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Covering spaces revisited

1. Introduction. In this paper we formulate a covering space analogue of the so-called K-theory constructed from vector bundles.

We begin in Section 2 by introducing sum and product operations for coverings. Sum is merely disjoint union while product is the well-known "fiber product". Since connected coverings of a space correspond to conjugacy classes of subgroups of its fundamental group, it is natural to ask for a group theoretic interpretation of product. Very crudely, product corresponds to intersection of subgroups as we show in Section 3.

Having developed our covering space preliminaries we proceed in Section 4 to the K-ish-like functor k and compute its structure. Next, the associated reduced functor $\tilde{k}$ is defined by the usual process in Section 5 and is shown to be non-representable and non-half-exact. In the next section we show that it fails to satisfy the wedge axiom as well. Finally, in Section 7 we introduce the notion of based coverings and modify $\tilde{k}$ to obtain $\tilde{k}'$, a non-half-exact functor which actually does satisfy the wedge axiom.

2. Operations on coverings. Let $\xi = \{E, p, X\}$ be a covering as defined in Chapter 2 of [3] with the exception that for convenience we shall assume (always) that the base space X is pathwise connected. As a consequence all fibers $p^{-1}(x) \subset E$, $x \in X$, have the same cardinal number which we will call the *rank* of ξ. If $rk(\xi) = n$, ξ is then an *n-fold* covering. We say that ξ is *connected* if the total space $E = E(\xi)$ is pathwise connected.

If $f: Y \to X$ is a (continuous) map there is induced a covering $f^*\xi$ over Y with $rk(f^*\xi) = rk(\xi)$, $E(f^*\xi) = \{(y, e) | f(y) = p(e)\} \subset Y \times E(\xi)$, and $p(f^*\xi)$ the restriction of the projection map onto the first factor. See p. 98 of [3] for further details.

For $i = 1, 2$, let $\xi_i = \{E_i, p_i, X_i\}$ be an n_i-fold covering and define their *external sum* by

$$\xi_1 \hat{\oplus} \xi_2 = \{E_1 \times X_2 + X_1 \times E_2, p, X_1 \times X_2\},$$

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where $+$ denotes disjoint union and p is either $p_1 \times 1$ or $1 \times p_2$ (1 is the identity map) depending on which portion of the total space we are dealing with. Then $p^{-1}(x_1, x_2)$ is homeomorphic to $p_1^{-1}(x_1)+p_2^{-1}(x_2)$ and $\xi_1 \hat{\oplus} \xi_2$ is an (n_1+n_2)-fold covering. *External product* is defined by

$$\xi_1 \hat{\otimes} \xi_2 = \{E_1 \times E_2, p_1 \times p_2, X_1 \times X_2\}.$$

The fiber over (x_1, x_2) is $p_1^{-1}(x_1) \times p_2^{-1}(x_2)$ and $\xi_1 \hat{\otimes} \xi_2$ is an $n_1 n_2$-fold covering. Notice that if $U_i \subset X_i$ is evenly covered by p_i, then $U_1 \times U_2$ is evenly covered in both constructions verifying that we indeed have coverings.

The following arithmetic properties are easily established:

(2.1) *Both* $\hat{\oplus}$ *and* $\hat{\otimes}$ *are associative.*

(2.2) *Let* $T: Y \times X \approx X \times Y$ *be the transposition homeomorphism. If* ξ *is a covering of* X *and* η *a covering of* Y, *then* $T^*(\xi \hat{\oplus} \eta) \cong \eta \hat{\oplus} \xi$ *and analogously for* $\hat{\otimes}$.

(2.3) *If* f *and* g *are maps to* X *and* Y *respectively, then* $(f \times g)^*(\xi \hat{\oplus} \eta) \cong f^* \xi \hat{\oplus} g^* \eta$ *and similarly for* $\hat{\otimes}$.

Of course the distributive laws don't make sense since the two sides of the formula have different base space, so we now pass from the external to the internal operations. In detail, if ξ and η are coverings of X let $\Delta : X \to X \times X$ be the diagonal map $\Delta(x) = (x, x)$ and define

$$\xi \oplus \eta = \Delta^*(\xi \hat{\otimes} \eta), \quad \xi \otimes \eta = \Delta^*(\xi \hat{\oplus} \eta).$$

The sum operation is very simple indeed: $E(\xi \oplus \eta) \approx E(\xi)+E(\eta)$; thus each covering is uniquely a sum of connected coverings. The product operation is more interesting and most of this paper is devoted to its study.

