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A note on strict K-monotonicity of some symmetric function spaces

Dedicated to Professor Julian Musielak on his 85-th birthday in friendship and high esteem

Abstract. We discuss some sufficient and necessary conditions for strict K-monotonicity of some important concrete symmetric spaces. The criterion for strict monotonicity of the Lorentz space $\Gamma_{p,w}$ with $0 < p < \infty$ is given.

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1. Introduction. Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of reals and positive integers, respectively. As usual $S(X)$ (resp. $B(X)$) stands for the unit sphere (resp. the closed unit ball) of a Banach space $(X, \|\cdot\|_X)$.

Let $L^0 = L^0(I, m)$ be a set of all (equivalence classes of) extended real valued Lebesgue measurable functions on I, where $I = [0, 1)$ or $I = [0, \infty)$ and m is the Lebesgue measure on the real line. For $x \in L^0$ we denote its *distribution function* by

$$d_x(\lambda) = m\{t \in I : |x(t)| > \lambda\}, \qquad \lambda \geq 0,$$

and its *decreasing rearrangement* by

$$x^*(t) = \inf\{\lambda > 0 : d_x(\lambda) \leq t\}, \quad t \geq 0.$$

Given $x \in L^0$ we denote the *maximal function* of x^* by

$$x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)ds.$$

It is well known that $x^* \le x^{**}$, x^{**} is non-increasing and subadditive, i.e.

$$(1) \qquad (x+y)^{**} \le x^{**} + y^{**}$$

for any $x, y \in L^0$. For the properties of d_x, x^* and x^{**}, the reader is referred to [1, 21].

A Banach lattice $(E, \|\cdot\|_E)$ is called a *Banach function space* (or a *Köthe function space*) if it is a sublattice of L^0 satisfying the following conditions

(1) If $x \in L^0$, $y \in E$ and $|x| \le |y|$ a.e., then $x \in E$ and $\|x\|_E \le \|y\|_E$.

(2) There exists a strictly positive $x \in E$.

The set $E_+ = \{x \in E : x \ge 0\}$ is called the *positive cone of* E. A Banach function space is said to be *strictly monotone* ($E \in (SM)$ for short) if for any $x, y \in E_+ \setminus \{0\}$ such that $x \le y$ and $y \ne x$, we have $\|x\|_E < \|y\|_E$.

A point $x \in E$ is said to have an *order continuous norm* if for any sequence (x_n) in E such that $0 \le x_n \le |x|$ and $x_n \to 0$ m-a.e. we have $\|x_n\|_E \to 0$. A Köthe space E is called *order continuous* ($E \in (OC)$ for short) if every element of E has an order continuous norm (see [17, 22, 26]). As usual E_a stands for the subspace of order continuous elements of E. Recall that a Banach function space E has the *Fatou property* if for any sequence (x_n) such that $0 \le x_n \in E$ for all $n \in \mathbb{N}$, $x \in L^0$, $x_n \uparrow x$ a.e. with $\sup_{n\in\mathbb{N}} \|x_n\|_E < \infty$, we have $x \in E$ and $\|x_n\|_E \uparrow \|x\|_E$.

A Banach function space E is said to be *symmetric* or *rearrangement invariant* if for every $x \in L^0$ and $y \in E$ with $d_x = d_y$, we have $x \in E$ and $\|x\|_E = \|y\|_E$. For any symmetric Banach function space E denote by ϕ_E its *fundamental function*, that is $\phi_E(t) = \|\chi_{[0,t)}\|_E$ for any $t \in I$ (see [1, 21]). It is well known that every fundamental function is quasi-concave, i.e. $\phi_E(0) = 0$, $\phi_E(t)$ is positive, non-decreasing and $t^{-1}\phi_E(t)$ is non-increasing for $t \in (0, m(I))$. It is well-known that quasi-concavity of fundamental function ϕ_E on I is equivalent to the fact that $\phi_E(t) \le \max(1, t/s)\phi_E(s)$ for all $s, t \in (0, m(I))$. Moreover, for each fundamental function ϕ_E, there is an equivalent, concave function $\widetilde{\phi}_E$, defined by $\widetilde{\phi}_E(t) := \inf_{s\in(0,m(I))}(1 + \frac{t}{s})\phi_E(s)$. Then $\phi_E(t) \le \widetilde{\phi}_E(t) \le 2\phi_E(t)$ for all $t \in I$.

For each symmetric function space E with the concave fundamental function ϕ_E there are the smallest and the largest symmetric space with the same fundamental function, namely the Lorentz space Λ_{ϕ_E} and the Marcinkiewicz space M_{ϕ_E} that will be defined below.

