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On the equation $a^x+b^y=c^z$

1. Introduction. In 1956 it was proved by Sierpiński in [8] that the equation $3^x+4^y=5^z$ has in positive integers x, y, z only one solution, namely $\langle x, y, z\rangle = \langle 2, 2, 2\rangle$.

Jeśmanowicz ([3]) has formulated the following conjecture: If $a^2+b^2=c^2$, then the equation $a^x+b^y=c^z$ has in positive integers x, y, z exactly one solution $x=y=z=2$.

It was proved by L. Jeśmanowicz that this conjecture holds for the following equations $5^x+12^y=13^z$, $7^x+24^y=25^z$, $9^x+40^y=41^z$, $11^x+60^y=61^z$.

Later, in [5] Ko Chao proved that the equation $a^x+b^y=c^z$ for $a=2n+1$, $b=2n(n+1)$, $c=2n(n+1)+1$ has in positive integers x, y, z only one solution $x=y=z=2$ if the following conditions are satisfied:

(i) $n\equiv 1, 4, 5, 9, 10 \pmod{12}$,

(ii) $n\equiv 1 \pmod 2$ and there exist prime number p and positive integer s such that $2n+1=p^s$.

(iii) $n\not\equiv 3 \pmod 4$ and there exists prime number $p\equiv 3 \pmod 4$ such that $2n+1\equiv 0 \pmod p$.

Demjanenko ([2]) showed that the conjecture of L. Jeśmanowicz is true for $a=2n+1$, $b=2n(n+1)$, $c=2n(n+1)+1$.

Let us remark that triples $\langle 2n+1, 2n(n+1), 2n(n+1)+1\rangle$ considered above we can get from the Pythagorean triples $a=(m^2-n^2)l$, $b=2mnl$, $c=(m^2+n^2)l$ in the case $m-n=1$ and $l=1$.

Next, it was proved by Józefiak ([4]) that if

$$a=2^{2r}p^{2s}-1,\qquad b=2^{r+1}p^s,\qquad c=2^{2r}p^{2s}+1, \tag{1}$$

where r, s are positive integer numbers and p is a prime number, then the equation

$$a^x+b^y=c^z \tag{2}$$

has exactly one solution in positive integer numbers $x=y=z=2$.

Let

$$a = (2m)^2 - 1, \quad b = 2(2m), \quad c = (2m)^2 + 1; \tag{3}$$

then we remark that in the case $m = 2^{r-1} p^s$, $r \geqslant 1$, $s \geqslant 1$, we get the numbers of the form (1).

We will prove the following theorem:

THEOREM 1. *Let* $a = (2m)^2 - 1$, $b = 2(2m)$, $c = (2m)^2 + 1$; *then the equation* $a^x + b^y = c^z$ *has exactly one solution* $x = y = z = 2$ *in positive integers* x, y, z.

Mąkowski ([6]) has proved that the equation

$$13^x - 3^y = 10 \tag{4}$$

has exactly two solutions in positive integers x, y, namely $\langle x, y\rangle = \langle 1, 1\rangle$, $\langle 3, 7\rangle$, and conjectured that the equation

$$13^x - 3^y = 10^z \tag{5}$$

has no solutions in positive integers if $z > 1$.

Chidambaraswamy ([1]) has proved that this conjecture is true.

In 1969 Perisatri ([7]) has proved the following theorem:

If $a \equiv 13 \pmod{20}$ and $b \equiv 3 \pmod{20}$, then the equation $a^x - b^y = 10^z$, $z \neq 0$ has no solutions in non-negative integers.

This theorem is a more general version of Mąkowski's conjecture proved by Chidambaraswamy.

In Section 3 of this paper we will present a simple proof of conjecture of A. Mąkowski. Furthermore we will prove the following theorem:

THEOREM 2. *Let* $y > 1$ *and* $2^\alpha + 1 = p$, *where p denotes a prime number. Then the equation*

$$(2^\alpha - 1)^x + (2(2^\alpha + 1))^y = (3 \cdot 2^\alpha + 1)^z \tag{6}$$

has no solutions in positive integers x, y, z.

2. Proof of Theorem 1. Suppose that the equation

$$(4m^2 - 1)^x + (4m)^y = (4m^2 + 1)^z \tag{2.1}$$

has a solution in positive integer numbers x, y, z. From (2.1) we get

$$4m \mid 1 - (-1)^x. \tag{2.2}$$

If x is an odd number, then (2.2) implies that $4m \mid 2$ which is impossible. Then $x = 2x_1$ and by (2.1) we have

$$4m^2 \mid (4m)^y. \tag{2.3}$$

If $y = 1$, then from (2.3) we get $m = 1$ and therefore equation (2.1) can be reduced to the form $3^x + 4 = 5^z$.

