\right)^{1/p} : \lambda \in E^*, \ \|\lambda\| = 1\right\} \tag{1}$$

and the (p,q)*-summing power-norm* by

$$\|\boldsymbol{x}\|_n^{(p,q)} = \sup\left\{\left(\sum_{j=1}^{n} |\langle x_j, \lambda_j\rangle|^q\right)^{1/q} : \boldsymbol{\lambda} \in (E^*)^n, \mu_{p,n}(\boldsymbol{\lambda}) \leqslant 1\right\}. \tag{2}$$

It is easy to see that for the dual space E^* one has

$$\mu_{p,n}(\boldsymbol{\lambda}) = \sup\left\{\left(\sum_{i=1}^{n} |\langle x, \lambda_j\rangle|^p\right)^{1/p} : x \in E, \|x\| = 1\right\}$$

for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (E^*)^n$.

An equivalent formulation of the weak p-summing power-norm is given by

$$\mu_{p,n}(\boldsymbol{x}) = \|T_{\boldsymbol{x}} : (\mathbb{C}^n, \|\cdot\|_{p'}) \to E\| = \sup\left\{\left\|\sum_{k=1}^{n} z_k x_k\right\| : \left(\sum_{k=1}^{n} |z_k|^{p'}\right)^{1/p'} = 1\right\}$$

(with the obvious modification for $p = 1$), where $T_{\boldsymbol{x}}$ stands for the operator $T_{\boldsymbol{x}}(\boldsymbol{z}) = \sum_{k=1}^{n} z_k x_k$. Actually, the map $\boldsymbol{x} \mapsto T_{\boldsymbol{x}}$ is an isometric linear isomorphism from $(E^n, \mu_{p,n})$ onto $\mathcal{B}((\mathbb{C}^n, \|\cdot\|_{p'}), E)$.

Concerning the (p,q)-summing power-norm let us mention that in the case $p \leqslant q$ it is actually a multinorm and has been studied deeply (see [3, 5]). It was shown to be connected to the theory of (q,p)-summing operators (see [8,10,12]) via the following key result (see [5, Theorem 2.6]) which establishes that

$$\|\boldsymbol{x}\|_n^{(p,q)} = \pi_{q,p}(T'_{\boldsymbol{x}} : E^* \to c_0)$$

for $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$, where $T'_{\boldsymbol{x}}$ stands for the adjoint of $T_{\boldsymbol{x}}$.

Computing the precise value of $\|\boldsymbol{x}\|_n^{(p,q)}$ is rather difficult in general. For instance, in the case $E = \ell_r$ for $1 \leqslant r \leqslant \infty$ and $1 \leqslant p, q < \infty$ it is known that

$$\|(\delta_1, \ldots, \delta_n)\|_n^{(p,q)} = n^{\left(\frac{1}{q} - \left(\frac{1}{p} - \frac{1}{r}\right)^+\right)^+}, \tag{3}$$

where (δ_k) stands for the canonical basis of ℓ_r and $a^+ = \max\{a, 0\}$. This result was first proved for $p \leqslant q$ (see [5, Example 2.16]) and later extended with a different proof for all values $1 \leqslant p, q \leqslant \infty$ (see [2, Proposition 2.13]).

With the help of (p,q)-summing multinorms Dales and Polyakov (see [7, Theorem 3.33]) found that $\|\boldsymbol{x}\|_n^{\max} = \|\boldsymbol{x}\|_n^{(1,1)}$. The characterization of those (p,q)-summing multinorms which are equivalent to the minimum multinorm based on an ℓ_r-space was studied in [5]. It was shown (see [5, Theorem 3.9] (or [3, Theorem 1.11] for a different proof) that in the case $E = \ell_r$, $1 \leqslant r < \infty$, there exists $C \geqslant 1$ such that

$$\|\boldsymbol{x}\|_n^{\min} \leqslant \|\boldsymbol{x}\|_n^{(p,q)} \leqslant C\|\boldsymbol{x}\|_n^{\min} \tag{4}$$

for all $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in (\ell_r)^n$ if and only if $\frac{1}{p} - \frac{1}{q} \geqslant \frac{1}{r}$.

In this paper we shall generalize (1) and (2) and introduce power-norms defined by means of Orlicz sequence spaces or symmetric sequence spaces. We shall formulate the analogues of (3) and (4) in this setting and obtain alternative proofs of the known results for the (p,q)-summing power-norms.

As in the case of the (p,q)-summing power-norms one should expect a connection with the theory of the (X_1, X_2)-summing operators, where X_1 and X_2 are symmetric sequence spaces (in particular, Orlicz spaces). We shall also show that the geometric properties of the Banach space E actually plays a role in the equivalence appearing in (4). The reader should be aware that the notions of the (X_1, X_2)-summing operators with respect to symmetric sequence spaces have previously been used for other purposes (for instance see [9] or [15]). Let us recall here the notion of the $(\Phi, 1)$-summing operator and the Φ-Orlicz property of a Banach space E introduced by L. Maligranda and M. Mastyło (see [15]). Given the Orlicz function Φ and two Banach spaces E and F, a bounded linear operator $T\colon E \to F$ is called $(\Phi,1)$-*summing* whenever $(\|Tx_k\|) \in \ell_\Phi$ for any weakly summable sequence $(x_k) \in \ell_1^w(E)$, where $\ell_1^w(E)$ stands for the space of sequences in E such that

$$\|(x_k)\|_{\ell_1^w(E)} = \sup_{\|\lambda\|_{E^*} \leqslant 1} \sum_k \big|\langle x_k, \lambda\rangle\big| < \infty.$$

It is not difficult to see that T is $(\Phi,1)$-summing if and only if there exists a constant $C > 0$ such that

$$\sum_{k=1}^{n} \big|\langle Tx_k, \lambda_k\rangle\big| \leqslant C\left(\sum_{k=1}^{n} \Phi^*\big(\|\lambda_k\|\big)\right)\mu_{1,n}(\boldsymbol{x})$$

for all $n \in \mathbb{N}$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (E^*)^n$, where Φ^* is the complementary function of Φ. A Banach space E is said to satisfy the Φ-*Orlicz property* whenever the identity map $Id\colon E \to E$ is $(\Phi,1)$-summing.

This paper contains four sections besides this introduction. In the first one we recall the basic definitions and results on Orlicz sequence spaces to be used in the rest of the paper. In Section 3 we define the notion of the (Φ, Ψ)-summing power-norm and study the connection between the spaces having the Φ-Orlicz property and those satisfying the analogue of (4) for Orlicz functions. Section 4 is devoted to recalling some facts on symmetric sequence spaces and studying the notion of the (X_1, X_2)-summing power-norm for symmetric sequence spaces X_1 and X_2. Finally we present an extension of (3) for (X_1, X_2)-summing power-norms in the last section.

2. Preliminaries on Orlicz spaces

We shall say that $\Phi\colon [0,\infty) \to [0,\infty)$ is an *Orlicz function* whenever it is convex and takes value zero only at zero (it is called a non-degenerated Orlicz function for instance in [14]). In particular Φ is continuous, increasing, $\lim_{t\to\infty}\Phi(t) = \infty$, and $\Phi(t)/t$ is non-decreasing. Hence

$$\Phi(st) \leqslant s\Phi(t), \quad 0 < s \leqslant 1, t \geqslant 0$$

and

$$\Phi(s) + \Phi(t) \leqslant \Phi(s+t), \quad s,t \geqslant 0.$$

We denote by Φ^* *the complementary function* (also called the *Young conjugate*) of Φ defined by

$$\Phi^*(s) = \sup\{ts - \Phi(t) : t \geqslant 0\}, \quad s \geqslant 0.$$

In particular

$$st \leqslant \Phi(t) + \Phi^*(s), \quad t,s \geqslant 0.$$

Observe that with this definition we allow $\Phi^*(s) = \infty$ for some values of s (for instance in the case $\Phi(t) = t$). We shall be interested only in the case when Φ^* is also a (non-degenerated) Orlicz function so we restrict ourselves to the class of Orlicz functions satisfying

$$\lim_{t\to\infty}\frac{\Phi(t)}{t} = \infty \quad \text{and} \quad \lim_{t\to 0^+}\frac{\Phi(t)}{t} = 0. \tag{5}$$

We shall denote by $\mathcal{O}$ the class of Orlicz functions satisfying (5). Every Orlicz function Φ, being increasing and convex, has a non-negative and non-decreasing right derivative $\phi(t)$ for every $t > 0$ and $\Phi(t) = \int_0^t \phi(s)ds$. For functions $\Phi \in \mathcal{O}$ one has that $\phi(0) = 0$ and $\lim_{t\to\infty}\phi(t) = \infty$, and if ψ stands for the right-inverse of ϕ, that is,

$$\psi(u) = \sup\{t : \phi(t) \leqslant u\}, \quad u \geqslant 0,$$

which turns out to be right-continuous, non-decreasing with $\psi(0) = 0$ and $\psi(u) > 0$ for $u > 0$, one obtains that $\Phi^*(s) = \int_0^s \psi(u)du$ for $s \geqslant 0$ and

$$s\psi(s) = \Phi\big(\psi(s)\big) + \Phi^*(s), \quad s > 0.$$

Recall that an Orlicz function Φ is said to satisfy the Δ_2-*condition* with constant $C > 0$ whenever

$$\Phi(2t) \leqslant C\Phi(t), \quad t > 0.$$

Since $2\Phi(t) \leqslant \Phi(2t)$ for $t > 0$ then the Δ_2-condition implies $C \geqslant 2$.

If we assume that $C = 2$ we automatically have, by convexity, that $\Phi(2t) = 2\Phi(t)$ and $\Phi(t+s) = \Phi(t) + \Phi(s)$. Hence $\Phi(ns) = n\Phi(s)$ and $\Phi(\frac{t}{m}) = \frac{\Phi(t)}{m}$ for all $n, m \in \mathbb{N}$, $s, t > 0$ and, using that Φ is continuous, we obtain that $\Phi(t) = t\Phi(1)$ for $t > 0$.

Therefore we have that Φ satisfies the Δ_2-condition with constant $C = 2$ if and only if $\Phi(t) = \Phi(1)t$ for $t > 0$.