For any symmetric space E we have $L^1 \cap L^\infty \subset E \subset L^1 + L^\infty$, where

$$L^1 \cap L^\infty = \left\{x : x \in L^1 \text{ and } x \in L^\infty\right\}$$

and $L^1 + L^\infty$ is the space which consists of all functions x in L^0 that are representable as a sum $x = y + z$ of functions y in L^1 and z in L^∞. The spaces $L^1 \cap L^\infty$ and $L^1 + L^\infty$ are equipped with the norms

$$\|x\|_{L^1 \cap L^\infty} = \max\left\{\|x\|_{L^1}, \|x\|_{L^\infty}\right\}$$

and

$$\|x\|_{L^1 + L^\infty} = \inf\left\{\|y\|_1 + \|z\|_\infty : y + z = x,\ y \in L^1,\ z \in L^\infty\right\},$$

respectively.

The relation $\prec$ is defined for any x, y in $L^1 + L^\infty$ by

$$x \prec y \Leftrightarrow x^{**}(t) \leq y^{**}(t) \text{ for all } t > 0.$$

Recall that a symmetric space E is *K-monotone* (KM for short) or has the *majorant property* if for any $x \in L^1 + L^\infty$ and $y \in E$ such that $x \prec y$, we have $x \in E$ and $\|x\|_E \leq \|y\|_E$.

It is well known that a symmetric space is K-monotone iff it is exact interpolation space between L^1 and L^∞. Moreover, symmetric spaces with Fatou property as well as separable symmetric spaces are K-monotone (see [21]).

A symmetric space E is called *strictly K-monotone* (SKM for short) if for any $x, y \in E$ such that $x \prec y$ and $x^* \neq y^*$ we have $\|x\|_E < \|y\|_E$.

There is proved in [3] (Proposition 2.1) that every separable symmetric space E with the Kadec-Klee property is strictly K-monotone. Moreover, in separable Lorentz spaces, strict K-monotonicity is equivalent to the Kadec-Klee property (see [3], Theorem 2.11).

H. Hudzik, A. Kamińska and M. Mastyło showed in [12] the following lemma.

Lemma 1.1 *Every symmetric rotund and K-monotone space E is strictly K-monotone.*

The goal of this note is to discuss some sufficient and necessary conditions for strict K-monotonicity of some important concrete symmetric spaces. By the way we conclude that the converse of Lemma 1.1 is not true in general.

2. Results. Let φ be an *Orlicz function*, i.e. $\varphi : \mathbb{R} \to [0, \infty]$, φ is convex, even, vanishing and continuous at zero, left continuous on $(0, \infty)$ and not identically equal to zero. Denote

$$a_\varphi = \sup \{u \geq 0 : \varphi(u) = 0\} \text{ and } b_\varphi = \sup \{u \geq 0 : \varphi(u) < \infty\}.$$

We write $\varphi > 0$ when $a_\varphi = 0$ and $\varphi < \infty$ if $b_\varphi = \infty$. Denote by p the right hand side derivative of φ with the domain restricted to the interval $[0, \infty)$. An Orlicz function φ is said to be *strictly convex* ($\varphi \in (SC)$ for short) if the inequality

$$\varphi\left(\frac{u+v}{2}\right) < \frac{1}{2}\varphi(u) + \frac{1}{2}\varphi(v)$$

holds for any $u, v \in [0, \infty)$ such that $u \neq v$. Define on L^0 a convex semimodular I_φ by

$$I_\varphi(x) = \begin{cases} \|\varphi \circ x\|_E & \text{if } \varphi \circ x \in E, \\ \infty & \text{otherwise,} \end{cases}$$

where $(\varphi \circ x)(t) = \varphi(x(t)), t \in T$. By the Calderón-Lozanovskiĭ space E_φ we mean

$$E_\varphi = \{x \in L^0 : I_\varphi(cx) < \infty \ \text{ for some } \ c > 0\}$$

equipped with so called *Luxemburg-Nakano norm* defined by

$$\|x\|_{E_\varphi} = \inf \{\lambda > 0 : I_\varphi(x/\lambda) \leq 1\}.$$

If $E = L^1$, then E_φ is the classical *Orlicz function space* L^φ equipped with the Luxemburg-Nakano norm. If E is a Lorentz function space, then E_φ is the corresponding *Orlicz-Lorentz function space* equipped with the Luxemburg-Nakano norm (see [10, 11, 18]). On the other hand, if $\varphi(u) = u^p, 1 \le p < \infty$, then E_φ is the *p-convexification* $E^{(p)}$ of E with the norm $\|x\|_{E^{(p)}} = \||x|^p\|_E^{1/p}$. We still assume that E is symmetric and consequently E_φ is also symmetric. We also assume that E has the Fatou property, whence E_φ has also Fatou property, whence E_φ is K-monotone Banach space.