Using the above properties of the external operations one can easily prove the following result:

PROPOSITION (2.4). *The operations of sum and product are associative and commutative, and the distributive laws hold. The trivial* 1*-fold covering* $O^1 = \{X, 1, X\}$ *is a multiplicative identity. Moreover, if* f *is a map to* X, *then* $f^*(\xi \oplus \eta) \cong f^* \xi \oplus f^* \eta$ *and similarily for* $\otimes$.

3. Relation with the fundamental group. For simplicity assume that X is a (connected simplicial or CW) complex. Then the set of isomorphism classes of *connected* coverings of X is in one-one correspondence with the set of conjugacy classes of subgroups of the fundamental group $\pi(X)$ of X in the following way: given covering $\xi = \{E, p, X\}$, choose a base point $e_0 \in E$ over the base point $x_0 \in X$; then ξ corresponds to the conjugacy class of $p_* \pi(E, e_0) \subset \pi(X, x_0)$. Varying e_0 in $p^{-1}(x_0)$ produces the various members of the conjugacy class. For a general reference see, for example, Chapter 5 of [2], Theorems (6.6) and (10.2) in particular.

Although one could classify coverings of X by using collections of conjugacy classes in $\pi(X)$ in the obvious way, little insight would be obtained. However, the product notion formulates nicely in terms of $\pi(X)$ as the next result shows.

THEOREM 3.1. *Let $\xi = \{E, p, X\}$ and $\eta = \{F, q, X\}$ be connected coverings of a complex X. Now*

$$\xi \otimes \eta \cong \zeta_1 \oplus \ldots \oplus \zeta_k$$

in a unique way with each ζ_i a connected covering of X. If $(x_0, e_0, f_0) \in E(\xi\otimes\eta) \subset X \times E \times F$ is base point for $E(\zeta_i)$, then ζ_i corresponds to the conjugacy class of the subgroup $p_*\pi(E, e_0) \cap q_*\pi(F, f_0)$ of $\pi(X, x_0)$ and $rk(\zeta_i)$ is the index $[\pi(X) : p_*\pi(E) \cap q_*\pi(F)]$.

Because of distributivity (3.1) can be used to obtain information on $\xi\otimes\eta$ even if ξ and η are not connected.

Before proving (3.1) we deduce a number of interesting consequences. First observe that if $\pi(X)$ is abelian each conjugacy class reduces to a single subgroup and the images of p_* and q_* are independent of the choice of e_0 and f_0. Hence all the ζ_i are the same. Also, $rk(\xi) = [\pi(X) : p_*\pi(E)]$ (see p. 162 of [2]) and we obtain from (3.1)

THEOREM (3.2). *Let X be a complex with $\pi(X)$ abelian. If ξ and η are connected coverings of X there exists a connected covering ζ of X such that*

$$\xi \otimes \eta \cong k\zeta,$$

where $k = rk(\zeta) rk\eta / rk\zeta$. Moreover, if ξ corresponds to subgroup H of $\pi(X)$ and η to $K \subset \pi(X)$, then ζ corresponds to $H \cap K$ and $rk(\zeta) = [\pi(X) : H \cap K]$.

If $\eta = \zeta$, then $H = K$ and we obtain

COROLLARY (3.3). *If ζ is a connected covering of a complex X with $\pi(X)$ abelian, then*

$$\zeta\otimes\zeta \cong rk(\zeta)\zeta.$$

Proof of (3.1). We must establish the facts about ζ_i. Write $D = E(\zeta\otimes\eta)$, $r = p(\xi\otimes\eta)$, $d_0 = (x_0, e_0, f_0)$, $H = p_*\pi(E, e_0)$, $K = q_*\pi(F, f_0)$ and $M = r_*\pi(D, d_0)$. It suffices to show that $M = H \cap K$ since ζ_i corresponds to the conjugacy class of M and $rk(\zeta_i) = [\pi(X, x_0) : M]$ by [2], p. 162.

