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On symmetric spaces containing isomorphic copies of Orlicz sequence spaces

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Summary. Let an Orlicz function N be $(1+\varepsilon)$-convex and $(2-\varepsilon)$-concave at zero for some $\varepsilon > 0$. Then the function $1/N^{-1}(t)$, $0 < t \leqslant 1$, belongs to a separable symmetric space X with the Fatou property, which is an interpolation space with respect to the couple (L_1, L_2), whenever X contains a strongly embedded subspace isomorphic to the Orlicz sequence space l_N. On the other hand, we find necessary and sufficient conditions on such an Orlicz function N under which a sequence of mean zero independent functions equimeasurable with the function $1/N^{-1}(t)$, $0 < t \leqslant 1$, spans, in the Marcinkiewicz space $M(\varphi)$ with $\varphi(t) := t/N^{-1}(t)$, a strongly embedded subspace isomorphic to the Orlicz sequence space l_N.

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Dedicated to Henryk Hudzik
on the occasion of his 70th birthday.

1. Introduction

Whereas the class of all subspaces of $L_1 = L_1(0,1)$ is so rich that it still does not have any reasonable description, far more information is available on subspaces of L_1 isomorphic to Orlicz spaces. First of all, an arbitrary subspace of L_1 isomorphic to an Orlicz sequence space $l_N \neq l_1$ can always be given by the span of appropriate sequence of independent identically distributed random variables. The latter fact was discovered in the case $N(t) = t^q$, $1 < q < 2$, by Kadec in 1958 [21]. More precisely, he proved that for arbitrary

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$1 \leqslant p < q < 2$ there exists a symmetrically distributed function $f \in L_p$ (a q-stable random variable) such that the sequence $\{f_k\}_{k=1}^{\infty}$ of independent copies of f spans in L_p a subspace isomorphic to l_q.

This direction of study was taken further by Bretagnolle and Dacunha-Castelle (see [14–16]). In particular, Dacunha-Castelle showed that for every mean zero $f \in L_p = L_p(0,1)$ the sequence $\{f_k\}_{k=1}^{\infty}$ of its independent copies is equivalent in L_p, $1 \leqslant p < 2$, to the unit vector basis of some Orlicz sequence space l_N [16, Theorem 1, p.X.8]. Moreover, Bretagnolle and Dacunha-Castelle proved that an Orlicz function space $L_N = L_N[0,1]$ can be isomorphically embedded into the space L_p, $1 \leqslant p < 2$, if and only if N is equivalent to an Orlicz function that is p-convex and 2-concave at zero [15, Theorem IV.3]. It should be mentioned that later some of these results were independently rediscovered by Braverman [11, 12].

The papers [11, 12, 14–16] exploit methods which depend heavily on techniques related to the theory of random processes. In contrast to that, in more recent papers [6] and [8], an approach based on methods and ideas from the interpolation theory of operators was used. In particular, [6, Theorem 9] and [8, Theorem 1.1] imply the following: *Let $1 \leqslant p < 2$ and let the Orlicz function N be $(p+\varepsilon)$-convex and $(2-\varepsilon)$-concave at zero for some $\varepsilon > 0$. If L_p contains a subspace isomorphic to the Orlicz sequence space l_N, then the function $1/N^{-1}(t)$, $0 < t \leqslant 1$, belongs to L_p.* The main aim of the present paper is to extend the above result from L_p-spaces to the more general class of interpolation symmetric spaces with respect to the couple (L_1, L_2) (Theorem 3.1). Note that in the case when $N(t) = t^{1/q}$, $1 < q < 2$, a similar result was proved by Raynaud [29] for *every* separable symmetric space, by using a completely different approach based on the profound theorem of Dacuncha--Castelle and Krivine on structure of l_q-subspaces of L_1 from [17].

In the final part of the paper, a result, which in a sense is converse of Theorem 3.1, is obtained. We find necessary and sufficient conditions on an Orlicz function N, $(1+\varepsilon)$-convex and $(2-\varepsilon)$-concave at zero for some $\varepsilon > 0$, under which a sequence of mean zero independent functions equimeasurable with the function $1/N^{-1}(t)$, $0 < t \leqslant 1$, spans the Orlicz sequence space l_N in every symmetric space X such that $X \supset M(\varphi)$ ($M(\varphi)$ being the Marcinkiewicz space generated by the function $\varphi(t) := t/N^{-1}(t)$) (Theorem 3.8).

2. Preliminaries

Recall the basic definitions from the theory of symmetric spaces (its detailed exposition can be found in the books [9, 23, 24]).