We say an Orlicz function φ satisfies *condition* $\Delta_2(\infty)$ if there exist $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) < \infty$ and the inequality $\varphi(2u) \leqslant K\varphi(u)$ holds for all $u \in [u_0, \infty)$. If there exists $K > 0$ such that $\varphi(2u) \leqslant K\varphi(u)$ for all $u \geqslant 0$, then we say that φ satisfies *condition* $\Delta_2(\mathbb{R}_+)$. We write for short $\varphi \in \Delta_2(\infty)$, $\varphi \in \Delta_2(\mathbb{R}_+)$, respectively.

For a Köthe function space E and an Orlicz function φ we say that φ satisfies *condition* Δ_2^E ($\varphi \in \Delta_2^E$ for short) if:

1) $\varphi \in \Delta_2(\infty)$ whenever $I = [0, 1)$;

2) $\varphi \in \Delta_2(\mathbb{R}_+)$ whenever $I = [0, \infty)$ (see [11]).

Proposition 2.1 *(i) Suppose* $E_a \neq \{0\}$. *If* $E_\varphi \in (SKM)$, *then* $\varphi > 0$ *and* $\varphi \in \Delta_2^E$.

(ii) If $\varphi > 0$, $\varphi \in \Delta_2^E$ *and* $E \in (SKM)$, *then* $E_\varphi \in (SKM)$.

(iii) If $\varphi \in \Delta_2^E$, $E \in (SM)$ *and* $\varphi \in (SC)$, *then* $E_\varphi \in (SKM)$.

Proof. (i) In order to prove that $\varphi > 0$ and $\varphi \in \Delta_2^E$ we need first to show the equality $b_\varphi = \infty$. Let $b_\varphi < \infty$. Since $E_a \neq \{0\}$, the fundamental function $\phi_E(t) = \|\chi_{(0,t)}\|_E$, $t \in I$, has the Darboux property on $[0, \|\chi_I\|_E)$ (see [7]). Consider the following cases.

$a)$ Assume that $\varphi(b_\varphi) < \infty$. Then there are numbers $a < b_\varphi$ and $t_1, t_2 \in I$ with $t_1 < t_2$ such that

$$\varphi(b_\varphi)\|\chi_{[0,t_1)}\|_E + \varphi(a)\|\chi_{[t_1,t_2)}\|_E < 1.$$

Define

$$x = b_\varphi\chi_{[0,t_1)} + a\chi_{[t_1,t_2)} \quad \text{and} \quad y = b_\varphi\chi_{[0,t_1)} + \frac{a}{2}\chi_{[t_1,t_2)}.$$

Obviously, $x = x^*$, $y = y^*$, $x^* \neq y^*$ and $y^{**} \le x^{**}$. Since $y \le x$, we have

$$I_\varphi(y) \le I_\varphi(x) = \|\varphi \circ x\|_E \le \varphi(b_\varphi)\|\chi_{[0,t_1)}\|_E + \varphi(a)\|\chi_{[t_1,t_2)}\|_E < 1.$$

Moreover,

$$\begin{aligned} I_\varphi\left(\frac{x}{\lambda}\right) &\ge I_\varphi\left(\frac{y}{\lambda}\right) = \left\|\varphi \circ \left(\frac{b_\varphi}{\lambda}\chi_{[0,t_1)} + \frac{a}{2\lambda}\chi_{[t_1,t_2)}\right)\right\|_E \\ &\ge \left\|\varphi \circ \left(\frac{b_\varphi}{\lambda}\chi_{[0,t_1)}\right)\right\|_E = \varphi\left(\frac{b_\varphi}{\lambda}\right)\|\chi_{[0,t_1)}\|_E = \infty \end{aligned}$$

for any $\lambda \in (0,1)$. Hence, by the definition of the norm $\|\cdot\|_{E_\varphi}$, we conclude that $\|x\|_{E_\varphi} = \|y\|_{E_\varphi} = 1$. Consequently, $E_\varphi \notin (SKM)$.