2.1. Proposition. *Let Φ satisfy the Δ_2-condition with constant C.*

(i) *If $A > C$ then there exists $\gamma > 1$ such that*

$$\Phi(\gamma t) \leqslant A\Phi\left(\frac{t}{2}\right), \quad t > 0.$$

(ii) *If $\beta = \log_2 C \geqslant 1$ then*

$$u\Phi(t) \leqslant \Phi(tu) \leqslant Cu^\beta \Phi(t), \quad u \geqslant 1,\ t \geqslant 0. \tag{6}$$

$$C^{-1}v^\beta \Phi(s) \leqslant \Phi(sv) \leqslant v\Phi(s), \quad 0 < v \leqslant 1, t \geqslant 0. \tag{7}$$

Proof. (i). If $A \geqslant C^2$ it suffices to take $\gamma = 2$. Assuming $C < A < C^2$, let $0 < \theta < 1$ such that $A = (1-\theta)C + \theta C^2$ and set $\gamma = 1 + \theta > 1$. Now

$$\Phi(\gamma t) = \Phi\big((1-\theta)t + \theta 2t\big) \leqslant (1-\theta)\Phi(t) + \theta\Phi(2t) \leqslant A\Phi\left(\frac{t}{2}\right).$$

(ii). Since $u\Phi(t) \leqslant \Phi(ut)$ for any $u \geqslant 1$ and $t \geqslant 0$, selecting $m \in \mathbb{N}$ so that $2^{m-1} \leqslant u \leqslant 2^m$ and using the Δ_2-condition, with $C = 2^\beta$, we have

$$\Phi(ut) \leqslant C^m \Phi(t) \leqslant 2^{\beta m}\Phi(t) = 2^\beta u^\beta \Phi(t).$$

This shows (6). Finally (7) follows from (6) with the change $v = \frac{1}{u}$ and $s = tu$. □

For each Orlicz function Φ, we define ℓ_Φ as the space all sequences of complex numbers (z_k) such that there exists $\rho > 0$ satisfying $\rho_\Phi(\frac{(z_k)}{\rho}) < \infty$, where

$$\rho_\Phi((z_k)) = \sum_{k=0}^{\infty} \Phi\big(|z_k|\big).$$

We equip the space ℓ_Φ with the Luxemburg norm

$$\|(z_k)\|_\Phi^L = \inf\Big\{\rho > 0 : \rho_\Phi\Big(\Big(\frac{z_k}{\rho}\Big)\Big) \leqslant 1\Big\}.$$

Note that $\rho_\Phi((z_k)) \leqslant 1$ if and only if $\|(z_k)\|_\Phi^L \leqslant 1$.

2.2. Notation. For each $n \in \mathbb{N}$ and $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, we write

$$\rho_{n,\Phi}(\boldsymbol{z}) = \sum_{k=1}^{n} \Phi\big(|z_k|\big).$$

2.3. Proposition. *Let $(E, \|\cdot\|) = (\mathbb{C}, |\cdot|)$ and let Φ be an Orlicz function with $\Phi(1) = 1$. For each $n \in \mathbb{N}$ and $\boldsymbol{z} = \sum_{k=1}^{n} z_k \delta_k$ where (δ_k) stands for the canonical basis of c_0, we write*

$$\ell_{n,\Phi}^{L}(\boldsymbol{z}) = \Big\| \sum_{k=1}^{n} z_k \delta_k \Big\|_{\Phi}^{L}.$$

Then $((\mathbb{C}^n, \ell_{n,\Phi}^{L}) : n \in \mathbb{N})$ is a power-norm based on $\mathbb{C}$.

Proof. Note that $\ell_{n,\Phi}^{L}(\cdot)$ is a norm in $\mathbb{C}^n$ for all $n \in \mathbb{N}$ and that $\ell_{1,\Phi}^{L}(z) = |z|/\Phi^{-1}(1) = |z|$ for $z \in \mathbb{C}$. Clearly $\rho_{n,\Phi}(z_{\sigma(1)}, \dots, z_{\sigma(n)}) = \rho_{n,\Phi}(\boldsymbol{z})$ and $\rho_{n+1,\Phi}(z_1, \dots, z_n, 0) = \rho_{n,\Phi}(\boldsymbol{z})$, for any $\boldsymbol{z} = (z_1, \dots, z_n)$ and permutation σ. Hence (P1) and (P2) hold. Now (P3) follows using that Φ is increasing since we have $\rho_{n,\Phi}(\boldsymbol{\alpha z}) \leqslant \rho_{n,\Phi}(A\boldsymbol{z})$ for $A = \max_{1\leqslant k\leqslant n} |\alpha_k|$ where $\boldsymbol{\alpha z} = (\alpha_1 z_1, \dots, \alpha_n z_n)$. This gives $\ell_{n,\Phi}^{L}(\boldsymbol{\alpha z}) \leqslant A\ell_{n,\Phi}^{L}(\boldsymbol{z})$ for any $\boldsymbol{z}, \boldsymbol{\alpha} \in \mathbb{C}^n$. □

We can equip the space ℓ_Φ with other equivalent norms, such as the Amemiya norm

$$\|(z_k)\|_{\Phi}^{A} = \inf\Big\{\rho > 0 \,:\, \frac{1}{\rho}\big(1 + \rho_\Phi((\rho z_k)\big) \leqslant 1\Big\}$$

or the Orlicz norm

$$\|(z_k)\|_{\Phi}^{O} = \sup\Big\{\Big|\sum_{k=1}^{\infty} z_k w_k\Big| \,:\, \rho_{\Phi^*}\big((w_k)\big) \leqslant 1\Big\}.$$

It is easy to see that $\|(z_k)\|_{\Phi}^{L} \leqslant \|(z_k)\|_{\Phi}^{O} \leqslant 2\|(z_k)\|_{\Phi}^{L}$ and $\|(z_k)\|_{\Phi}^{O} \leqslant \|(z_k)\|_{\Phi}^{A}$. It was shown by H. Hudzik and L. Maligranda in [11] that actually $\|(z_k)\|_{\Phi}^{O} = \|(z_k)\|_{\Phi}^{A}$.

It is elementary to see that

$$\Big|\sum_{k=1}^{\infty} w_k z_k\Big| \leqslant \big\|(z_k)\big\|_{\Phi}^{O} \big\|(w_k)\big\|_{\Phi^*}^{L}$$

for any $(z_k) \in \ell_\Phi$ and $(w_k) \in \ell_{\Phi^*}$, and actually $(\ell_\Phi)^* = \ell_{\Phi^*}$ whenever Φ^* is a finite-valued function.

Using the Orlicz norm we can also define a power-norm based on $\mathbb{C}$.

2.4. Proposition. *Let $(E, \|\cdot\|) = (\mathbb{C}, |\cdot|)$ and let $\Phi \in \mathcal{O}$ such that $\Phi^*(1) = 1$. For each $n \in \mathbb{N}$ and $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ we denote*

$$\ell_{n,\Phi}^{O}(\boldsymbol{z}) = \sup\Big\{\Big|\sum_{k=1}^{n} z_k w_k\Big| \,:\, \rho_{n,\Phi^*}(\boldsymbol{w}) \leqslant 1\Big\}.$$

Then $((\mathbb{C}^n, \ell_{n,\Phi}^{O}) : n \in \mathbb{N})$ is a power-norm based on $\mathbb{C}$.

Proof. Note that $\ell^O_{n,\Phi}(\cdot)$ is a norm in $\mathbb{C}^n$ for all $n \in \mathbb{N}$ and $\ell^O_{1,\Phi}(z) = |z|(\Phi^*)^{-1}(1) = |z|$ for $z \in \mathbb{C}$. Clearly, denoting $\boldsymbol{z}_\sigma = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$ for $\boldsymbol{z} \in \mathbb{C}^n$ and permutation σ, we have

$$\begin{aligned}\ell^O_{n,\Phi}(\boldsymbol{z}_\sigma) &= \sup\left\{\left|\sum_{k=1}^n z_{\sigma(k)} w_k\right| : \rho_{n,\Phi^*}(\boldsymbol{w}) \leqslant 1\right\} \\ &= \sup\left\{\left|\sum_{k=1}^n z_k w_{\sigma^{-1}(k)}\right| : \rho_{n,\Phi^*}(\boldsymbol{w}) \leqslant 1\right\} \\ &= \ell^O_{n,\Phi}(\boldsymbol{z}).\end{aligned}$$

Hence (P1) holds.

Since $\rho_{n,\Phi^*}(\boldsymbol{w}) \leqslant \rho_{n+1,\Phi^*}(\boldsymbol{w}, w_{n+1})$ where $(\boldsymbol{z}, z_{n+1}) = (z_1, \dots, z_n, z_{n+1})$ we have

$$\begin{aligned}\ell^O_{n,\Phi}((\boldsymbol{z}, 0)) &= \sup\left\{\left|\sum_{k=1}^n z_k w_k\right| : \rho_{n+1,\Phi^*}((\boldsymbol{w}, w_{n+1})) \leqslant 1\right\} \\ &= \sup\left\{\left|\sum_{k=1}^n z_k w_k\right| : \rho_{n,\Phi^*}(\boldsymbol{w}) \leqslant 1\right\} \\ &= \ell^O_{n,\Phi}(\boldsymbol{z}).\end{aligned}$$

Hence (P2) holds.

Let $\boldsymbol{z}, \boldsymbol{\alpha} \in \mathbb{C}^n$, $A = \max_{1 \leqslant k \leqslant n} |\alpha_k|$. Then

$$\begin{aligned}\ell^O_{n,\Phi}(\boldsymbol{\alpha z}) &\leqslant \sup\left\{\sum_{k=1}^n \left|\alpha_k z_k w_k\right| : \rho_{n,\Phi^*}(\boldsymbol{w}) \leqslant 1\right\} \\ &\leqslant A \sup\left\{\sum_{k=1}^n \left|z_k w_k\right| : \rho_{n,\Phi^*}(\boldsymbol{w}) \leqslant 1\right\} \\ &= A \sup\left\{\left|\sum_{k=1}^n z_k \xi_k\right| : \rho_{n,\Phi^*}(\boldsymbol{\xi}) \leqslant 1\right\} \\ &= A \ell^O_{n,\Phi}(\boldsymbol{z}).\end{aligned}$$

Hence (P3) holds and the proof is finished. □

Let us consider a notion that generalizes the concept of complementary function and which will be useful for our purposes.