By the Sierpiński's result ([8]) the last equation has no solutions in positive integers. Thus we get $y > 1$.

From (2.1) we obtain $16m^2 \mid 4m^2(z+2x_1)$ and $4 \mid z+2x_1$. Then we have $z = 2z_1$. We can present (2.1) in the form

$$2^{2y}m^y = \big((1+4m^2)^{z_1}+(1-4m^2)^{x_1}\big)\big((1+4m^2)^{z_1}-(1-4m^2)^{x_1}\big). \tag{2.4}$$

Let

$$u=(1+4m^2)^{z_1}, \qquad v=(1-4m^2)^{x_1} \tag{2.5}$$

and

$$m = 2^{s-1}m_1, \qquad \text{where } s \geqslant 1,\ (m_1, 2) = 1.$$

It is easy to see that

$$(u+v, m_1) = 1. \tag{2.6}$$

From (2.4) and (2.5) we get

$$(u+v)(u-v) = 2^{(s+1)y}m_1^y. \tag{2.7}$$

From (2.7) and (2.6) we obtain

$$u+v \mid 2^{(s+1)y}. \tag{2.8}$$

Therefore,

$$u+v = 2^k, \qquad \text{where } 1 \leqslant k \leqslant (s+1)y. \tag{2.9}$$

By (2.9) and (2.7) it follows that

$$u-v = 2^{(s+1)y-k}m_1^y. \tag{2.10}$$

From (2.9) and (2.5) we get

$$(1+4m^2)^{z_1}+(1-4m^2)^{x_1} = 2^k. \tag{2.11}$$

If $k > 1$, then by (2.11) it follows that $4 \mid 2$, which is not true. Hence $k = 1$ and therefore (2.9) and (2.10) imply that

$$u+v=2, \qquad u-v = 2^{(s+1)y-1}m_1^y. \tag{2.12}$$

From (2.12) and (2.5) we obtain

$$(1+4m^2)^{z_1} = 1+2^{sy+y-2}m_1^y, \qquad (1-4m^2)^{x_1} = 1-2^{sy+y-2}m_1^y. \tag{2.13}$$

If $x_1 = 2x_2$, then by (2.13) we get $2^{(s+1)y-2}m_1^y < 1$, which is impossible.

Therefore, x_1 is an odd number. Hence by (2.13) we have

$$(4m^2-1)^{x_1} = 2^{(s+1)y-2}m_1^y-1. \tag{2.14}$$

Since $m = 2^{s-1} m_1$, we have $2^{(s+1)y-2} m_1^y = 2^{2(y-1)} m^y$ and therefore by (2.14) it follows that

$$(2.15) \qquad (4m^2-1)^{x_1} = 2^{2(y-1)} m^y - 1.$$

For $y > 2$ by (2.15) we get $2^3 m^3 \mid 4m^2 x_1$; hence $2m \mid x_1$, which is impossible because x_1 is an odd number. Therefore, $y = 2$. Hence by (2.15) we get

$$(2.16) \qquad (4m^2-1)^{x_1} = 2^2 m^2 - 1 = 4m^2 - 1.$$

From (2.16) we obtain that $x_1 = 1$ and therefore $x = 2x_1 = 2$. Since $x = y = 2$, by (2.1) it follows that $z = 2$ and the proof is completed.

3. Simple proof of a conjecture of A. Mąkowski. Suppose that the equation

$$(3.1) \qquad 13^x - 3^y = 10^z$$

has a solution in positive integer numbers x, y, z, where $z > 1$. Let us remark that for $z > 1$ we have

$$(3.2) \qquad 10^z \equiv 0 \pmod 4.$$

From (3.1) and (3.2) we get

$$(3.3) \qquad 13^x - 3^y \equiv 1^x - (-1)^y \equiv 0 \pmod 4$$

and therefore $y \equiv 0 \pmod 2$. Similarly, $10^z \equiv 0 \pmod 5$ implies $13^x - 3^y \equiv 3^x - 3^y \equiv 0 \pmod 5$ and therefore we have $x \equiv y \pmod 4$. Since $y \equiv 0 \pmod 2$, we have $x \equiv 0 \pmod 2$.

By (3.1) for $x = 2x_1$, $y = 2y_1$ it follows that

$$(3.4) \qquad 2^z 5^z = (13^{x_1} - 3^{y_1})(13^{x_1} + 3^{y_1}).$$

It is easy to see that $(13^{x_1} - 3^{y_1}, 13^{x_1} + 3^{y_1}) = 2$. Hence by (3.4) we obtain

$$(3.5) \qquad 13^{x_1} - 3^{y_1} = 2, \qquad 13^{x_1} + 3^{y_1} = 2^{z-1} 5^z,$$

or

$$(3.6) \qquad 13^{x_1} - 3^{y_1} = 2^{z-1}, \qquad 13^{x_1} + 3^{y_1} = 2 \cdot 5^z.$$

Since $13^{x_1} - 3^{y_1} \equiv 1 \pmod 3$, (3.5) is impossible.