b) Let $\varphi(b_\varphi) = \infty$. For any $n \in \mathbb{N}$ define $u_n = \left(1 - \frac{1}{2^n}\right) b_\varphi$. Then $(\varphi(u_n))$ is an increasing sequence tending to infinity. By the Darboux property (see [7]), we conclude that $\lim_{t\longrightarrow 0^+} \|\chi_{[0,t)}\|_E = 0$. Take a decreasing sequence (s_n) such that

$$s_n < \frac{1}{2^n} \text{ and } \varphi(u_n)\|\chi_{[0,s_n)}\|_E < \frac{1}{2^n}$$

for every $n \in \mathbb{N}$. Let $t_n = \sum_{k=n}^{\infty} s_k$ for every $n \in \mathbb{N}$. Setting

$$x = \sum_{n=1}^{\infty} u_n \chi_{[t_{n+1},t_n)} \text{ and } y = \sum_{n=2}^{\infty} u_n \chi_{[t_{n+1},t_n)},$$

we have

$$I_\varphi(y) \le I_\varphi(x) = \|\varphi \circ x\|_E \le \sum_{n=1}^{\infty} \varphi(u_n)\|\chi_{[t_{n+1},t_n)}\|_E = \sum_{n=1}^{\infty} \varphi(u_n)\|\chi_{[0,s_n)}\|_E < 1$$

and

$$\begin{aligned} I_\varphi\left(\frac{x}{\lambda}\right) &\ge I_\varphi\left(\frac{y}{\lambda}\right) = \left\|\varphi \circ \left(\frac{y}{\lambda}\right)\right\|_E = \left\|\varphi \circ \left(\sum_{n=2}^{\infty} \frac{u_n}{\lambda}\chi_{[t_{n+1},t_n)}\right)\right\|_E \\ &\ge \left\|\varphi \circ \left(\frac{u_k}{\lambda}\chi_{[t_{k+1},t_k)}\right)\right\|_E = \varphi\left(\frac{u_k}{\lambda}\right)\|\chi_{[0,s_k)}\|_E \end{aligned}$$

for any $\lambda \in (0,1)$ and any $k \in \mathbb{N}$. Obviously, there is $k_\lambda \in \mathbb{N}$ such that $\frac{u_k}{\lambda} > b_\varphi$ for any $k \ge k_\lambda$. Consequently, $\varphi\left(\frac{u_k}{\lambda}\right) = \infty$ for each $k \ge k_\lambda$. Hence $I_\varphi(x/\lambda) = I_\varphi(y/\lambda) = \infty$ for any $\lambda \in (0,1)$. Therefore, by the definition of the norm $\|\cdot\|_{E_\varphi}$, we conclude that $\|x\|_{E_\varphi} = \|y\|_{E_\varphi} = 1$.

On the other hand, it is easy to see that $x^* \ne y^*$ and $y^{**} \le x^{**}$. Thus $E_\varphi \notin (SKM)$.

To prove that $\varphi > 0$, suppose conversely that $a_\varphi > 0$. Since $b_\varphi = \infty$, there exists $b > a_\varphi$ such that $\varphi(b)\left\|\chi_{[0,1/2)}\right\|_E = 1$. Setting

$$x = b\chi_{[0,\frac{1}{2})} + a_\varphi\chi_{[\frac{1}{2},1)} \quad \text{and} \quad y = b\chi_{[0,\frac{1}{2})} + \frac{a_\varphi}{2}\chi_{[\frac{1}{2},1)},$$

we get $x^* \ne y^*$ and $y^{**} \le x^{**}$. Moreover, $I_\varphi(y) = I_\varphi(x) = 1$ and $\min\left\{I_\varphi\left(\frac{x}{\lambda}\right), I_\varphi\left(\frac{y}{\lambda}\right)\right\} > 1$ for any $\lambda \in (0,1)$. Hence, by the definition of the norm $\|\cdot\|_{E_\varphi}$, we have $\|x\|_{E_\varphi} = \|y\|_{E_\varphi} = 1$. Therefore, $E_\varphi \notin (SKM)$.

Suppose $\varphi \notin \Delta_2^E$. We discuss only the case $I = [0,1)$. Then $\varphi \notin \Delta_2(\infty)$ and we find an element $x = \sum_{i=1}^{\infty} u_i\chi_{A_i}$, where the sequence (u_i) increases to ∞, (A_i) is a sequence of Lebesgue measurable pairwise disjoint sets, $I_\varphi(x) < 1$ and $\|x\|_{E_\varphi} = 1$ (see Theorem 1 in [9]). Taking $y = \sum_{i=2}^{\infty} u_i\chi_{A_i}$ we get $x^{**} \ge y^{**}$, $x^* \ne y^*$ and $\|y\|_{E_\varphi} = 1$. Thus $E_\varphi \notin (SKM)$.