2.5. Definition. Let ψ, Φ be Orlicz functions such that

$$\lim_{t\to\infty} \frac{\Phi(t)}{\psi(t)} = \infty \quad \text{and} \quad \lim_{t\to 0^+} \frac{\Phi(t)}{\psi(t)} = 0. \tag{H}$$

We define *the complementary function of* Φ *with respect to* ψ by the formula

$$\Phi_\psi(s) = \sup\{\psi(ts) - \Phi(t) : t > 0\}.$$

2.6. Remark. Assumption H implies that

$$0 \leqslant \Phi_\psi(s) < \infty, \quad s \geqslant 0$$

and

$$\ell_\psi \subseteq \ell_\Phi.$$

Indeed, we have that for each $s > 1$ there exists $t_s > 1$ so that

$$\psi(st) - \Phi(t) \leqslant \psi(st) - \frac{1}{s}\Phi(st) < 0, \quad t \geqslant t_s.$$

Hence, using continuity of ψ and Φ,

$$\Phi_\psi(s) = \sup\{\psi(ts) - \Phi(t) : 0 \leqslant t \leqslant t_s\} < \infty.$$

On the other hand, it is known (see [14, page 139]) that $\ell_\psi \subseteq \ell_\Phi$ with continuous embedding if and only if there exists $C_0, t_0 > 0$ such that

$$\Phi(t) \leqslant C_0\psi(t), \quad 0 \leqslant t \leqslant t_0. \tag{8}$$

Since (8) follows from $\lim_{t\to 0^+} \frac{\Phi(t)}{\psi(t)} = 0$, one gets $\ell_\psi \subseteq \ell_\Phi$.

2.7. Proposition. *Let Φ, ψ be Orlicz functions satisfying H. Then Φ_ψ is an Orlicz function.*

Proof. Denoting by Ψ_t for each $t > 0$ the convex function $\Psi_t(u) = \psi(tu) - \Phi(t)$, one gets that $\Phi_\psi(s) = \sup_{t>0} \Psi_t(s)$ is convex. Clearly $\Phi_\psi(0) = 0$. From H there exists $t_0 > 0$ such that that $\psi(t) - \Phi(t) > 0$ for $0 < t < t_0$. Since Φ_ψ is increasing, it suffices to show that $\Phi_\psi(s) > 0$ for s small enough. Let $0 < s < \min\{t_0, 1\}$ and observe that

$$\Phi_\psi(s) \geqslant \psi(s) - \Phi(1) \geqslant \psi(s) - \Phi(s) > 0.$$

The proof is then complete. □

2.8. Example. Let $1 < p < q < \infty$ with $\frac{1}{p} + \frac{1}{q} < 1$ and set $\psi(t) = \frac{t^p}{p}$ and $\Phi(t) = \frac{t^q}{q}$. Then $\Phi_\psi(t) = \frac{t^r}{r}$ where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

Indeed, it suffices to observe that

$$\Phi_\psi(s) = \sup\left\{\frac{s^p t^p}{p} - \frac{t^q}{q} : t \geqslant 0\right\} = \frac{s^p t(s)^p}{p} - \frac{t(s)^q}{q}$$

where $t(s)$ is the solution of the equation $s^p t^{p-1} = t^{q-1}$, that is, $t(s) = s^{\frac{p}{p-q}}$. Hence $\Phi_\psi(s) = \frac{1}{r}s^r$.

2.9. Proposition. *Let ψ, Φ be Orlicz functions satisfying H such that ψ satisfies the Δ_2-condition with constant C and*

$$\inf_{t>0} \frac{\Phi(2t)}{\Phi(t)} > C.$$

Then Φ_ψ satisfies the Δ_2-condition.

Proof. Let $A = \inf_{t>0} \frac{\Phi(2t)}{\Phi(t)} > C$. We know from (i) in Proposition 2.1 that there exists $\gamma > 1$ so that $\psi(\gamma t) \leqslant A\psi(\frac{t}{2})$ for all $t > 0$. Then

$$\begin{aligned} \Phi_\psi(s) &\geqslant \sup\Big\{\psi(st) - \frac{1}{A}\Phi(2t) \,:\, t > 0\Big\} \\ &= \frac{1}{A}\sup\Big\{A\psi\Big(\frac{s}{2}u\Big) - \Phi(u) \,:\, u > 0\Big\} \\ &\geqslant \frac{1}{A}\sup\big\{\psi(\gamma su) - \Phi(u) \,:\, u > 0\big\} \\ &= \frac{1}{A}\Phi_\psi(\gamma s). \end{aligned}$$

Let $m \in \mathbb{N}$ be such that $1 < \gamma \leqslant 2^{1/m}$. Hence $\Phi_\psi(2s) \leqslant A^m \Phi_\psi(s)$ and then Φ_ψ satisfies the Δ_2-condition with constant A^m. □

2.10. Corollary. *Let $\Phi \in \mathcal{O}$ with*

$$\inf_{t>0} \frac{\Phi(2t)}{\Phi(t)} > 2. \tag{9}$$

then Φ^* is an Orlicz function satisfying the Δ_2-condition.

3. (Φ, Ψ)-summing norms

3.1. Definition. Let $(E, \|\cdot\|)$ be a normed space and Φ be an Orlicz function. For each $n \in \mathbb{N}$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (E^*)^n$ we define

$$\mu^L_{\Phi,n}(\boldsymbol{\lambda}) = \sup_{\|x\|=1} \inf\Big\{\rho > 0 \,:\, \sum_{k=1}^n \Phi\Big(\Big|\Big\langle x, \frac{\lambda_k}{\rho}\Big\rangle\Big|\Big) \leqslant 1\Big\}$$

3.2. Remark. Of course we have

$$\mu^L_{\Phi,n}(\boldsymbol{\lambda}) = \sup\big\{\ell^L_{n,\Phi}(\langle x, \boldsymbol{\lambda}\rangle) \,:\, \|x\| \leqslant 1\big\}, \tag{10}$$

where $\langle x, \boldsymbol{\lambda}\rangle = (\langle x, \lambda_1\rangle, \ldots, \langle x, \lambda_n\rangle)$.

3.3. Definition. Let $(E, \|\cdot\|)$ be a normed space and let $\Phi \in \mathcal{O}$. For each $n \in \mathbb{N}$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (E^*)^n$ we define

$$\mu^O_{\Phi,n}(\boldsymbol{\lambda}) = \sup\Big\{\Big\|\sum_{k=1}^n z_k \lambda_k\Big\|' \,:\, \rho_{n,\Phi^*}(\boldsymbol{z}) \leqslant 1\Big\}.$$

3.4. Remark. Clearly we have

$$\mu^O_{\Phi,n}(\boldsymbol{\lambda}) = \sup\big\{\ell^O_{n,\Phi}(\langle x, \boldsymbol{\lambda}\rangle) \,:\, \|x\| \leqslant 1\big\}. \tag{11}$$

Hence

$$\mu^L_{\Phi,n}(\boldsymbol{\lambda}) \leqslant \mu^O_{\Phi,n}(\boldsymbol{\lambda}) \leqslant 2\mu^L_{\Phi,n}(\boldsymbol{\lambda}).$$

Making use of Proposition 2.3 and Proposition 2.4 together with (10) and (11) we arrive at the following result.

3.5. Proposition. *If Φ is an Orlicz function with $\Phi(1) = 1$ (respectively, $\Phi \in \mathcal{O}$ with $\Phi^*(1) = 1$) then the sequence $((E^*)^n, \mu^L_{\Phi,n}(\cdot) : n \in \mathbb{N})$ (respectively, $((E^*)^n, \mu^O_{\Phi,n}(\cdot) : n \in \mathbb{N})$) defines a power-norm based on E^*.*

Note that if φ is convex with $\varphi(0) = 0$, for each $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, $z_{n+1} \in \mathbb{C}$, and $|\alpha| + |\beta| \leqslant 1$ one has that

$$\rho_{n+1,\varphi}(z_1, \dots, z_{n-1}, \alpha z_n, \beta z_n) \leqslant \rho_{n,\varphi}(\boldsymbol{z}), \tag{12}$$

$$\rho_{n,\varphi}(z_1, \dots, z_{n-1}, \alpha z_n + \beta z_{n+1}) \leqslant \rho_{n+1,\varphi}(\boldsymbol{z}, z_{n+1}). \tag{13}$$

Using these estimates we can easily get the following facts.

3.6. Proposition. *Let $n \in \mathbb{N}$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (E^*)^n$, $(\boldsymbol{\lambda}, \lambda_{n+1}) \in (E^*)^{n+1}$, and $(\alpha, \beta) \in \mathbb{C}^2$ such that $|\alpha| + |\beta| = 1$. Then*

$$\mu_{\Phi,n}(\lambda_1, \dots, \lambda_{n-1}, \alpha\lambda_n + \beta\lambda_{n+1}) \leqslant \mu_{\Phi,n+1}(\boldsymbol{\lambda}, \lambda_{n+1}) \tag{14}$$

$$\mu_{\Phi,n+1}(\lambda_1, \dots, \lambda_{n-1}, \alpha\lambda_n, \beta\lambda_n) \leqslant \mu_{\Phi,n}(\boldsymbol{\lambda}) \tag{15}$$

where $\mu_{\Phi,n}$ stands for either $\mu^O_{\Phi,n}$ or $\mu^L_{\Phi,n}$.