Let $I = [0,1]$ or $[0,\infty)$. By $x^*(s)$ we denote the *non-increasing left-continuous rearrangement* of the absolute value of the measurable function $x = x(t)$, $t \in I$, i.e.,

$$x^*(s) := \inf\{\tau > 0 : m\{t \in I : |x(t)| > \tau\} < s\},$$

where m is the Lebesgue measure. Two measurable functions $x(t)$ and $y(t)$, $t \in I$, are called *equimeasurable* if $x^*(s) = y^*(s)$, $s \in I$.

A Banach function space X on I is said to be *symmetric* if from $y = y(t) \in X$ and $x^*(t) \leqslant y^*(t)$, $t \in I$, it follows that $x = x(t) \in X$ and $\|x\|_X \leqslant \|y\|_X$.

We begin with defining some classes of symmetric spaces. Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. Then the space $L_{p,q}$ is defined as the set of all measurable functions on I for which the following quasi-norm is finite:

$$\|x\|_{p,q} := \left(\frac{q}{p}\int_I \left(t^{1/p}x^*(t)\right)^q \frac{dt}{t}\right)^{1/q}, \quad 1 \leqslant q < \infty,$$

and

$$\|x\|_{p,\infty} := \sup_{t\in I, t\neq 0} t^{1/p}x^*(t).$$

Replacing in the preceding formulas $x^*(t)$ with $x^{**}(t) := \frac{1}{t}\int_0^t x^*(s)\,ds$, we get an equivalent symmetric norm in $L_{p,q}$, for every $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. We have $L_p = L_{p,p}$ and $L_{p,q_1} \subset L_{p,q_2}$ if $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$.

Another natural generalization of L_p-spaces are *Orlicz spaces* (see [22, 24]). Let $N(u)$ be an *Orlicz function*, that is, an increasing convex function on $[0,\infty)$ such that $N(0) = 0$. The Orlicz space L_N consists of all measurable functions $x(t)$ on I such that the function $N(|x(t)|/\rho) \in L_1$ for some $\rho > 0$. It is equipped with the *Luxemburg norm*

$$\|x\|_{L_N} := \inf\left\{\rho > 0 : \int_I N\left(\frac{|x(t)|}{\rho}\right)dt \leqslant 1\right\}.$$

In particular, if $N(u) = u^p$, $1 \leqslant p < \infty$, we obtain usual L_p-spaces.

Let φ be an increasing concave function on I with $\varphi(0) = 0$. The Marcinkiewicz space $M(\varphi)$ consists of all measurable functions $x(t)$ on I such that

$$\|x\|_{M(\varphi)} := \sup_{\substack{s\in I\\ s\neq 0}} \frac{1}{\varphi(s)}\int_0^s x^*(t)dt < \infty.$$

In particular, $L_{p,\infty} = M(t^{1/p})$, $1 < p < \infty$.

For a symmetric space X on I, the *Köthe dual space* (or *associated space*) X' consists of all measurable functions y such that

$$\|y\|_{X'} := \sup_{\substack{x\in X\\ \|x\|_X\leqslant 1}} \int_I |x(t)y(t)|\,dt$$

is finite. Then, X' equipped with the norm $\|\cdot\|_{X'}$ is a symmetric space. Moreover, $X \subset X''$ continuously with constant 1, and the isometric equality $X = X''$ holds if and only if the norm in X has the *Fatou property*, that is, if the conditions $0 \leqslant x_n \nearrow x$ a.e. on I and $\sup_{n\in\mathbf{N}} \|x_n\| < \infty$ imply that $x \in X$ and $\|x_n\| \nearrow \|x\|$. In particular, all Orlicz and Marcinkiewicz spaces have the Fatou property.

Next, we will mainly consider symmetric spaces on $[0,1]$. In this case, L_∞ is the smallest and L_1 the largest symmetric space [23, Theorem II.4.1].

The *dilation operator* $\sigma_\tau x(t) := x(t/\tau) \cdot \chi_{[0,\min(1,\tau)]}(t)$, $\tau > 0$, is bounded in any symmetric space X on $[0,1]$ (throughout the paper, χ_E is the characteristic function of a set E). Moreover, $\|\sigma_\tau\|_{X\to X} \leqslant \max(1,\tau)$ (see [23, Theorem 2.4.5]). The function $\|\sigma_\tau\|_{X\to X}$ is semi-multiplicative, and hence one may define the *upper and lower Boyd indices* of X:

$$\alpha_X = \lim_{\tau\to 0+} \frac{\ln \|\sigma_\tau\|_{X\to X}}{\ln \tau} \quad \text{and} \quad \beta_X = \lim_{\tau\to+\infty} \frac{\ln \|\sigma_\tau\|_{X\to X}}{\ln \tau}.$$

Note that $0 \leqslant \alpha_X \leqslant \beta_X \leqslant 1$ [23, § 2.1] and $\alpha_{L_p} = \beta_{L_p} = 1/p$, $1 \leqslant p \leqslant \infty$.