(*ii*) Take $x, y \in E_\varphi$, $x^{**} \le y^{**}$ and $x^* \ne y^*$. Without loss of generality we may assume that $\|y\|_{E_\varphi} = 1$. We have

$$\int_0^t x^*(s)ds \le \int_0^t y^*(s)ds$$

for all $t \in I$. By property 18^o from [21], page 100, (see also [1, page 56, Proposition 3.6]),

$$\int_0^t x^*(s)z(s)ds \leq \int_0^t y^*(s)z(s)ds$$

for all $t \in I$ and any z, provided $z = z^*$. Take $z = \frac{\varphi \circ x^*}{x^*}\chi_{\operatorname{supp} x^*}$ that is non-increasing as the composition of the non-decreasing function $\frac{\varphi(u)}{u}$ and the non-increasing function x^*. Therefore,

$$\int_0^t \varphi\left(x^*(s)\right)ds \leq \int_0^t \varphi\left(y^*(s)\right)ds$$

for any $t \in I$. Set $u = \varphi \circ x$ and $v = \varphi \circ y$. Then $v \in E$ and $\|v\|_E = 1$, by $\varphi \in \Delta_2^E$. Moreover, by $\varphi > 0$,

$$u^* = (\varphi \circ x)^* = \varphi \circ x^* \neq \varphi \circ y^* = (\varphi \circ y)^* = v^*.$$

Furthermore,

$$u^{**}(t) = \frac{1}{t}\int_0^t \varphi\left(x^*(s)\right)ds \leq \frac{1}{t}\int_0^t \varphi\left(y^*(s)\right)ds = v^{**}(t)$$

for each $t \in I$. By strict K-monotonicity of E, we have $u \in E$ and $I_\varphi(x) = \|u\|_E < 1$. Finally, by $\varphi \in \Delta_2^E$, we obtain $\|x\|_{E_\varphi} < 1$.

(iii). By Corollary 2.8 [18], $E_\varphi \in (SC)$. Now, applying Lemma 1.1, we conclude that $E_\varphi \in (SKM)$.

Remark 2.2 The implication in Proposition 2.1 (ii) cannot be reversed, i.e. the condition $E \in (SKM)$ is not necessary for $E_\varphi \in (SKM)$. The suitable counterexample is given in Remark 2.11.

Corollary 2.3 *Let $p > 1$. If $E \in (SKM)$, then its p-convexification $E^{(p)} \in (SKM)$.*

Notice that Proposition 2.1 (ii) is a generalization of Theorem 14 from [12]. Take

$$E = \Lambda_\phi = \left\{x \in L^0 : \|x\|_{\Lambda_\phi} = \int_I x^*(t)\phi'(t)dt < \infty\right\}$$

where ϕ is concave, increasing function with $\phi(0) = \phi(0+) = 0$. If ϕ' is strictly decreasing then $\Lambda_\phi \in (SKM)$ by Theorem 2.11 from [3]. Therefore, Theorem 14 in [12] follows from our Proposition 2.1 (ii). Moreover, the assumptions $\varphi > 0$ and $\varphi < \infty$ are stated apriori in [12] and we proved that these conditions are necessary for $E_\varphi \in (SKM)$.

The space

$$M_\phi = \left\{x \in L^0 : \|x\|_{M_\phi} = \sup_{t \in I}\phi(t)x^{**}(t) < \infty\right\},$$

where ϕ is quasi-concave function on I, is called the *Marcinkiewicz function space.* M_ϕ is a symmetric Banach function space on I with the fundamental function $\phi_{M_\phi}(t) = \phi(t)$. Moreover, for any symmetric Banach function space E we have $E \overset{1}{\hookrightarrow} M_{\phi_E}$ since

$$x^{**}(t) \le \frac{1}{t}\,\|x^*\|_E \|\chi_{[0,t]}\|_{E'} = \|x\|_E \frac{1}{\phi_E(t)}$$

for any $t \in I$ (see, for example, [1] or [21]).

LEMMA 2.4 *The Marcinkiewicz function space M_ϕ is not strictly K-monotone for any quasi-concave function ϕ.*

Proof. For any Banach function spaces E, F, we have that $\phi_{E\odot F} = \phi_E \phi_F$, where $E \odot F$ denotes the pointwise product of E and F, i.e.

$$E \odot F = \{xy : x \in E \text{ and } y \in F\}$$

equipped with a quasi norm defined by the formula

$$\|z\|_{E\odot F} = \inf\left\{\|x\|_E \|y\|_F : z = xy,\ x \in E,\ y \in F\right\}.$$