First, $M \subset H \cap K$. To see this restrict the commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{h} & E \times F \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p\times q} \\ X & \xrightarrow{\Delta} & X \times X \end{array},$$

where $h(x, e, f) = (e, f)$, to $E(\zeta_i) \subset D$ and apply the functor π. Since π commutes with cartesian product we obtain the commutative diagram

$$\begin{array}{ccc} \pi(E(\zeta_i), d_0) & \xrightarrow{h_*} & \pi(E, e_0) \times \pi(F, f_0) \\ {\scriptstyle r_*}\downarrow & & \downarrow{\scriptstyle p_* \times q_*} \\ M & & H \times K \\ \cap & & \cap \\ \pi(X, x_0) & \xrightarrow{\Delta_*} & \pi(X, x_0) \times \pi(X, x_0) \end{array}$$

But r_*, p_*, q_* are monomorphisms (Theorem 4.1, p. 154, of [2]), are onto M, H, K respectively, and Δ_* is the diagonal homomorphism. Hence $M \subset H$ and $M \subset K$.

Finally, we show that $M \supset H \cap K$. If $\alpha \in H \cap K$, let $\lambda : S^1 \to E$, $\mu : S^1 \to F$ be base point preserving maps such that $\alpha = p_*[\lambda] = q_*[\mu]$, where $[\lambda] \in \pi(E, e_0)$ denotes the class of λ. Let $G : S^1 \times I \to X$ be a based homotopy from $p\lambda$ to $q\mu$. Applying the covering homotopy theorem ([3], p. 67) to the commutative diagram

$$\begin{array}{ccc} S^1 \times \{0\} & \xrightarrow{\lambda} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 \times I & \xrightarrow{G} & X \end{array}$$

we obtain a based homotopy $G' : S^1 \times I \to E$ such that $G'i = \lambda$ and $pG' = G$. Setting $\lambda'(z) = G'(z, 1)$, we have G' as a based homotopy $\lambda \simeq \lambda'$ and $p\lambda' = q\mu$. Then $\sigma : S^1 \to X \times E \times F$ defined by $\sigma(z) = (q\mu(z), \lambda'(z), \mu(z))$ is a based map with $\sigma(S^1) \subset D$. Since r is projection onto the first factor, we have

$$\alpha = [q\mu] = [r\sigma] = r_*[\sigma] \in M$$

and the proof is complete.

4. The functor k. It will be convenient to introduce a unique 0-fold covering, namely the *empty covering* $\emptyset$ whose total space is the empty set $\emptyset$. Then $\emptyset$ is an additive identity for the sum operation.

Let $\mathrm{Cov}_n(X)$ be the set of isomorphism classes of n-fold coverings of X, $0 \leqslant n < \infty$, and let $\mathrm{Cov}(X)$ be their union.

By a *semiring* we shall mean an algebraic system satisfying the axioms of a commutative associative ring with identity except for the existence of additive inverses. Then (2.4) may be reformulated as follows:

(4.1) Cov *is a contravariant semiring valued functor from the category* $\mathscr{C}$ *of connected spaces.*

The contravariant functor $k : \mathscr{C} \to \mathscr{R}$ to the category $\mathscr{R}$ of rings is defined as the ring completion of Cov (see p. 103 of [1]). In detail, define $k(X) = \mathrm{Cov}(X) \times \mathrm{Cov}(X)/\sim$, where $\sim$ is the equivalence relation

$(\xi, \eta) \sim (\xi', \eta')$ iff there exists $\zeta \in \mathrm{Cov}(X)$ such that $\xi \oplus \eta' \oplus \zeta \cong \xi' \oplus \eta \oplus \zeta$; since $\oplus$ is disjoint union, $\sim$ becomes: $(\xi, \eta) \sim (\xi', \eta')$ iff $\xi \oplus \eta' \cong \xi' \oplus \eta$. Write $\mathrm{cls}(\xi, \eta)$ for the equivalence class of (ξ, η) and define

$$(4.2) \qquad \begin{aligned} \mathrm{cls}(\xi, \eta) + \mathrm{cls}(\varrho, \sigma) &= \mathrm{cls}(\xi \oplus \varrho, \eta \oplus \sigma), \\ \mathrm{cls}(\xi, \eta)\mathrm{cls}(\varrho, \sigma) &= \mathrm{cls}(\xi \otimes \varrho \oplus \eta \otimes \sigma, \eta \otimes \varrho \oplus \xi \otimes \sigma). \end{aligned}$$

Then $k(X)$ becomes a commutative associative ring with identity; moreover, $\mathrm{cls}(\xi, \xi) = 0$ and $-\mathrm{cls}(\xi, \eta) = \mathrm{cls}(\eta, \xi)$. As for naturality, if $f: Y \to X$ set $f^* \mathrm{cls}(\xi, \eta) = \mathrm{cls}(f^* \xi, f^* \eta)$. In this way k becomes a functor.