Suppose X is a symmetric space on $[0,1]$. A closed subspace B of X is said to be *strongly embedded* in X if, in B, convergence in the L_1-norm is equivalent to convergence in the X-norm (cf. [1, Definition 6.4.4]).

Let (X_0, X_1) be a Banach couple (i.e., X_0 and X_1 are Banach spaces linearly and continuously embedded into a common Hausdorff topological vector space). A Banach space X is called an *interpolation space* with respect to (X_0, X_1) (in short, $X \in I(X_0, X_1)$) if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ and every linear operator bounded in $X_0 + X_1$ and in X_i, $i = 0,1$, acts boundedly in X.

Given Banach couple (X_0, X_1) the *Peetre K-functional* $K(t,x;X_0,X_1)$ is defined for $x \in X_0 + X_1$ and $t > 0$ by

$$K(t,x;X_0,X_1) = \inf\big\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\big\}.$$

In particular, $K(1,x;X_0,X_1)$ is the norm in the Banach space $X_0 + X_1$.

Interpolation in the Banach couple (X_0, X_1) is *described by the real K-method of interpolation* if from $x, y \in X_0 + X_1$ and the inequality

$$K(t,y;X_0,X_1) \leqslant K(t,x;X_0,X_1) \quad \text{for all } t > 0$$

it follows that there exists a linear operator $T\colon X_0 + X_1 \to X_0 + X_1$ such that $Tx = y$. For a detailed exposition of the interpolation theory of operators, see [9, 10, 24].

As in the function case, to any Orlicz function N we associate the Orlicz sequence space l_N of all sequences of scalars $a = (a_n)_{n=1}^\infty$ such that

$$\sum_{n=1}^\infty N\left(\frac{|a_n|}{\rho}\right) < \infty$$

for some $\rho > 0$. When equipped with the norm

$$\|a\|_{l_N} := \inf\left\{\rho > 0 : \sum_{n=1}^\infty N\left(\frac{|a_n|}{\rho}\right) \leqslant 1\right\},$$

l_N is a Banach space. Clearly, if $N(t) = t^p$, $p \geqslant 1$, then the Orlicz space l_N is the familiar space l_p. Moreover, the sequence $\{e_n\}_{n=1}^{\infty}$ given by

$$e_n = (\underbrace{0, \dots, 0}_{n-1 \text{ times}}, 1, 0, \dots)$$

is a Schauder basis in every Orlicz space l_N, provided that N satisfies the Δ_2-condition at zero, i.e., there are $u_0 > 0$ and $C > 0$ such that $N(2u) \leqslant CN(u)$ for all $0 < u < u_0$.

Let $1 \leqslant p < q < \infty$. Given an Orlicz function N, we say that N is *p-convex* (resp. *q-concave*) *at zero* if the map $t \mapsto N(t^{1/p})$ (resp. $t \mapsto N(t^{1/q})$) is convex (resp. concave). In what follows, without loss of generality, we assume that $N(1) = 1$ and that $N\colon [0, \infty) \to [0, \infty)$ is a bijection.

For a fixed $f \in L_1(0,1)$, every $k \in \mathbb{N}$, and $t > 0$ we set

$$\overline{f}_k(t) := \begin{cases} f(t-k+1), & t \in [k-1, k), \\ 0, & \text{otherwise}. \end{cases}$$

Finally, positive functions (quasi-norms) f and g are said to be equivalent (we write $f \asymp g$) if there exists a positive finite constant C such that $C^{-1} f \leqslant g \leqslant Cf$.

3. Results

The main goal of this paper is to prove the following result.

3.1. Theorem. *Let X be a separable symmetric space on $[0,1]$ such that $X + L_2 \in I(L_1, L_2)$ and let the Orlicz function N be $(1{+}\varepsilon)$-convex and $(2{-}\varepsilon)$-concave at zero for some $\varepsilon > 0$. If X contains a strongly embedded subspace isomorphic to the Orlicz space l_N, then the function $1/N^{-1}$ belongs to the space X''.*

For the proof of this theorem we need an auxiliary assertion.