By the well-known Lozanovskiĭ factorization theorem, for any Banach function space E we have that $L^1 \equiv E \odot E'$, where E' denotes the Köthe dual of E. Then, by Theorem 2 from [20],

$$t = \phi_{L^1}(t) = \phi_{E\odot E'}(t) = \phi_E(t)\phi_{E'}(t)$$

for any $t \in I$. Taking $E = \Lambda_\phi$ in the above inequality, we get $\phi_{E'}(t) = \frac{t}{\phi(t)}$, so this function must be quasi-concave. Consequently, its derivative is non-increasing (the same we may conclude from the well known equality $(\Lambda_\phi)' = M_{t/\varphi(t)}$). Let $a \in (0,1)$. Denote

$$x(t) = \begin{cases} \left(\frac{t}{\phi(t)}\right)' & \text{for } t < a, \\ 0 & \text{for } t \ge a \end{cases}$$

and

$$y(t) = \left(\frac{t}{\phi(t)}\right)'$$

for any $t \in I$. Then $x = x^*$. Moreover,

$$\int_0^t x^*(s)ds = \frac{t}{\phi(t)}$$

for any $t < a$, whence

$$x^{**}(t) = \begin{cases} \frac{1}{\phi(t)} & \text{for } t < a, \\ \frac{a}{t\phi(a)} & \text{for } t \ge a. \end{cases}$$

Thus

$$\|x\|_{M_\phi} = \max\left\{1, \sup_{t\ge a} \frac{a\phi(t)}{t\phi(a)}\right\} = 1,$$

because $\phi(t)/t$ is non-increasing. Clearly, $\|y\|_{M_\phi} = 1$. Since $x^{**} \leq y^{**}$ and $x^* \neq y^*$, by the definition of strict K-monotonicity, the proof is finished.

By Lemma 2.4 and the transposition of Lemma 1.1, we get immediately.

Corollary 2.5 *The Marcinkiewicz function space M_ϕ is not rotund.*

Corollary 2.5 is also an immediate consequence of results obtained by A. Kamińska and A.M. Parrish in [16]. Namely, they proved that the only extreme points of the unit ball $S(M_\phi)$ are $x \in S(M_\phi)$ such that $x^*(t) = \left[\frac{t}{\phi(t)}\right]'$ for all $t \in I$.

Consider also another Marcinkiewicz space M_ϕ^* than the space M_ϕ defined above, as

$$M_\phi^* = M_\phi^*(I) = \{x \in L^0(I) : \|x\|_{M_\phi^*} = \sup_{t\in I} \phi(t)x^*(t) < \infty\}.$$

The Marcinkiewicz space M_ϕ^* need not be a Banach space and always we have $M_\phi \overset{1}{\hookrightarrow} M_\phi^*$. Moreover, $M_\phi^* \overset{C}{\hookrightarrow} M_\phi$ if and only if

$$\int_0^t \frac{1}{\phi(s)}\,ds \leq C\frac{t}{\phi(t)}$$

for all $t \in I$ (see [20]). In general, M_ϕ^* is quasi-Banach function space.

Lemma 2.6 *Let $I = [0,1)$ or $I = [0,\infty)$. The Marcinkiewicz function space M_ϕ^* is not strictly K-monotone for any quasi-concave function ϕ.*

Proof. It is enough to replace the function $\frac{t}{\phi(t)}$ by $\frac{1}{\phi(t)}$ in the proof of Lemma 2.4.

The following result, related also to Lemma 1.1, describes relationship between strict monotonicity and strict K-monotonicity.

Theorem 2.7 *If $(E, \|\cdot\|)$ a symmetric space is strictly K-monotone and has property that $x^*(\infty) = 0$ for every $x \in E$, then E is strictly monotone.*

Proof Let $x, y \in E$, $0 \leq x \leq y$ and $x \neq y$. Since $x^*(\infty) = 0$, by Lemma 3.2 [13], we get $x^* \leq y^*$ and $x^* \neq y^*$. Since $x^{**} \leq y^{**}$, $x^* \neq y^*$ and, by strict K-monotonicity of $\|\cdot\|$, we obtain $\|x\| < \|y\|$. ■

Remark 2.8 Notice that the reverse conclusion does not hold, in other words even uniform monotonicity does not imply strict K-monotonicity. Indeed, considering $L^1[0,1]$ and taking $x = 2\chi_{[0,1/2]}$ and $y = \chi_{[0,1]}$ we obtain $x^* \neq y^*$, $y^{**} \leq x^{**}$ and $\|x\|_{L^1} = \|y\|_{L^1}$. Recall also that if $E \in (OC)$, then $x^*(\infty) = 0$ for any $x \in E$. Moreover, the converse is not true (see [6], Lemma 2.5 and Remark 2.1).