EXAMPLE (4.3). *If X is simply connected, $k(X) \cong \boldsymbol{Z}$.*

The proof is elementary.

Define $\psi: \mathrm{Cov}(X) \to k(X)$ be sending ξ to $\mathrm{cls}(\xi, \emptyset)$. Because of (4.2), ψ is a semiring homomorphism which is natural in X. Also, $\psi(\xi) = \psi(\eta)$ implies that $\xi \otimes \emptyset \cong \eta \otimes \emptyset$ and we obtain

(4.4) $\psi: \mathrm{Cov} \to k$ *is an injective natural transformation of semiring valued functors.*

This result allows us to consider $\mathrm{Cov}(X)$ as a subsemiring of $k(X)$. Further,

$$\mathrm{cls}(\xi, \eta) = \mathrm{cls}(\xi, \emptyset) + \mathrm{cls}(\emptyset, \eta) = \mathrm{cls}(\xi, \emptyset) - \mathrm{cls}(\eta, \emptyset) = \xi - \eta,$$

so we may think of $k(X)$ as being obtained from $\mathrm{Cov}(X)$ by merely throwing in negatives of coverings.

THEOREM (4.5). *If X is a complex with abelian fundamental group, then $k(X)$ is additively the free abelian group on the collection of subgroups of $\pi(X)$ having finite index. Multiplication of generators is given as follows: the product of subgroups H and K is*

$$\frac{[\pi(X) : H][\pi(X) : K]}{[\pi(X) : H \cap K]} H \cap K.$$

Proof. Now $\mathrm{Cov}(X) \subset k(X)$ and $\mathrm{Cov}(X)$ is additively generated by isomorphism classes of connected coverings, each of which corresponds to a subgroup of $\pi(X)$ of finite index. Since there are no additive relations in $\mathrm{Cov}(X)$ the stated additive structure of $k(X)$ follows. As for the multiplicative structure, simply apply (3.2).

Remark. Using (3.1) instead of (3.2), we can similarly prove a result for the case $\pi(X)$ non-abelian. However, its statement is unduly cumbersome.

5. The reduced functor $\tilde{k}$. The semiring homomorphism $rk: \mathrm{Cov}(X) \to \boldsymbol{Z}$ assigning the rank to each covering extends to a ring homomorphism $rk: k(X) \to \boldsymbol{Z}$ by $rk(\mathrm{cls}(\xi, \eta)) = rk\xi - rk\eta$. Its kernel $\tilde{k}(X)$ is thus an ideal in $k(X)$. Since $rk(f^*\xi) = rk(\xi)$, $\tilde{k}$ becomes a contravariant functor $\tilde{k}: \mathscr{C} \to \mathscr{R}$ with

$$\tilde{k}(X) = \{\mathrm{cls}(\xi, \eta) \mid rk\xi = rk\eta\}.$$

Example (5.1). *If X is simply connected, then $\tilde{k}(X) = 0$.*

Remark. Up to this point our theory is formally identically to K-theory but now we start observing differences. For example, $\tilde{k}(X)$ is *not* isomorphic to the set of stable equivalence classes formed from $\mathrm{Cov}(X)$. To see this notice that the set of stable classes with the usual addition has additive inverses iff a "Whitney inverse theorem" holds for coverings; by the definition of addition, the latter is not valid.

For convenience we now restrict the domain category to $\mathscr{W}$, the category of connected pointed finite CW complexes. Recall that a contravariant functor $F: \mathscr{W} \to \mathscr{R}$ is *representable* (on $\mathscr{W}$) if there exists a based *classifying space B* such that F is functorially isomorphic to $[B]$, where $[X, B]$ is the set of based homotopy classes of maps from X to B. Although most geometrically constructed contravariant functors are representable we have