Let $1 \leqslant p < 2$, and let N and Q be Orlicz functions that are p-convex and 2-concave at zero and satisfy the following conditions:

$$\lim_{u \to 0+} Q(u) u^{-p} = 0$$

and for some $K > 0$

$$N(u) \leqslant KQ(u), \quad 0 < u \leqslant 1. \tag{1}$$

By [16, Theorem 1, p.X.8] (see also [6, Theorem 9]), there exist sequences $\{f_n\}$ and $\{g_n\}$ of mean zero independent identically distributed functions which in L_p are equivalent to the unit vector bases of l_N and l_Q, respectively. We set $f := f_i^*$ and $g := g_i^*$, $i = 1, 2, \dots$

3.2. Proposition. *Let $1 \leqslant p < 2$, and let the functions N, Q, f, and g satisfy the above conditions.*

(i) *If Y is a symmetric space such that $Y \in I(L_p, L_2)$, then from $g \in Y$ it follows that $f \in Y$.*

(ii) *If the function N is $(p+\varepsilon)$-convex and $(2-\varepsilon)$-concave at zero for some $\varepsilon > 0$ and X is a symmetric space such that $X + L_2 \in I(L_p, L_2)$, then $g \in X$ implies $f \in X$.*

Proof. As said above, we assume that $N(1) = Q(1) = 1$.

First, by [8, Proposition 2.4], we have

$$\frac{1}{N^{-1}(t)} \asymp \left(\frac{1}{t}\int_0^t f(s)^p ds\right)^{1/p} + \left(\frac{1}{t}\int_t^1 f(s)^2 ds\right)^{1/2}, \quad 0 < t \leqslant 1, \tag{2}$$

and

$$\frac{1}{Q^{-1}(t)} \asymp \left(\frac{1}{t}\int_0^t g(s)^p ds\right)^{1/p} + \left(\frac{1}{t}\int_t^1 g(s)^2 ds\right)^{1/2}, \quad 0 < t \leqslant 1. \tag{3}$$

Moreover, by the well-known Holmstedt formula [20], the K-functional for the couple $(L_p[0,1], L_2[0,1]), 1 \leqslant p < 2$, satisfies the following:

$$K(t, x; L_p, L_2) \asymp \left(\int_0^{t^{\frac{2p}{2-p}}} x^*(u)^p du\right)^{1/p} + t\left(\int_{t^{\frac{2p}{2-p}}}^1 x^*(u)^2 du\right)^{1/2}, \quad 0 < t \leqslant 1,$$

with constants independent of $x \in L_p$ and $0 < t \leqslant 1$. Therefore, (2) and (3) can be rewritten as follows

$$\frac{t^{1/p}}{N^{-1}(t)} \asymp K\left(t^{\frac{2p}{2-p}}, f; L_p, L_2\right), \quad 0 < t \leqslant 1,$$

and

$$\frac{t^{1/p}}{Q^{-1}(t)} \asymp K\left(t^{\frac{2p}{2-p}}, g; L_p, L_2\right), \quad 0 < t \leqslant 1.$$

Since inequality (1) and concavity of the inverse function N^{-1} imply that

$$Q^{-1}(u) \leqslant N^{-1}(Ku) \leqslant KN^{-1}(u), \quad 0 < u \leqslant 1,$$

for some $C > 0$ we obtain

$$K(s, f; L_p, L_2) \leqslant CK(s, g; L_p, L_2), \quad 0 < s \leqslant 1.$$

Clearly, the latter inequality holds for all $s > 0$. So, since interpolation in the Banach couple (L_p, L_2) is described by the real K-method of interpolation [28], from $Y \in I(L_p, L_2)$ and $g \in Y$, we infer $f \in Y$, and part (i) is proved.

Now, let us prove (ii). First of all, as above, we have $f \in X + L_2$. Furthermore, by the hypothesis concerning to the function N and by [8, Theorem 1.1], for sufficiently small $t > 0$,

$$f(t) \asymp \frac{1}{N^{-1}(t)}. \tag{4}$$

We will show that

$$N(t) \geqslant t^r, \quad 0 < t \leqslant 1, \tag{5}$$

where $r = 2-\varepsilon$. Indeed, since N is r-concave, it follows that the function $N(t^{1/r})$ is concave, and therefore

$$N\big((us)^{1/r}\big) \geqslant uN(s^{1/r}), \quad 0 < u \leqslant 1,$$

whence

$$N(tv) \geqslant t^r N(v), 0 < t, \quad v \leqslant 1.$$

Since $N(1) = 1$, we obtain (5).