It remains to consider the system (3.6). Let us notice that for $z \geqslant 1$ we have $2 \cdot 5^z \equiv 2 \pmod 8$ and therefore

$$(3.7) \qquad 13^{x_1} + 3^{y_1} \equiv 2 \pmod 8.$$

Since $13^{x_1} \equiv 5^{x_1} \pmod 8$ and $5^{x_1} \equiv 1, 5 \pmod 8$, $3^{y_1} \equiv 1, 3 \pmod 8$, by (3.7) it follows that $5^{x_1} \equiv 1 \pmod 8$ for $x_1 = 2x_2$ and $3^{y_1} \equiv 1 \pmod 8$ for $y_1 = 2y_2$.

Putting $x_1 = 2x_2$ and $y_1 = 2y_2$ in the first equation of the system (3.6), we obtain

$$(3.8) \qquad (13^{x_2} - 3^{y_2})(13^{x_2} + 3^{y_2}) = 2^{z-1}.$$

But $(13^{x_2} - 3^{y_2}, 13^{x_2} + 3^{y_2}) = 2$ and, therefore, from (3.8) we get

$$(3.9) \qquad 13^{x_2} - 3^{y_2} = 2, \quad 13^{x_2} + 3^{y_2} = 2^{z-2}.$$

Now, we can remark that $13^{x_2} - 3^{y_2} \equiv 1 \pmod 3$ and therefore system (3.9) does not hold. The proof is complete.

4. Proof of Theorem 2. Suppose that equation (6) has a solution in positive integers x, y, z such that $y > 1$. First, we can remark that for $y > 1$ we have $(2(2^\alpha + 1))^y \equiv 0 \pmod 4$ and in virtue of (6) we get

$$(4.1) \qquad (3 \cdot 2^\alpha + 1)^z - (2^\alpha - 1)^x \equiv 0 \pmod 4.$$

Now, we will consider two cases: (i) $\alpha = 1$, (ii) $\alpha \geqslant 2$. Let $\alpha \geqslant 2$; then from (4.1) we obtain $1 \equiv (-1)^x \pmod 4$, and hence $x = 2x_1$.

We remark that $(2(2^\alpha + 1))^y \equiv 0 \pmod{(2^\alpha + 1)}$ and, therefore, by (6) we get

$$(4.2) \qquad (3 \cdot 2^\alpha + 1)^z - (2^\alpha - 1)^x \equiv 0 \pmod{(2^\alpha + 1)}.$$

From (4.2) we get

$$(4.3) \qquad (-2)^z \equiv (-2)^x \pmod{(2^\alpha + 1)}.$$

It is easy to see that in multiplicative group of residues $\bmod (2^\alpha + 1)$ the number 2 has the order equal to 2α. Therefore we obtain

$$(4.4) \qquad (-2)^z = ((-1) \cdot 2)^z \equiv 2^{(\alpha+1)z} \pmod{(2^\alpha + 1)}.$$

For $x = 2x_1$, we have

$$(4.5) \qquad (-2)^x \equiv 2^x \pmod{(2^\alpha + 1)}.$$

From (4.4) and (4.5) we get

$$(4.6) \qquad 2^x \equiv 2^{(\alpha+1)z} \pmod{(2^\alpha + 1)}.$$

In the theory of group of the following theorem is well known. Let G be a multiplicative group and $a \in G$, or $a = k$. Then we have

$$a^s = a^l \quad \text{iff} \quad k \mid l - s.$$

From this theorem and (4.6) we obtain

$$(4.7) \qquad (\alpha + 1) z \equiv x \pmod{2\alpha}.$$

On the other hand, be the assumption that $2^\alpha + 1 = p$, where p is a prime number and $\alpha \geqslant 2$ we get that $\alpha = 2\alpha_1$. Therefore, by (4.7) it follows that $z = 2z_1$. In this case equation (6) can be written in the form

$$(4.8) \qquad 2^y p^y = ((3 \cdot 2^\alpha + 1)^{z_1} + (2^\alpha - 1)^{x_1})((3 \cdot 2^\alpha + 1)^{z_1} - (2^\alpha - 1)^{x_1}).$$

It is easy to see that $(3 \cdot 2^\alpha + 1, 2^\alpha - 1) = 1$ and, therefore, we have

$$(4.9) \qquad ((3 \cdot 2^\alpha + 1)^{z_1} + (2^\alpha - 1)^{x_1}, (3 \cdot 2^\alpha + 1)^{z_1} - (2^\alpha - 1)^{x_1}) = 2.$$