REMARK 2.9 We claim that Theorem 2.7 is false when the assumption that $x^*(\infty) = 0$ for any $x \in E$ is omitted. Let's focus on $E = \Lambda_\phi$ where ϕ is strictly concave and $\phi(\infty) < \infty$. It is easy to observe that the proof of the implication $(i) \Rightarrow (ii)$ of Theorem 2.11 [3] does not require the assumption that for each $x \in E$ we have $x^*(\infty) = 0$, and so Λ_ϕ is strictly K-monotone. Finally, by assumption $\phi(\infty) < \infty$ and, by Lemma 3.1 [18], it follows that the Lorentz space Λ_ϕ is not strictly monotone, which proves our claim.

Suppose w is a measurable nonnegative weight function defined on I and $0 < p < \infty$. Consider now the space

$$\Gamma_{p,w} = \left\{ x \in L^0 : \|x\|_{\Gamma_{p,w}} = \left(\int_I (x^{**}(t))^p w(t)dt \right)^{1/p} < \infty \right\}.$$

In order to $\Gamma_{p,w} \neq \{0\}$ we need to assume that w is from class D_p that is

$$W(s) := \int_0^s w(t)dt < \infty \quad \text{and} \quad W_p(s) := s^p \int_s^\alpha t^{-p} w(t)dt < \infty$$

for all $0 < s \in I$, where $\alpha = m(I)$. It is well known that $\left(\Gamma_{p,w}, \|\cdot\|_{\Gamma_{p,w}}\right)$ is a symmetric Banach (quasi Banach) function space when $p \geq 1$ (if $0 < p < 1$), respectively. Moreover, $\Gamma_{p,w}$ has the Fatou property. The spaces $\Gamma_{p,w}$ were introduced by A.P. Calderón in [2]. He was inspired by the classical Lorentz spaces

$$\Lambda_{p,w} = \left\{ x \in L^0 : \|x\|_{\Lambda_{p,w}} = \left(\int_I (x^*(t))^p w(t)dt \right)^{1/p} < \infty \right\}$$

introduced by G.G. Lorentz in [23]. The spaces $\Lambda_{p,w}$ are p-convexification of the Lorentz spaces Λ_ϕ (defined above) with $\phi' = w$. $\Gamma_{p,w}$ is an interpolation space between L^1 and L^∞ yielded by the Lions-Peetre K-method [1, 21]. Obviously, $\Gamma_{p,w} \subset \Lambda_{p,w}$. The reverse inclusion $\Lambda_{p,w} \subset \Gamma_{p,w}$ holds iff $w \in B_p$ (cf. [15]). Moreover, the spaces $\Gamma_{p,w}$ and $\Lambda_{p,w}$ are also related by Sawyer's result (Theorem 1 in [24]; see also [25]), which states that the Köthe dual of $\Lambda_{p,w}$, for $1 < p < \infty$ and $\int_0^\infty w(t)dt = \infty$, coincides with the space $\Gamma_{p',\widetilde{w}}$, where $1/p + 1/p' = 1$ and $\widetilde{w}(t) = \left(t / \int_0^t w(s)ds\right)^{p'} w(t)$.

The following result characterizes conditions under which the Lorentz space $\Gamma_{p,w}$ is strictly K-monotone. The deliberated property in $\Gamma_{p,w}$ is expressed in notion of $W(u) = \int_0^u w(s)ds$. Observe that W is strictly increasing if and only if for any $(a,b) \subset I$ we have $m((a,b) \cap \operatorname{supp}(w)) > 0$.

THEOREM 2.10 *Let $0 < p < \infty$. The space $\Gamma_{p,w}$ has strictly K-monotone quasi-norm $\|\cdot\|_{\Gamma_{p,w}}$ if and only if W is strictly increasing.*

PROOF NECESSITY. Suppose conversely that there exists $(\beta, \gamma) \subset I$ with $\beta < \gamma$ such that

$$m((\beta,\gamma) \cap \operatorname{supp}(w)) = 0. \tag{2}$$

Define

$$f = \chi_{\left(0, \frac{\beta+\gamma}{2}\right)}, \qquad g = \chi_{(0,\beta)} + \frac{1}{2}\chi_{(\beta,\gamma)}.$$

Then $f = f^* \neq g = g^*$. Observe that

$$f^{**}(t) = \chi_{(0, \frac{\beta+\gamma}{2})}(t) + \frac{\beta+\gamma}{2t}\chi_{[\frac{\beta+\gamma}{2},\alpha)}(t)$$

and

$$g^{**}(t) = \chi_{(0,\beta]}(t) + \frac{t+\beta}{2t}\chi_{(\beta,\gamma)}(t) + \frac{\beta+\gamma}{2t}\chi_{[\gamma,\alpha)}(t),$$