Theorem (5.2). *$\tilde{k}$ is not representable on $\mathscr{W}$.*

Actually, we shall prove something stronger. Following A. Dold we say that a contravariant functor $F: \mathscr{W} \to \mathscr{R}$ is *half exact* if for every sequence $A \overset{i}{\subset} X \overset{q}{\to} X/A$ in $\mathscr{W}$, where X/A is the space obtained from X by identifying A to a point, the induced sequence $F(A) \leftarrow F(X) \leftarrow F(X/A)$ is exact. For example, every representable functor is half exact; this is a simple exercise with the homotopy extension theorem. Thus (5.2) follows from

Theorem (5.3). *$\tilde{k}$ is not half exact.*

Proof. Let $* = 1 \in S^1$ be the base point, and set $X = S^1 \times S^1$, $A = S^1 \times \{*\}$ with $(*, *)$ as base point. We shall show that the sequence

$$\tilde{k}(X/A) \overset{q^*}{\to} \tilde{k}(X) \overset{i^*}{\to} \tilde{k}(A)$$

is not exact.

Let ξ be the usual 2-fold cover $p: E = S^1 \to S^1$ and write $\xi \times \xi$ in place of $\xi \hat{\otimes} \xi$. We assert that $\xi \times \xi$ does *not* belong to the image of $q^*: \mathrm{Cov}(X/A) \to \mathrm{Cov}(X)$. Suppose instead that $\xi \times \xi \simeq q^*\zeta$, where $\zeta = \{E', p', X/A\}$ is a 4-fold covering. Equivalently, we would have a commutative diagram

$$\begin{array}{ccc} E \times E & \xrightarrow{h} & E' \\ \downarrow{\scriptstyle p \times p} & & \downarrow{\scriptstyle p'} \\ X = S^1 \times S^1 & \xrightarrow{q} & X/A \end{array}$$

in which the map h is bijective on fibers. Restricting the diagram to A gives

$$\begin{array}{ccc} E\times(2\text{ points}) & \xrightarrow{h} & (4\text{ points}) \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p'} \\ A = S^1\times\{*\} & \xrightarrow{q} & \{*\} \end{array}$$

But E is connected and h is continuous, so h is not onto contrary to bijectivity of h on fibers.

Returning to our above sequence observe that $\xi\times\xi$ and $\xi\times O^2$ are non-isomorphic 4-fold covers of X which restrict to isomorphic covers of A. Since $\operatorname{cls}(\xi\times\xi,\ \xi\times O^2)\in\tilde{k}(X)$ is non-zero but maps by i^* to 0 in $\tilde{k}(A)$, it suffices to show that this element is not in the image of q^*. Suppose instead that $\operatorname{cls}(\xi\times\xi,\ \xi\times O^2) = q^*\operatorname{cls}(\zeta,\eta)$. Writing $\zeta = \zeta_1\oplus\ldots\oplus\zeta_r$ in terms of components we obtain

$$q^*\zeta_1\oplus\ldots\oplus q^*\zeta_r\oplus\xi\times O^2 \cong \xi\times\xi\oplus q^*\eta.$$

Because of our choice of spaces each $q^*\zeta_i$ is connected as is $\xi\times\xi$. It follows that $\xi\times\xi\cong q^*\zeta_i$ for some i contrary to the above paragraph, a contradiction which completes the proof.

6. Behavior on wedges. The *wedge sum* $X\vee Y$ of spaces $X, Y\in\mathscr{W}$ is obtained from the disjoint union $X+Y$ by identifying base points of X and Y. There is then the natural homomorphism $\Psi:\tilde{k}(X\vee Y)\to\tilde{k}(X)\oplus\tilde{k}(Y)$ defined by sending $\operatorname{cls}(\xi,\eta)$ to $\operatorname{cls}(\xi|X,\eta|X)\oplus\operatorname{cls}(\xi|Y,\eta|Y)$. Moreover,

(6.1) *Φ is an epimorphism.*

Proof. Given $\operatorname{cls}(\alpha,\beta)\in\tilde{k}(X)$ and $\operatorname{cls}(\gamma,\delta)\in\tilde{k}(Y)$, with $m = rk(\alpha) = rk(\beta)$ and $n = rk(\gamma) = rk(\delta)$, form an m-fold cover $\alpha\vee O^m$ of $X\vee Y$ in the evident way. Setting $\xi = \alpha\vee O^m\oplus O^n\vee\gamma$, $\eta = \beta\vee O^m\oplus O^n\vee\delta$, we find $\operatorname{cls}(\xi,\eta)\in\tilde{k}(X\vee Y)$ is mapped by Φ to

$$\operatorname{cls}(\alpha\oplus O^n,\ \beta\oplus O^n)\oplus\operatorname{cls}(O^m\oplus\gamma,\ O^m\oplus\delta) = \operatorname{cls}(\alpha,\beta)\oplus\operatorname{cls}(\gamma,\delta).$$