Proof. The case $\mu^L_{\Phi,n}$ follows directly from (12) and (13). To see the case $\mu^O_{\Phi,n}$ note that for each $\rho_{n,\Phi^*}(\boldsymbol{z}) \leqslant 1$ and $\rho_{n+1,\Phi^*}(\boldsymbol{w}) \leqslant 1$, using (12) and (13) again, we have

$$\Big\|\sum_{k=1}^{n-1} z_k\lambda_k + z_n(\alpha\lambda_n + \beta\lambda_{n+1})\Big\|' \leqslant \mu^O_{\Phi,n+1}(\boldsymbol{\lambda}, \lambda_{n+1}),$$

and

$$\Big\|\sum_{k=1}^{n-1} w_k\lambda_k + (w_n\alpha + w_{n+1}\beta)\lambda_n\Big\|' \leqslant \mu^O_{\Phi,n}(\boldsymbol{\lambda}).$$

These estimates give (14) and (15). □

3.7. Proposition. *Let Φ, ψ be Orlicz functions satisfying H. Then*

$$\mu^O_{(\Phi_\psi)^*,n}(\boldsymbol{z\lambda}) \leqslant 3\ell^L_{n,\psi^*}(\boldsymbol{z})\mu^L_{\Phi,n}(\boldsymbol{\lambda}) \tag{16}$$

and

$$\mu^O_{\psi,n}(\boldsymbol{z\lambda}) \leqslant 3\ell^L_{n,\Phi_\psi}(\boldsymbol{z})\mu^L_{\Phi,n}(\boldsymbol{\lambda}) \tag{17}$$

for all $n \in \mathbb{N}$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (E^*)^n$, $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ where $\boldsymbol{z\lambda} = (z_1\lambda_1, \dots, z_n\lambda_n)$.

Proof. Assume that $\mu^L_{\Phi,n}(\boldsymbol{\lambda}) \leqslant 1$ and $\ell^L_{n,\psi^*}(\boldsymbol{z}) \leqslant 1$. Since

$$uvw \leqslant \psi^*(u) + \Phi(v) + \Phi_\psi(w), \quad u, v, w \geqslant 0$$

then for each $x \in E$ with $\|x\| = 1$ and $\rho_{n,\Phi_\psi}(\boldsymbol{w}) \leqslant 1$ we have that

$$\Big|\sum_{k=1}^n w_k z_k \langle x, \lambda_k\rangle\Big| \leqslant \rho_{n,\psi^*}(\boldsymbol{z}) + \rho_{n,\Phi_\psi}(\boldsymbol{w}) + \rho_{n,\Phi}\big(\langle x, \boldsymbol{\lambda}\rangle\big).$$

Therefore, using that $\rho_{n,\psi^*}(\boldsymbol{z}) \leqslant 1$ and $\rho_{n,\Phi}(\langle x, \boldsymbol{\lambda}\rangle) \leqslant 1$, we obtain $\ell^O_{n,(\Phi_\psi)^*}(\langle x, \boldsymbol{z\lambda}\rangle) \leqslant 3$ for any $\|x\| = 1$. This gives (16).

A similar argument shows (17) and the proof is complete. □

3.8. Definition. Let $(E, \|\cdot\|)$ be a normed space, Φ an Orlicz function, and $\Psi \in \mathcal{O}$. For each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$ we define

$$\|\boldsymbol{x}\|_n^{(\Phi,\Psi)} = \sup\big\{\ell^O_{n,\Psi}(\langle \boldsymbol{x}, \boldsymbol{\lambda}\rangle) : \mu^L_{\Phi,n}(\boldsymbol{\lambda}) \leqslant 1\big\} \tag{18}$$

where $\langle \boldsymbol{x}, \boldsymbol{\lambda}\rangle = (\langle x_1, \lambda_1\rangle, \ldots, \langle x_n, \lambda_n\rangle)$.

We write $\|\boldsymbol{x}\|_n^{(p,\Psi)}$ for $\Phi(t) = t^p$ and $\|\boldsymbol{x}\|_n^{(\Phi,q)}$ for $\Psi(t) = t^q$.

Of course, to define $\|\boldsymbol{x}\|_n^{(\Phi,\Psi)}$ we might have chosen other possibilities such as

$$\sup\big\{\ell^L_{n,\Psi}(\langle \boldsymbol{x}, \boldsymbol{\lambda}\rangle) : \mu^L_{\Phi,n}(\boldsymbol{\lambda}) \leqslant 1\big\},$$
$$\sup\big\{\ell^L_{n,\Psi}(\langle \boldsymbol{x}, \boldsymbol{\lambda}\rangle) : \mu^O_{\Phi,n}(\boldsymbol{\lambda}) \leqslant 1\big\}$$

or

$$\sup\big\{\ell^O_{n,\Psi}(\langle \boldsymbol{x}, \boldsymbol{\lambda}\rangle) : \mu^O_{\Phi,n}(\boldsymbol{\lambda}) \leqslant 1\big\}$$

all of them being equivalent.

Notice that the norm in (18) is given by

$$\|\boldsymbol{x}\|_n^{(\Phi,\Psi)} = \sup\left\{\Big|\sum_{k=1}^n \langle x_k, z_k \lambda_k\rangle\Big| : \rho_{n,\Psi^*}(\boldsymbol{z}) \leqslant 1, \mu^L_{\Phi,n}(\boldsymbol{\lambda}) \leqslant 1\right\}. \tag{19}$$

The next result is elementary but we include the proof for the sake of completeness.

3.9. Proposition. *Let $(E, \|\cdot\|)$ be a Banach space, Φ an Orlicz function, and $\Psi \in \mathcal{O}$ with $\Psi^*(1) = \Phi(1) = 1$. Then the sequence $(\|\cdot\|_n^{(\Phi,\Psi)} : n \in \mathbb{N})$ defines a power-norm based on E.*

Proof. For each $n \in \mathbb{N}$, $\|\cdot\|_n^{(\Phi,\Psi)}$ is a norm on E^n and $\|\boldsymbol{x}\|_n^{(\Phi,\Psi)}$ coincides with the norm of the operator $\mathrm{T}_{\boldsymbol{x}}$ from $((E^*)^n, \mu^L_{\Phi,n})$ into $(\mathbb{C}^n, \ell^O_{n,\Psi})$ given by $\mathrm{T}_{\boldsymbol{x}}(\boldsymbol{\lambda}) = \langle \boldsymbol{x}, \boldsymbol{\lambda}\rangle$. For $n = 1$, $x \in E$ and $\lambda \in E^*$ we have

$$\mu^L_{\Phi,1}(\lambda) = \|\lambda\|/\Phi^{-1}(1) = \|\lambda\|, \quad \|x\|_1^{(\Phi,\Psi)} = \|x\|(\Psi^*)^{-1}(1)\Phi^{-1}(1) = \|x\|.$$

Let $\boldsymbol{x} \in E^n$, $\boldsymbol{\alpha} \in \mathbb{C}^n$, σ be a permutation of $\{1, \ldots, n\}$ and denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Then

$$\begin{aligned} \|\boldsymbol{x}_\sigma\|_n^{(\Phi,\Psi)} &= \sup\left\{\left|\sum_{k=1}^n \langle x_{\sigma(k)}, z_k\lambda_k\rangle\right| : \rho_{n,\Psi^*}(\boldsymbol{z}) \leqslant 1, \mu_{\Phi,n}^L(\boldsymbol{\lambda}) \leqslant 1\right\} \\ &= \sup\left\{\left|\sum_{k=1}^n \langle x_k, z_{\sigma^{-1}(k)}\lambda_{\sigma^{-1}(k)}\rangle\right| : \rho_{n,\Psi^*}(\boldsymbol{z}) \leqslant 1, \mu_{\Phi,n}^L(\boldsymbol{\lambda}) \leqslant 1\right\} \\ &= \|\boldsymbol{x}\|_n^{(\Phi,\Psi)}. \end{aligned}$$

This shows (P1).

Now using that $\rho_{n+1,\Psi^*}(\boldsymbol{z}, 0) = \rho_{n,\Psi^*}(\boldsymbol{z})$ and $\rho_{n,\Psi^*}(\boldsymbol{z}) \leqslant \rho_{n+1,\Psi^*}(\boldsymbol{z}, z_{n+1})$ we have

$$\begin{aligned} \|(\boldsymbol{x}, 0)\|_{n+1}^{(\Phi,\Psi)} &= \sup\left\{\left|\sum_{k=1}^n \langle x_k, z_k\lambda_k\rangle\right| : \rho_{n+1,\Psi^*}(\boldsymbol{z}) \leqslant 1, \mu_{\Phi,n+1}^L(\boldsymbol{\lambda}) \leqslant 1\right\} \\ &= \sup\left\{\left|\sum_{k=1}^n \langle x_k, z_k\lambda_k\rangle\right| : \rho_{n,\Psi^*}(\boldsymbol{z}) \leqslant 1, \mu_{\Phi,n}^L(\boldsymbol{\lambda}) \leqslant 1\right\} \\ &= \|\boldsymbol{x}\|_n^{(\Phi,\Psi)}. \end{aligned}$$

This gives (P2).

Finally, denoting $\boldsymbol{\alpha x} = (\alpha_1 x_1, \ldots, \alpha_n x_n)$, (P3) follows trivially since

$$\begin{aligned} \|\boldsymbol{\alpha x}\|_n^{(\Phi,\Psi)} &= \sup\left\{\left|\sum_{k=1}^n \langle \alpha_k x_k, z_k\lambda_k\rangle\right| : \rho_{n,\Psi^*}(\boldsymbol{z}) \leqslant 1, \ \mu_{\Phi,n}^L(\boldsymbol{\lambda}) \leqslant 1\right\} \\ &\leqslant \left(\max_{1\leqslant k\leqslant n} |\alpha_k|\right) \sup\left\{\sum_{k=1}^n \left|\langle x_k, z_k\lambda_k\rangle\right| : \rho_{n,\Psi^*}(\boldsymbol{z}) \leqslant 1, \mu_{\Phi,n}^L(\boldsymbol{\lambda}) \leqslant 1\right\} \\ &\leqslant \left(\max_{1\leqslant k\leqslant n} |\alpha_k|\right) \sup\left\{\left|\sum_{k=1}^n \langle x_k, \xi_k\lambda_k\rangle\right| : \rho_{n,\Psi^*}(\boldsymbol{\xi}) \leqslant 1, \mu_{\Phi,n}^L(\boldsymbol{\lambda}) \leqslant 1\right\} \\ &= \left(\max_{1\leqslant k\leqslant n} |\alpha_k|\right) \|\boldsymbol{x}\|_n^{(\Phi,\Psi)}. \end{aligned}$$

The proof is then complete. □

3.10. Proposition. *Let E be a Banach space and let ψ, Φ be Orlicz functions satisfying H. Then*

$$\|\boldsymbol{x}\|_n^{(\Phi,\psi)} \leqslant 3\|\boldsymbol{x}\|_n^{((\Phi_\psi)^*,1)}$$

for each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$.