for all $t > 0$, whence $f^{**} \geq g^{**}$. Moreover, by equality (2), we get

$$\begin{aligned} \|f\|_{\Gamma_{p,w}}^p &= \int_0^\alpha \left(\chi_{(0,\frac{\beta+\gamma}{2})}(t) + \frac{\beta+\gamma}{2t}\chi_{[\frac{\beta+\gamma}{2},\alpha)}(t)\right)^p w(t)dt \\ &= \int_0^\beta w(t)dt + \int_\gamma^\alpha \left(\frac{\beta+\gamma}{2}\right)^p \frac{w(t)}{t^p}dt \end{aligned}$$

and

$$\begin{aligned} \|g\|_{\Gamma_{p,w}}^p &= \int_0^\alpha \left(\chi_{(0,\beta]}(t) + \frac{t+\beta}{2t}\chi_{(\beta,\gamma)}(t) + \frac{\beta+\gamma}{2t}\chi_{[\gamma,\alpha)}(t)\right)^p w(t)dt \\ &= \int_0^\beta w(t)dt + \int_\gamma^\alpha \left(\frac{\beta+\gamma}{2}\right)^p \frac{w(t)}{t^p}dt. \end{aligned}$$

Consequently, $\|f\|_{\Gamma_{p,w}} = \|g\|_{\Gamma_{p,w}}$ which provides that $\Gamma_{p,w}$ is not strictly K-monotone.

Sufficiency. Assume for the contrary that W is strictly increasing and $\Gamma_{p,w}$ is not strictly K-monotone. Then there exist $f, g \in \Gamma_{p,w}$ such that $f^{**} \leq g^{**}$, $f^* \neq g^*$ and $\|f\|_{\Gamma_{p,w}} = \|g\|_{\Gamma_{p,w}}$. Notice that $2f^{**} \leq (f^* + g^*)^{**} \leq 2g^{**}$ and, by K-monotonicity of $\|\cdot\|_{\Gamma_{p,w}}$, we get

$$2\|f^*\|_{\Gamma_{p,w}} \leq \|f^* + g^*\|_{\Gamma_{p,w}} \leq 2\|g^*\|_{\Gamma_{p,w}}.$$

Since $\|f^*\|_{\Gamma_{p,w}} = \|g^*\|_{\Gamma_{p,w}}$, it follows that

$$\int_I \left(\left(\frac{f^* + g^*}{2}\right)^{**p}(t) - f^{**p}(t)\right) w(t)dt = 0.$$

On the other hand, $m((a,b) \cap \operatorname{supp}(w)) > 0$ for any $(a,b) \subset (0,\alpha)$ and

$$\left(\frac{f^* + g^*}{2}\right)^{**}(t) - f^{**}(t) = \frac{1}{2t}\int_0^t (g^*(s) - f^*(s))\,ds = 0$$

for all $t > 0$, which implies that $f^* = g^*$ and this contradiction completes the proof. ∎

REMARK 2.11 (*i*)The converse of Lemma 1.1 is not true in general. It is enough to consider $\Gamma_{p,w}[0,\infty)$ with $\int_0^\infty w(t)dt < \infty$ and the function $W(u) = \int_0^u w(t)dt$ being strictly increasing. By Theorem 2.10, $\Gamma_{p,w}[0,\infty) \in (SKM)$ and, by Theorem 3.2 from [5], we conclude, that $\Gamma_{p,w}[0,\infty)$ is not rotund.

(*ii*) Note that a strictly K-monotone function space may contain an isometric copy of l^∞. It is enough to consider the space $\Gamma_{p,w}[0,\infty)$ with W being strictly increasing and $W(\infty) < \infty$. Then, by Theorem 2.10, $\Gamma_{p,w}[0,\infty)$ has strictly K-monotone quasi-norm $\|\cdot\|_{\Gamma_{p,w}}$, although, by Proposition 2.1 from [5], it contains an order-isometric copy of l^∞. Clearly, l^∞ is not strictly K-monotone. Since strict K-monotonicity is established on the cone of decreasing rearrangements of $\Gamma_{p,w}[0,\infty)$, possessing of order-isometric copy of l^∞ does not contradict the discussed property.

(*iii*) The condition $E \in (SKM)$ is not necessary for $E_\varphi \in (SKM)$ (cf. Proposition 2.1 (*ii*)). It is enough to take $E = \Gamma_{p,w}[0,\infty)$ with $\int_0^\infty w(t)dt = \infty$, W being not strictly increasing and $\varphi \in (SC)$. Then, by Theorem 2.2 in [5], $\Gamma_{p,w}[0,\infty) \in (SM)$ and, by Theorem 2.10, $\Gamma_{p,w}[0,\infty) \notin (SKM)$. In view of Proposition 2.1 (*iii*), $E_\varphi \in (SKM)$, because $\varphi \in (SC)$.

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