(6.2) *Injectivity of Φ is equivalent to the following condition: if $\xi,\eta\in\operatorname{Cov}(X\vee Y)$ are such that $\xi|X\cong\eta|X$, $\xi|Y\cong\eta|Y$, then $\xi\cong\eta$.*

We assert that Φ *is is not injective.* Indeed, we have

EXAMPLE (6.3). Let $\zeta = \{S^1, p, S^1\}$ be the usual 3-fold covering with $p(z) = z^3$. The fiber over the base point $* = 1\in S^1$ is then the set $\{a = 1,\ b = \exp(2\pi i/3),\ c = b^2\}$. Form 3-fold covering ξ of $S^1\vee S^1$ by gluing two copies of ζ together matching the fibers by the identity function. To define η we put in a twist in the matching process by identifying a with a, b with c, and c with b. Certainly the restrictions of ξ and η to either copy of S^1 in the wedge are isomorphic but we claim that ξ and η

are *not* isomorphic. To see this suppose instead the there were a homeomorphism $f: E(\xi) \approx E(\eta)$ such that $p(\xi) = p(\eta)f: E(\xi) \to S^1 \vee S^1$. Restricting to either copy of S^1 we get an automorphism of ζ, and these are classified by the elements of

$$\pi(S^1, *)/p_*\pi(S^1, *) \simeq \mathbf{Z}/3\mathbf{Z} \simeq \mathbf{Z}_3;$$

see Corollary 7.4, p. 163, of [2], for example. Thus an automorphism must induce a cyclic permutation on the fiber $\{a, b, c\}$. Suppose f induced the identity permutation over the first copy of S^1 in $S^1 \vee S^1$; over the second copy f would induce the *odd* permutation $a \to a$, $b \to c$, $c \to b$, a contradiction. Similar arguments yield contradictions for the remaining two cases of posible behavior of f on the first copy of ζ. Thus f does not exist and the proof is complete.

Remark. It is easily shown ([4], (2.1)) that half exactness implies that Φ is an isomorphism. Hence (6.3) gives an alternate proof of (5.3). However, the earlier proof of (5.3) is still needed since we will make reference to it in the next section.

7. Based coverings and $\tilde{k}'$. Our aim here is to modify the definition of $\tilde{k}$ so as to make Φ injective. Reflection on (6.2) and (6.3) leads one to introduce the following notion. A *based n-fold covering* of $X \in \mathscr{W}$ is a pair (ξ, i), where $\xi \in \mathrm{Cov}_n(X)$ and i is a bijection of $\mathbf{Z}_n$ with the fiber of ξ over the base point of X. Based n-fold coverings (ξ, i) and (η, j) of X are isomorphic if there is an isomorphism $f: \xi \simeq \eta$ such that $j = fi$. Using these notions we define functions Cov', k' and $\tilde{k}'$ in the obvious way. Moreover, we have

THEOREM (7.1). *$\tilde{k}': \mathscr{W} \to \mathscr{R}$ is a contravariant functor which is not half exact yet satisfies the wedge axiom $\Phi: \tilde{k}'(X \vee Y) \simeq \tilde{k}'(X) \oplus \tilde{k}'(Y)$.*

Since the wedge axiom is such an easy consequence of half exactness, this result is of some interest. As for its validity, a look at the proofs of (5.3) and (6.1) shows that they also work for $\tilde{k}'$, so it remains only to verify the condition of (6.2). But if (ξ, i) and (η, j) are coverings of $X \vee Y$ which are isomorphic when restriced to both X and Y, then the isomorphisms are compatible over the base point of $X \vee Y$ and therefore can be glued together to give $(\xi, i) \simeq (\eta, j)$; notice that this was not the case in example (6.3).

References

[1] D. Husemoller, *Fibre bundles*, New York 1966.
[2] W. Massey, *Algebraic topology, an introduction*, New York 1967.
[3] E. Spanier, *Algebraic topology*, New York 1966.
[4] R. W. West, *On the representability of half exact functors over non-connected CW complexes*, Ill. J. Math. 11 (1967), p. 64-70.