Proof. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (E^*)^n$ and $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ with $\mu_{\Phi,n}^L(\boldsymbol{\lambda}) \leqslant 1$ and $\rho_{n,\psi^*}(\boldsymbol{z}) \leqslant 1$. Then, due to (16) we have

$$\mu^L_{(\Phi_\psi)^*,n}(\boldsymbol{z\lambda}) \leqslant \mu^O_{(\Phi_\psi)^*,n}(\boldsymbol{z\lambda}) \leqslant 3.$$

Hence

$$\begin{aligned}\Big|\sum_{k=1}^{n} z_k\langle x_k,\lambda_k\rangle\Big| &\leqslant \sum_{k=1}^{n}\big|\langle x_k, z_k\lambda_k\rangle\big| \\ &\leqslant \|\boldsymbol{x}\|_n^{((\Phi_\psi)^*,1)}\mu^L_{(\Phi_\psi)^*,n}(\boldsymbol{z\lambda}) \\ &\leqslant 3\|\boldsymbol{x}\|_n^{((\Phi_\psi)^*,1)}.\end{aligned}$$

The proof is complete using now (19). □

3.11. Proposition. *Let $(E,\|\cdot\|)$ be a normed space and $\Psi\in\mathcal{O}$ with $\Psi^*(1)=1$. Then the sequence $(\|\cdot\|_n^{(1,\Psi)}:n\in\mathbb{N})$ defines a multinorm based on E.*

Proof. Due to Proposition 3.9 we only need to check property (M). Given $\boldsymbol{x}=(x_1,\dots,x_{n-1},x_n)\in E^n$ we shall show that

$$\Big|\sum_{k=1}^{n-1}\langle x_k, z_k\lambda_k\rangle + \langle x_n, z_n\lambda_n + z_{n+1}\lambda_{n+1}\rangle\Big| \leqslant \|\boldsymbol{x}\|_n^{(1,\Psi)} \tag{20}$$

for all $\mu^L_{1,n+1}(\boldsymbol{\lambda})\leqslant 1$ and $\rho_{n+1,\Psi^*}(\boldsymbol{z})\leqslant 1$.

Given $\boldsymbol{\lambda}\in(E^*)^{n+1}$ and $\boldsymbol{z}\in\mathbb{C}^{n+1}$ with $\mu^L_{1,n+1}(\boldsymbol{\lambda})\leqslant 1$ and $\rho_{n+1,\Psi^*}(\boldsymbol{z})\leqslant 1$, we first select $\alpha>0$ such that

$$\Psi^*(\alpha)=\Psi^*(|z_{n+1}|)+\Psi^*(|z_n|).$$

Hence denoting

$$\tilde z_k = z_k,\quad \tilde\lambda_k=\lambda_k,\quad k=1,\dots,n-1$$

and

$$\tilde z_n=\alpha,\quad \tilde\lambda_n=\frac{z_n}{\alpha}\lambda_n+\frac{z_{n+1}}{\alpha}\lambda_{n+1}$$

we have

$$\sum_{k=1}^{n-1}\langle x_k, z_k\lambda_k\rangle + \langle x_n, z_n\lambda_n+z_{n+1}\lambda_{n+1}\rangle = \sum_{k=1}^{n}\langle x_k,\tilde z_k\tilde\lambda_k\rangle.$$

Notice that $\rho_{n,\Psi^*}(\tilde{\boldsymbol{z}})=\rho_{n+1,\Psi^*}(\boldsymbol{z})\leqslant 1$ and, using that $\max\{|z_{n+1}|,|z_n|\}\leqslant\alpha$, we have

$$\mu^L_{1,n}\Big(\lambda_1,\dots,\lambda_{n-1},\frac{z_n}{\alpha}\lambda_n+\frac{z_{n+1}}{\alpha}\lambda_{n+1}\Big)\leqslant 1.$$

Therefore we obtain (20). □

Recall from the introduction that the ψ-Orlicz property of a Banach space is equivalent to the existence of a constant $C>0$ such that

$$\ell^L_{n,\psi}(\|x_1\|,\dots,\|x_n\|)\leqslant C\mu_{1,n}(\boldsymbol{x})$$

for any $\boldsymbol{x}\in E^n$.

3.12. Theorem. *Let $\Phi \in \mathcal{O}$ and let E be a Banach space. The following statements are equivalent:*

(i) *E^* has the Φ-Orlicz property.*

(ii) *There exists $C \geqslant 1$ such that*

$$\|\boldsymbol{x}\|_n^{min} \leqslant \|\boldsymbol{x}\|_n^{(1,\Phi)} \leqslant C\|\boldsymbol{x}\|_n^{min}$$

for any $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$.

Proof. (i) $\Rightarrow$ (ii). Let $\boldsymbol{x} \in E^n$ with $\max_{1\leqslant k\leqslant n} \|x_k\| = 1$, and let $\boldsymbol{\lambda} \in (E^*)^n$ with $\mu_{1,n}(\boldsymbol{\lambda}) \leqslant 1$. Hence

$$\ell^L_{n,\Phi}\big(\langle \boldsymbol{x}, \boldsymbol{\lambda}\rangle\big) \leqslant \ell^L_{n,\Phi}\big(\|\lambda_1\|, \dots, \|\lambda_n\|\big) \leqslant C.$$

Then $\|\boldsymbol{x}\|_n^{(1,\Phi)} \leqslant C\|\boldsymbol{x}\|_n^{min}$. Since the inequality $\|\boldsymbol{x}\|_n^{min} \leqslant \|\boldsymbol{x}\|_n^{(1,\Phi)}$ holds for any power--norm, the implication is complete.

(ii) $\Rightarrow$ (i). Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (E^*)^n$ with $\mu_{1,n}(\boldsymbol{\lambda}) \leqslant 1$. For each $\varepsilon > 0$, select $x_k \in E$ with $\|x_k\| = 1$ for $1 \leqslant k \leqslant n$ and $(1-\varepsilon)\|\lambda_k\|' \leqslant \langle x_k, \lambda_k\rangle \leqslant \|\lambda_k\|'$. Hence, by the assumption, we obtain

$$\ell^L_{n,\Phi}((1-\varepsilon)\|\lambda_1\|', \dots, (1-\varepsilon)\|\lambda_n\|') \leqslant C.$$

Therefore $\ell^L_{n,\Phi}((\|\lambda_1\|, \dots, \|\lambda_n\|)) \leqslant \frac{C}{1-\varepsilon}$. Finally, taking limits as $\varepsilon \to 0$ the Φ-Orlicz property of E^* is shown. □

3.13. Corollary. *Let $E = \ell_\Phi$ where $\Phi \in \mathcal{O}$ satisfies the Δ_2-condition and $\Phi^*(\sqrt{t})$ is concave, and let Ψ be an Orlicz function such that $\ell_2 \subseteq \ell_\Psi$ with continuous embedding. Then there exists $C \geqslant 1$ such that*

$$\|\boldsymbol{x}\|_n^{min} \leqslant \|\boldsymbol{x}\|_n^{(1,\Psi)} \leqslant C\|\boldsymbol{x}\|_n^{min} \tag{21}$$

for any $\boldsymbol{x} = (x_1, \dots, x_n) \in (\ell_\Phi)^n$.

Proof. Recall that $\ell_{\Phi^*} = (\ell_\Phi)^*$, and $\Phi^*(\sqrt{t})$ being concave implies that ℓ_{Φ^*} has cotype 2 (see [13]), and in particular

$$\left(\sum_{k=1}^n \|\lambda_k\|^2_{\Phi^*}\right)^{1/2} \leqslant C_1\mu_{1,n}(\boldsymbol{\lambda}).$$

for each $\boldsymbol{\lambda} \in (\ell_{\Phi^*})^n$. Let $\boldsymbol{x} \in (\ell_\Phi)^n$ and $\boldsymbol{\lambda} \in (\ell_{\Phi^*})^n$ and use the following estimates

$$\begin{aligned}
\ell^L_{n,\Psi}\big(\langle \boldsymbol{x}, \boldsymbol{\lambda}\rangle\big) &\leqslant \ell^L_{n,\Psi}\big(\|x_1\|_\Phi \|\lambda_1\|_{\Phi^*}, \dots, \|x_n\|_\Phi \|\lambda_n\|_{\Phi^*}\big) \\
&\leqslant \big(\max_{1\leqslant k\leqslant n} \|x_k\|_\Phi\big) \ell^L_{n,\Psi}\big(\|\lambda_1\|_{\Phi^*}, \dots, \|\lambda_n\|_{\Phi^*}\big) \\
&\leqslant C_0 \big(\max_{1\leqslant k\leqslant n} \|x_k\|_\Phi\big) \left(\sum_{k=1}^n \|\lambda_k\|^2_{\Phi^*}\right)^{1/2} \\
&\leqslant C_0 C_1 \big(\max_{1\leqslant k\leqslant n} \|x_k\|_\Phi\big) \mu_{1,n}(\boldsymbol{\lambda}).
\end{aligned}$$

Hence we obtain (21) from Theorem 3.12. □

Let $\mathcal{M}(\ell_\Phi, \ell_\psi)$ denote the space of pointwise multipliers between two Orlicz spaces, i.e.

$$\mathcal{M}(\ell_\Phi, \ell_\psi) = \big\{(z_k)_{k\in\mathbb{N}} : (w_k z_k) \in \ell_\psi \forall (w_k) \in \ell_\Phi\big\}$$

and

$$\|(z_k)\|_{\mathcal{M}(\ell_\Phi,\ell_\psi)} = \sup\big\{\|(w_k z_k)_k\|_{\ell_\psi} : \|(w_k)\|_{\ell_\Phi} \leqslant 1\big\}.$$

3.14. Lemma. *Let Φ, ψ be Orlicz functions satisfying H. Then*

$$\ell_{\Phi_\psi} \subseteq \mathcal{M}(\ell_\Phi, \ell_\psi)$$

with continuous embedding.