From (4.8) and (4.9) we obtain

$$(4.10)\quad (3\cdot 2^{\alpha}+1)^{z_1}-(2^{\alpha}-1)^{x_1}=2,\qquad (3\cdot 2^{\alpha}+1)^{z_1}+(2^{\alpha}-1)^{x_1}=2^{y-1}p^{y},$$

or

$$(4.11)\quad (3\cdot 2^{\alpha}+1)^{z_1}-(2^{\alpha}-1)^{x_1}=2^{y-1},\qquad (3\cdot 2^{\alpha}+1)^{z_1}+(2^{\alpha}-1)^{x_1}=2\cdot p^{y}.$$

For $\alpha=2\alpha_1$ and by $(2^{\alpha}-1)^{x_1}\equiv 0 \pmod 3$ we have $(3\cdot 2^{\alpha}+1)^{z_1}-(2^{\alpha}-1)^{x_1}\equiv 1 \pmod 3$ and, therefore, from (4.10) it follows that $2\equiv 1 \pmod 3$ which is impossible.

It remains to consider the system (4.11). Let us notice that

$$(4.12)\qquad (3\cdot 2^{\alpha}+1)^{z_1}\equiv (2^{\alpha}+1)^{z_1} \pmod{2^{\alpha+1}}$$

and

$$(4.13)\qquad 2p^{y}=2(2^{\alpha}+1)^{y}\equiv 2 \pmod{2^{\alpha+1}}.$$

From (4.11), (4.12) and (4.13) we obtain

$$(4.14)\qquad (2^{\alpha}+1)^{z_1}+(2^{\alpha}-1)^{x_1}\equiv 2 \pmod{2^{\alpha+1}}.$$

It is easy to show that

$$(4.15)\qquad (2^{\alpha}+1)^{z_1}\equiv 1,\quad 2^{\alpha}+1 \pmod{2^{\alpha+1}},$$

$$(4.16)\qquad (2^{\alpha}-1)^{x_1}\equiv 1,\quad 2^{\alpha}-1 \pmod{2^{\alpha+1}}.$$

By (4.14), (4.15) and (4.16) it follows that $(2^{\alpha}+1)^{z_1}\equiv 1 \pmod{2^{\alpha+1}}$ and $(2^{\alpha}-1)^{x_1}\equiv 1 \pmod{2^{\alpha+1}}$. Thus $x_1=2x_2$, and $z_1=2z_2$ and from the first equation of (4.11) we obtain

$$(4.17)\qquad \big((3\cdot 2^{\alpha}+1)^{z_2}-(2^{\alpha}-1)^{x_2}\big)\big((3\cdot 2^{\alpha}+1)^{z_2}+(2^{\alpha}-1)^{x_2}\big)=2^{y-1}.$$

Since

$$\big((3\cdot 2^{\alpha}+1)^{z_2}-(2^{\alpha}-1)^{x_2},(3\cdot 2^{\alpha}+1)^{z_2}+(2^{\alpha}-1)^{x_2}\big)=2,$$

by (4.17) we get

$$(4.18)\quad (3\cdot 2^{\alpha}+1)^{z_2}-(2^{\alpha}-1)^{x_2}=2,\qquad (3\cdot 2^{\alpha}+1)^{z_2}+(2^{\alpha}-1)^{x_2}=2^{y-2}.$$

Since $\alpha=2\alpha_1$ and $(2^{2\alpha_1}-1)^{x_2}\equiv 0 \pmod 3$, we have $(3\cdot 2^{\alpha}+1)^{z_2}-(2^{\alpha}-1)^{x_2}\equiv 1 \pmod 3$ and therefore by (4.18) it follows that $2\equiv 1 \pmod 3$ which is impossible.

In the case $\alpha\geqslant 2$ the proof is finished.

Let $\alpha=1$; then equation (6) can be reduced to the form

$$(4.19)\qquad 1+6^{y}=7^{z}.$$

For $z>1$ we have, from (4.19), $y>1$ and $9\mid 6^{y}$.

We remark that the number 7 belongs to exponent 3 (mod 9) and, therefore, we have $3 \mid z$. Thus we have $19 \mid 7^3 - 1 \mid 7^z - 1 = 6^y$ which is impossible.

The proof is thus completed.

Added in correction. Professor A. Schinzel in his letter (29.11.1983) has informed one of the authors that first theorem of our paper has been proved by Lu-Wen-Twan in the paper *On Pythagorean numbers* $4n^2-1$, $4n$, $4n^2+1$ (in Chinese), Acta Sci. Natur. Univ. Szechuan (1959), 39–42. We have not known this paper and to this time we have not any possibility to compare the methods of the proof of the above-mentioned result. Moreover, A. Mąkowski has remarked that some generalization of his conjecture was given by Toyoizumi in paper *On the equation* $a^x - b^y = (2p)^z$, Studia Math. 46 (1978), 113–115.

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