Proof. Let $n \in \mathbb{N}$, $\boldsymbol{z} \in \mathbb{C}^n$ with $\rho_{n,\Phi_\psi}(\boldsymbol{z}) \leqslant 1$. We shall see that if $\rho_{n,\Phi}(\boldsymbol{w}) \leqslant 1$ then $\rho_{n,\psi}(\frac{1}{4}\boldsymbol{zw}) \leqslant 1$ where $\boldsymbol{zw} = (z_1 w_1, \dots, z_n w_n)$. Indeed, since $\psi(ts) \leqslant \Phi(t) + \Phi_\psi(s)$,

$$\rho_{n,\psi}\big(\frac{1}{4}\boldsymbol{zw}\big) \leqslant \rho_{n,\Phi}\big(\frac{1}{2}\boldsymbol{z}\big) + \rho_{n,\Phi_\psi}\big(\frac{1}{2}\boldsymbol{w}\big) \leqslant \frac{1}{2}\rho_{n,\Phi}(\boldsymbol{z}) + \frac{1}{2}\rho_{n,\Phi_\psi}(\boldsymbol{w}) \leqslant 1.$$

This shows that $\|(z_k)\|_{\mathcal{M}(\ell_\Phi,\ell_\psi)} \leqslant 4\|(z_k)\|_{\ell_{\Phi_\psi}}$. □

We now borrow a result from [15].

3.15. Lemma (see [15, Theorem 2.3]). *Let Φ be a super-multiplicative Orlicz function such that $t \to \Phi(\sqrt{t})$ is convex. Then the space ℓ_Φ has the Φ-Orlicz property.*

3.16. Theorem. *Let $\Phi, \psi \in \mathcal{O}$ satisfying H with $\Phi(1) = 1$, $\Phi(\sqrt{t})$ convex, and $\Phi(ts) \geqslant \Phi(t)\Phi(s)$ for $t, s \geqslant 0$. Then*

$$\|\boldsymbol{x}\|_n^{(1,\psi)} \leqslant C \ell^L_{n,\Phi_\psi}\big(\|x_1\|_{\Phi^*}, \dots, \|x_n\|_{\Phi^*}\big) \tag{22}$$

for any $\boldsymbol{x} = (x_1, \dots, x_n) \in (\ell_{\Phi^*})^n$.

Proof. Let us first show that Φ satisfies (9). Since $\Phi(2t) \geqslant \Phi(2)\Phi(t)$ it suffices to see that $\Phi(2) > 2$. This follows using that $\Phi(2) \geqslant 2\Phi(\sqrt{2}) > 2\Phi(1) = 2$. Hence from Corollary 2.10 we conclude that Φ^* satisfies the Δ_2-condition and, in particular, we have $\ell_\Phi = (\ell_{\Phi^*})^*$. On the other hand Lemma 3.15 gives that ℓ_Φ has the Φ-Orlicz property. Hence for each $\boldsymbol{\lambda} \in (\ell_\Phi)^n$ we have

$$\ell^L_{n,\Phi}\big(\|\lambda_1\|_\Phi, \dots, \|\lambda_n\|_\Phi\big) \leqslant C\mu_{1,n}(\boldsymbol{\lambda}). \tag{23}$$

Therefore, combining Lemma 3.14 and (23) we have

$$\begin{aligned}
\ell^L_{n,\psi}\big(\langle \boldsymbol{x}, \boldsymbol{\lambda}\rangle\big) &\leqslant \ell^L_{n,\psi}\big(\|x_1\|_{\Phi^*}\|\lambda_1\|_\Phi, \dots, \|x_n\|_{\Phi^*}\|\lambda_n\|_\Phi\big) \\
&\leqslant \big\|\big(\|x_1\|_{\Phi^*}, \dots, \|x_n\|_{\Phi^*}\big)\big\|_{\mathcal{M}(\ell_\Phi,\ell_\psi)} \ell^L_{n,\Phi}\big(\|\lambda_1\|_\Phi, \dots, \|\lambda_n\|_\Phi\big) \\
&\leqslant C\ell^L_{n,\Phi_\psi}\big(\|x_1\|_{\Phi^*}, \dots, \|x_n\|_{\Phi^*}\big)\mu_{1,n}(\boldsymbol{\lambda}).
\end{aligned}$$

This gives (22). □

The behaviour at infinity of $\varphi^{max}_n(E)$, which stands for the "rate of growth" of the maximum multinorm, has been carefully analyzed in [7]. From (22) we obtain upper estimates for the "rate of growth" $\varphi_{\|\cdot\|_n}(E)$ whenever $E = \ell_\Psi$ is an Orlicz sequence space with certain properties and the multinorm $\|\cdot\|_n = \|\cdot\|^{(1,\psi)}_n$ for another Orlicz function adapted to Ψ.

3.17. Corollary. *Let $\Phi, \psi \in \mathcal{O}$ satisfying H with $\Phi(1) = 1$, $\Phi(\sqrt{t})$ convex, and $\Phi(ts) \geqslant \Phi(t)\Phi(s)$ for $t, s \geqslant 0$. Then $\varphi_{\|\cdot\|^{1,\psi}_n}(\ell_{\Phi^*}) \leqslant C\big((\Phi_\psi)^{-1}(\frac{1}{n})\big)^{-1}$.*

In particular, for $1 < p, q < \infty$ and $q > \max\{2, p\}$ we obtain $\varphi_{\|\cdot\|^{1,p}_n}(\ell_{q'}) \leqslant Cn^{p/q}$.

4. Summing power-norms on symmetric sequence spaces

Orlicz spaces are particular cases of the so-called symmetric sequence spaces. Recall that *a symmetric sequence space* $(\ell, \|\cdot\|)$ is a Banach space of sequences in $\mathbb{C}^{\mathbb{N}}$ such that

(A1) If $(z_k) \in \ell$ and $|w_k| \leqslant |z_k|$ for all $k \in \mathbb{N}$, then $(w_k) \in \ell$ and $\|(w_k)\| \leqslant \|(z_k)\|$.

(A2) If $(z_k) \in \ell$ and $\sigma\colon \mathbb{N} \to \mathbb{N}$ is a permutation, then $(z_{\sigma(k)}) \in \ell$ and $\|(z_{\sigma(k)})\| = \|(z_k)\|$.

In particular, any non-trivial symmetric sequence space ℓ satisfies

(1) $\ell_1 \subseteq \ell \subseteq \ell_\infty$ with continuity.

(2) $\delta_k \in \ell$ for all k and $\|\delta_k\| = \|\delta_1\|$ for all $j \in \mathbb{N}$, where $\delta_k(i) = \delta_{ki}$ for any $k, i \in \mathbb{N}$.

We say that a symmetric sequence space is *maximal* (see [12,14]) whenever, denoting $P_n((z_k)) = \sum_{k=1}^n z_k\delta_k = (z_k)_{k=1}^n$, it satisfies

(A3) $\|(z_k)\| = \sup_{n\in\mathbb{N}} \|P_n((z_k))\|$.

As usual we denote by ℓ' the associated sequence space given by

$$\ell' = \left\{ (w_k) \,:\, \sum_k |z_k w_k| < \infty \ \forall\ (z_k) \in \ell \right\}$$

with the norm $\|(w_k)\|' = \sup\{\sum_k |z_k w_k| : \|(z_k)\| = 1\}$.

The connection between power-norms and maximal symmetric sequence spaces is due to the following observation which follows from both definitions.

4.1. Proposition. *Let L_n be a norm on $\mathbb{C}^n$ for each $n \in \mathbb{N}$. Then $((\mathbb{C}^n, L_n) : n \in \mathbb{N})$ is a power-norm based on $\mathbb{C}$ if and only if*

$$\ell = \left\{ (z_k) \,:\, \sup_{n \in \mathbb{N}} L_n(z_1, \dots, z_n) < \infty \right\}$$

is a maximal symmetric sequence space with $\|\delta_1\| = 1$.

A way to generate examples of symmetric sequence spaces is the use of "pointwise multipliers". Let $\ell, \tilde{\ell}$ be symmetric sequence spaces. We define

$$\mathcal{M}(\ell, \tilde{\ell}) = \left\{ (z_k)_{k \in \mathbb{N}} \,:\, (w_k z_k) \in \tilde{\ell}\ \forall (w_k) \in \ell \right\}$$

and

$$\|(z_k)\|_{\mathcal{M}(\ell, \tilde{\ell})} = \sup\left\{ \|(w_k z_k)_k\|_{\tilde{\ell}} \,:\, \|(w_k)\|_{\ell} \leqslant 1 \right\}.$$

It is elementary to see that $\mathcal{M}(\ell, \tilde{\ell})$ is also a symmetric sequence space. In the case $\tilde{\ell} = \ell_1 = \{(z_k) : \sum_{k=1}^{\infty} |z_k| < \infty\}$ one has $\mathcal{M}(\ell, \ell_1) = \ell'$.

A simple consequence of Hölder's inequality gives, for $1 \leqslant p, q \leqslant \infty$, $1/s = (1/q - 1/p)^+$ and $a^+ = \max\{a, 0\}$

$$\mathcal{M}(\ell_p, \ell_q) = \ell_s. \tag{24}$$

Recall that the so-called "fundamental function" associated to a given symmetric sequence space $(\ell, \|\cdot\|)$ is defined by

$$\varphi_\ell(n) = \left\| \sum_{j=1}^{n} \delta_j \right\|.$$

We shall use the notation $\phi_{\ell, \tilde{\ell}}(n) = \varphi_{\mathcal{M}(\ell, \tilde{\ell})}(n)$ for the fundamental function of the space of multipliers between two symmetric sequence spaces ℓ and $\tilde{\ell}$, that is,

$$\phi_{\ell, \tilde{\ell}}(n) = \sup\left\{ \|(w_k)_{k=1}^n\|_{\tilde{\ell}} \,:\, \|(w_k)_{k=1}^n\|_{\ell} \leqslant 1 \right\}.$$

In particular, for $\ell = \ell_p$ and $\tilde{\ell} = \ell_q$, $1 \leqslant p, q \leqslant \infty$, one has that

$$\varphi_{\ell_p}(n) = n^{1/p}, \quad \phi_{\ell_p, \ell_q}(n) = n^{(1/q - 1/p)^+}.$$

It is known (see [14, page 118]) that for symmetric sequence spaces

$$\varphi_\ell(n)\varphi_{\ell'}(n) = n, \quad n \in \mathbb{N}.$$

In particular, for any Orlicz function $\Phi \in \mathcal{O}$ the fundamental function of $(\ell_\Phi, \|\cdot\|_\Phi^L)$ (respectively, $(\ell_{\Phi^*}, \|\cdot\|_{\Phi^*}^O)$ is given by

$$\varphi_{\ell_\Phi}(n) = \frac{1}{\Phi^{-1}(\frac{1}{n})} \quad \Big(\text{respectively, } \varphi_{\ell_{\Phi^*}}(n) = n\Phi^{-1}\Big(\frac{1}{n}\Big)\Big).$$

Throughout this section $(\ell, \|\cdot\|_\ell)$ stands for a maximal symmetric sequence space with $\|\delta_1\|_\ell = 1$, ℓ' stands for its associate space, and we use the notation

$$L_n(\boldsymbol{z}) = \|\sum_{k=1}^n z_k\delta_k\|_\ell, \quad L_n'(\boldsymbol{z}) = \sup\Big\{\sum_{k=1}^n |z_k w_k| \,:\, L_n(\boldsymbol{w}) \leqslant 1\Big\}.$$

Let us define the following sequence of norms on E^n.

4.2. Definition. Let $(E, \|\cdot\|)$ be a normed space and ℓ a maximal symmetric sequence space. For $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$ we define

$$\mu_{\ell,n}(\boldsymbol{x}) = \sup\Big\{L_n\Big(\langle x_1, \lambda\rangle, \ldots, \langle x_n, \lambda\rangle\Big) \,:\, \lambda \in E^*,\ \|\lambda\|' \leqslant 1\Big\}.$$

It is easy to check that for the dual space E^* and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (E^*)^n$ one has

$$\mu_{\ell,n}(\boldsymbol{\lambda}) = \sup\big\{L_n(\langle x, \lambda_1\rangle, \ldots, \langle x, \lambda_n\rangle) \,:\, \|x\| \leqslant 1\big\}.$$

In the case $E = \tilde{\ell}$, for a given maximal symmetric sequence space $\tilde{\ell}$, and a particular choice $\lambda_k = z_k\delta_k \in E^*$ for $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we observe that

$$\mu_{\ell,n}(z_1\delta_1, \ldots, z_n\delta_n) = \|\boldsymbol{z}\|_{\mathcal{M}(\tilde{\ell},\ell)}. \tag{25}$$

In particular

$$\mu_{\ell,n}(\delta_1, \ldots, \delta_n) = \phi_{\tilde{\ell},\ell}(n) = \phi_{\ell',\tilde{\ell}'}(n).$$

It is also elementary to show that for $\ell \subseteq \tilde{\ell}$ and $x_k \in \ell$ for $k = 1, \ldots, n$, one has

$$\mu_{\ell,n}(x_1, \ldots, x_n) \leqslant \phi_{\tilde{\ell},\ell}(n)\mu_{\tilde{\ell},n}(x_1, \ldots, x_n),$$

In particular for $1 \leqslant p \leqslant q < \infty$

$$\mu_{p,n}(x_1, \ldots, x_n) \leqslant n^{1/s}\mu_{q,n}(x_1, \ldots, x_n), \quad \frac{1}{s} = \Big(\frac{1}{p} - \frac{1}{q}\Big)^+.$$

4.3. Proposition. *Let E and ℓ be a normed space and a maximal symmetric sequence space with $L_1(\delta_1) = 1$ respectively. Then the sequence $(\mu_{\ell,n}(\cdot) : n \in \mathbb{N})$ defines a power-norm based on E^*. Moreover, for any maximal symmetric sequence space $E = \tilde{\ell}$ one has*

$$\mu_{\ell,n}(\boldsymbol{\xi\lambda}) \leqslant \min\bigl\{\|\boldsymbol{\xi}\|_{\mathcal{M}(\tilde{\ell},\ell)}\mu_{\tilde{\ell},n}(\boldsymbol{\lambda}),\ \tilde{L}_n(\boldsymbol{\xi})\mu_{\mathcal{M}(\tilde{\ell},\ell),n}(\boldsymbol{\lambda})\bigr\} \tag{26}$$

for all $n \in \mathbb{N}$, $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n) \in (E^*)^n$ and $\boldsymbol{\xi} = (\xi_1,\dots,\xi_n) \in \mathbb{C}^n$ where $\boldsymbol{\xi\lambda} = (\xi_1\lambda_1,\dots,\xi_n\lambda_n)$.

Proof. It is straightforward that the sequence $(\mu_{\ell,n}(\cdot) : n \in \mathbb{N})$ defines a power-norm based on E^*. Let $n \in \mathbb{N}$, $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n) \in (E^*)^n$, and $\boldsymbol{\xi} = (\xi_1,\dots,\xi_n) \in \mathbb{C}^n$. Then

$$\begin{aligned}\mu_{\ell,n}(\boldsymbol{\xi\lambda}) &= \sup\bigl\{L_n(\xi_1\langle x,\lambda_1\rangle,\dots,\xi_n\langle x,\lambda_n\rangle) : \|x\| = 1\bigr\}\\ &\leqslant \|\boldsymbol{\xi}\|_{\mathcal{M}(\tilde{\ell},\ell)} \sup\bigl\{\tilde{L}_n(\langle x,\lambda_1\rangle,\dots,\langle x,\lambda_n\rangle) : \|x\| = 1\bigr\}\\ &= \|\boldsymbol{\xi}\|_{\mathcal{M}(\tilde{\ell},\ell)}\mu_{\tilde{\ell},n}(\boldsymbol{\lambda}).\end{aligned}$$

Similarly, changing the roles between ξ_k and $\langle x,\lambda_k\rangle$ one gets the other estimate. This shows (26). □

4.4. Definition. Let ℓ and $\tilde{\ell}$ be maximal symmetric sequence spaces and let us write

$$L_n((z_1,\dots,z_n)) = \Bigl\|\sum_{k=1}^n z_k\delta_k\Bigr\|_\ell \quad\text{and}\quad \tilde{L}_n((z_1,\dots,z_n)) = \Bigl\|\sum_{k=1}^n z_k\delta_k\Bigr\|_{\tilde{\ell}},\qquad n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1,\dots,x_n) \in E^n$ we define

$$\|\boldsymbol{x}\|_n^{(\ell,\tilde{\ell})} = \sup\bigl\{\tilde{L}_n\bigl(\langle x_1,\lambda_1\rangle,\dots,\langle x_n,\lambda_n\rangle\bigr) : \mu_{\ell,n}(\boldsymbol{\lambda}) \leqslant 1\bigr\}.$$

It is elementary to check that $\|(x_1,\dots,x_n)\|_n^{(\ell,\tilde{\ell})}$ is given by the infimum of the constants C satisfying

$$\Bigl|\sum_{k=1}^n w_k\langle x_k,\lambda_k\rangle\Bigr| \leqslant C\tilde{L}'_n(\boldsymbol{w})\mu_{\ell,n}(\boldsymbol{\lambda}).$$

4.5. Proposition. *Let E be a normed space and let ℓ and $\tilde{\ell}$ be maximal symmetric sequence spaces with $L_1(\delta_1) = \tilde{L}_1(\delta_1) = 1$. Then the sequence $(\|\cdot\|_n^{(\ell,\tilde{\ell})} : n \in \mathbb{N})$ defines a power-norm based on E satisfying that*

$$\|\boldsymbol{x}\|_n^{(\ell,\mathcal{M}(\tilde{\ell},\ell))} \leqslant \|\boldsymbol{x}\|_n^{(\ell_1,\tilde{\ell}')} \tag{27}$$

for all $n \in \mathbb{N}$, $\boldsymbol{x} = (x_1,\dots,x_n) \in E^n$.

In particular, for $1 \leqslant p,q < \infty$ and $1/s = (1/p - 1/q)^+$

$$\|\boldsymbol{x}\|_n^{(p,q)} \leqslant \|\boldsymbol{x}\|_n^{(1,s')} \tag{28}$$

for all $n \in \mathbb{N}$, $\boldsymbol{x} = (x_1,\dots,x_n) \in E^n$.

Proof. It is straightforward that the sequence $(\|\cdot\|_n^{(\ell,\tilde{\ell})} : n \in \mathbb{N})$ defines a power-norm. Let $n \in \mathbb{N}$, $\boldsymbol{x} \in E^n$, $\boldsymbol{\lambda} \in (E^*)^n$, and set $\langle \boldsymbol{x}, \boldsymbol{\lambda} \rangle = (\langle x_1, \lambda_1 \rangle, \dots, \langle x_n, \lambda_n \rangle) \in \mathbb{C}^n$. Clearly one has

$$\begin{aligned}
\|\langle \boldsymbol{x}, \boldsymbol{\lambda} \rangle\|_{\mathcal{M}(\tilde{\ell},\ell)} &= \sup\big\{L_n(\langle x_1, \lambda_1 w_1 \rangle, \dots, \langle x_n, \lambda_n w_n \rangle) \;:\; \tilde{L}_n(\boldsymbol{w}) \leqslant 1\big\} \\
&= \sup\left\{\sum_{k=1}^n |\langle x_k, \lambda_k \rangle w_k z_k| \;:\; L_n'(\boldsymbol{z}) \leqslant 1, \tilde{L}_n(\boldsymbol{w}) \leqslant 1\right\} \\
&= \sup\big\{\tilde{L}_n'(\langle x_1, \lambda_1 z_1 \rangle, \dots, \langle x_n, \lambda_n z_n \rangle) \;:\; L_n'(\boldsymbol{z}) \leqslant 1\big\} \\
&\leqslant \|\boldsymbol{x}\|_n^{(\ell_1,\tilde{\ell}')} \sup\big\{\mu_{1,n}(\boldsymbol{z}\boldsymbol{\lambda}) \;:\; L_n'(\boldsymbol{z}) \leqslant 1\big\} \\
&\leqslant \|\boldsymbol{x}\|_n^{(\ell_1,\tilde{\ell}')} \mu_{\ell,n}(\boldsymbol{\lambda})
\end{aligned}$$

where the last inequality follows from the estimate $\mu_{1,n}(\boldsymbol{z}\boldsymbol{\lambda}) \leqslant L_n'(\boldsymbol{z})\mu_{\ell,n}(\boldsymbol{\lambda})$. Finally, taking $\ell = \ell^p$ and $\tilde{\ell} = \ell^s$, one has $\mathcal{M}(\tilde{\ell}, \ell) = \ell^q$ and (28) follows from (27). $\square$

5. Computing the fundamental function

Throughout this section X_1 and X_2 will stand for maximal symmetric sequence spaces with $\|\delta_1\|_{X_1} = \|\delta_1\|_{X_2} = 1$, and E will be also a maximal symmetric sequence space L with $\|\delta_1\|_L = 1$. Our aim is to compute $\|(\delta_1, \dots, \delta_n)\|_n^{(X_1,X_2)}$. As in the previous section we denote by L_n and $\tilde{L}_n$ the norms of X_1 and X_2 restricted to $\mathbb{C}^n$, respectively.

5.1. Proposition. *Let $\boldsymbol{x} = (x_1, \dots, x_n) \in L^n$ with $x_j = \sum_{i=1}^\infty x_{j,i}\delta_i \in L$ for $j \in \{1, \dots, n\}$. Then*

$$\tilde{L}_n(x_{1,1}, \dots, x_{n,n}) \leqslant \|\boldsymbol{x}\|_n^{(X_1,X_2)} \phi_{L,X_1}(n).$$

In particular

$$\big\|(\delta_1, \dots, \delta_n)\big\|_n^{(X_1,X_2)} \geqslant \frac{\varphi_{X_2}(n)}{\phi_{L,X_1}(n)}. \tag{29}$$

Proof. Take $\lambda_k = \delta_k \in L^* = L'$ for all k and notice that

$$\mu_{X_1,n}(\delta_1, \dots, \delta_n) = \sup\left\{\Big\|\sum_{k=1}^n z_k \delta_k\Big\|_{L'} \;:\; L_n'(\boldsymbol{z}) \leqslant 1\right\} = \phi_{X_1',L'}(n) = \phi_{L,X_1}(n).$$

Therefore

$$\|\boldsymbol{x}\|_n^{(X_1,X_2)} \geqslant \tilde{L}_n\Big(\Big\langle x_1, \frac{1}{\phi_{L,X_1}(n)}\delta_1\Big\rangle, \dots, \Big\langle x_n, \frac{1}{\phi_{L,X_1}(n)}\delta_n\Big\rangle\Big).$$

Hence

$$\tilde{L}_n(x_{1,1}, \dots, x_{n,n}) \leqslant \|\boldsymbol{x}\|_n^{(X_1,X_2)} \phi_{L,X_1}(n).$$

In particular, when applied to $x_j = \delta_j$, we obtain (29). $\square$

5.2. Lemma. *Let* $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (L')^n$ *with* $\lambda_j = \sum_{i=1}^{\infty} \lambda_{j,i}\delta_i \in L'$ *for* $j \in \{1, \dots, n\}$. *Then*

$$\mu_{X_1,n}(\lambda_{1,1}\delta_1, \dots, \lambda_{n,n}\delta_n) \leqslant \mu_{X_1,n}(\boldsymbol{\lambda}).$$

Proof. For each $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in (L')^n$, as in the introduction, we use the notation $T_{(\gamma_1,\dots,\gamma_n)}(\boldsymbol{z}) = \sum_{k=1}^{n} z_k\gamma_k$ for the operator from $(\mathbb{C}^n, \|\cdot\|_{X_1'})$ into L'. We shall write $S_{\boldsymbol{\gamma}} = P_n T_{\boldsymbol{\gamma}}$, that is,

$$S_{\boldsymbol{\gamma}}(\boldsymbol{z}) = \left(\sum_{k=1}^{n} z_k\gamma_{k,1}, \dots, \sum_{k=1}^{n} z_k\gamma_{k,n}\right).$$

Clearly we have

$$\mu_{X_1,n}(\lambda_{1,1}\delta_1, \dots, \lambda_{n,n}\delta_n) = \|S_{(\lambda_{1,1}\delta_1,\dots,\lambda_{n,n}\delta_n)}\|_{(\mathbb{C}^n,\|\cdot\|_{X_1'})\to(\mathbb{C}^n,\|\cdot\|_{L'})}$$

and $\|S_{\boldsymbol{\gamma}}\|_{(\mathbb{C}^n,\|\cdot\|_{X_1'})\to(\mathbb{C}^n,\|\cdot\|_{L'})} \leqslant \|T_{\boldsymbol{\gamma}}\|_{(\mathbb{C}^n,\|\cdot\|_{X_1'})\to L'} = \mu_{X_1,n}(\boldsymbol{\gamma})$.

For each $j \in \{1, \dots, n\}$ and $\rho_k = (1 - 2\delta_{k,j})\delta_k$ for $k \in \{1, \dots, n\}$ we denote by A_j the operator $S_{(\rho_1,\dots,\rho_n)}$, that is to say,

$$A_j(\boldsymbol{z}) = T_{(\delta_1,\dots,-\delta_j,\delta_{j+1},\dots,\delta_n)}(\boldsymbol{z}) = (z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n).$$

Obviously $\|A_j\|_{(\mathbb{C}^n,\|\cdot\|_{X_1'})\to(\mathbb{C}^n,\|\cdot\|_{X_1'})} = \|A_j\|_{(\mathbb{C}^n,\|\cdot\|_{L'})\to(\mathbb{C}^n,\|\cdot\|_{L'})} = 1$ for all $j \in \{1, \dots, n\}$.

Now for each $j \in \{1, \dots, n\}$ and each bounded operator $T\colon (\mathbb{C}^n, \|\cdot\|_{X_1'}) \to (\mathbb{C}^n, \|\cdot\|_{L'})$ we define

$$\Delta_j T = \frac{1}{2}(T + A_j T A_j).$$

It is straightforward to see that for each $j \in \{1, \dots, n\}$ and $\boldsymbol{\gamma} \in (L')^n$ we have that $\Delta_j S_{\boldsymbol{\gamma}} = S_{(\gamma_1^{(j)},\dots,\gamma_n^{(j)})}$ where

$$\gamma_k^{(j)} = \sum_{i\neq j} \gamma_{k,i}\delta_i \quad \text{for } k \neq j \quad \text{and} \quad \gamma_j^{(j)} = \gamma_{j,j}\delta_j.$$

In particular

$$\Delta_1 S_{\boldsymbol{\lambda}} = \frac{1}{2}(S_{\boldsymbol{\lambda}} + A_1 S_{\boldsymbol{\lambda}} A_1) = S_{(\lambda_{1,1}\delta_1, \sum_{j=2}^{n}\lambda_{2,j}\delta_j, \dots, \sum_{j=2}^{n}\lambda_{n,j}\delta_j)}.$$

Repeating the procedure we obtain

$$S_{(\lambda_{1,1}\delta_1,\dots,\lambda_{n,n}\delta_n)} = \Delta_n\Delta_{n-1}\cdots\Delta_1 S_{\boldsymbol{\lambda}}.$$

Hence, since $\|\Delta_j T\|_{(\mathbb{C}^n,\|\cdot\|_{X_1'})\to(\mathbb{C}^n,\|\cdot\|_{L'})} \leqslant \|T\|_{(\mathbb{C}^n,\|\cdot\|_{X_1'})\to(\mathbb{C}^n,\|\cdot\|_{L'})}$, we conclude that

$$\begin{aligned}
\mu_{X_1,n}(\lambda_{1,1}\delta_1, \dots, \lambda_{n,n}\delta_n) &= \|\Delta_n\Delta_{n-1}\cdots\Delta_1 S_{\boldsymbol{\lambda}}\|_{(\mathbb{C}^n,X_1')\to(\mathbb{C}^n,L')} \\
&\leqslant \|T_{\boldsymbol{\lambda}}\|_{(\mathbb{C}^n,X_1')\to L'} \\
&= \mu_{X_1,n}(\boldsymbol{\lambda}).
\end{aligned}$$

The proof is now complete. □

5.3. Theorem. *Let X_1, X_2, and L be maximal symmetric sequence spaces with $\|\delta_1\|_{X_1} = \|\delta_1\|_{X_2} = \|\delta_1\|_L = 1$. Then*

$$\|(z_1\delta_1, \dots, z_n\delta_n)\|_n^{(X_1,X_2)} = \|\boldsymbol{z}\|_{\mathcal{M}(\mathcal{M}(L,X_1),X_2)} \tag{30}$$

for all $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$.

In particular, for $L = \ell_r$ and $1 \leqslant p, q < \infty$

$$\|(\delta_1, \dots, \delta_n)\|_n^{(p,q)} = n^{(1/q-(1/p-1/r)^+)^+}. \tag{31}$$

Proof. Due to Lemma 5.2,

$$\begin{aligned}
&\|(z_1\delta_1, \dots, z_n\delta_n)\|_n^{(X_1,X_2)} = \\
&= \sup\left\{\tilde{L}_n(\langle z_1\delta_1, \lambda_1\rangle, \dots, \langle z_n\delta_n, \lambda_n\rangle) : \mu_{X_1,n}(\boldsymbol{\lambda}) \leqslant 1\right\} \\
&= \sup\left\{\tilde{L}_n(\langle z_1\delta_1, w_1\delta_1\rangle, \dots, \langle z_n\delta_n, w_n\delta_n\rangle) : \mu_{X_1,n}(w_1\delta_1, \dots, w_n\delta_n)) \leqslant 1\right\} \\
&= \sup\left\{\tilde{L}_n(\boldsymbol{zw}) : \mu_{X_1,n}(w_1\delta_1, \dots, w_n\delta_n)) \leqslant 1\right\}
\end{aligned}$$

where $\boldsymbol{zw} = (z_1w_1, \dots, z_nw_n)$.

From (25) we know that $\mu_{X_1,n}(w_1\delta_1, \dots, w_n\delta_n)) = \|\boldsymbol{w}\|_{\mathcal{M}(L,X_1)}$ and therefore we obtain (30).

To see (31), we use (30) and (24) to get

$$\|(\delta_1, \dots, \delta_n)\|_n^{(p,q)} = \phi_{\ell^s,\ell^q}(n) = n^{(1/q-1/s)^+}$$

for $1/s = (1/p - 1/r)^+$ □

5.4. Corollary. *Let $E = L$ and X_2 be maximal symmetric sequence spaces. Then*

$$\|(\delta_1, \dots, \delta_n)\|_n^{(\ell_1,X_2)} = \phi_{X_2',L}(n).$$

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