From (4) and (5), for some $c > 0$ and sufficiently small $t > 0$, we have

$$f(t) \geqslant ct^{-1/r}. \tag{6}$$

On the other hand, $f \in X + L_2$ implies $f = h_1 + h_2$, where $h_1 \in X, h_2 \in L_2$. In view of the inclusion $L_2 \subset L_{q,\infty}$ valid for every $q < 2$, choosing $q \in (r, 2)$, we obtain

$$h_2^*(t) \leqslant Ct^{-1/q}, \quad 0 < t \leqslant 1.$$

Hence, from [23, § II.2, Inequality (2.23), p. 67] it follows that

$$f(t) \leqslant h_1^*(t/2) + 2^{1/q}Ct^{-1/q}, \quad 0 < t \leqslant 1,$$

and so, by (6),

$$h_1^*(t/2) \geqslant f(t) - 2^{1/q}Ct^{-1/q} = f(t)\left(1 - 2^{1/q}C\frac{t^{-1/q}}{f(t)}\right) \geqslant f(t)\big(1 - 2^{1/q}Cc^{-1}t^{1/r-1/q}\big).$$

This and the inequality $q > r$ imply that

$$h_1^*(t/2) \geqslant \frac{1}{2}f(t)$$

for sufficiently small $t > 0$. Since $h_1 \in X$, we obtain $f \in X$. □

Proof of Theorem 3.1. By hypothesis, there is a sequence $\{h_k\}_{k=1}^\infty \subset X$, which in the spaces X and L_1 is equivalent to the unit vector basis $\{e_n\}_{n=1}^\infty$ in l_N. Thus, with constants independent of $(c_k) \in l_N$, we have

$$\Big\|\sum_{k=1}^\infty c_k h_k\Big\|_1 \asymp \Big\|\sum_{k=1}^\infty c_k h_k\Big\|_X \asymp \|(c_k)\|_{l_N}. \tag{7}$$

Clearly, since the function N is $(1+\varepsilon)$-convex and $(2-\varepsilon)$-concave at zero for some $\varepsilon > 0$, $\{e_n\}$ is a weakly null sequence in l_N. Therefore, from (7) it follows that $h_k \xrightarrow{w} 0$ in X.

Further, by a version of the (so-called) subsequence splitting property, proved in [5, Lemma 3.6], passing to a subsequence (but preserving the notation), we obtain

$$h_n = u_n + v_n + w_n, \quad n \in \mathbb{N},$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are sequences in X such that $u_n^* \leqslant g$, $g = g^* \in X''$, v_n are pairwise disjoint, $\lim_{n\to\infty} \|w_n\|_X = 0$, $u_n \xrightarrow{w} 0$, $v_n \xrightarrow{w} 0$. It is clear that $v_n \xrightarrow{w} 0$ in L_1 and, therefore, by disjointness, $\|v_n\|_1 \to 0$. Hence, the stability property of a basic sequence (see, for instance, [1, Theorem 1.3.9]) allows us to claim, passing again to some subsequence, that, in view of (7),

$$\Big\|\sum_{k=1}^{\infty} c_k u_k\Big\|_1 \asymp \|(c_k)\|_{l_N}. \tag{8}$$

Moreover, by the proof of [27, Theorem 4.5] (see also [3, Proposition 2.1]), there is a subsequence of $\{u_n\}$ (again we keep the notation) such that

$$u_n = x_n + y_n,$$

where $\{x_n\}$ is the sequence of martingale differences, $x_n \xrightarrow{w} 0$ in L_1, and $\|y_n\|_1 \to 0$. By [5, Lemma 5] (for results on comparison of norms of sums of martingale differences and their disjoint copies in general symmetric spaces, see [7]), we obtain

$$\Big\|\sum_{i=1}^{n} x_i\Big\|_1 \leqslant C_1 \Big\|\sum_{i=1}^{n} \overline{x_i}\Big\|_{(L_1+L_2)(0,\infty)}, \quad n \in \mathbb{N},$$

where $\overline{x_i}$ are pairwise disjoint copies of the functions x_i, $i = 1, 2, \ldots$ (see Preliminaries). Since $\overline{u_i} = \overline{x_i} + \overline{y_i}$ and $m(\operatorname{supp} \overline{y_i}) \leqslant 1$,

$$\|\overline{u_i} - \overline{x_i}\|_{(L_1+L_2)(0,\infty)} = \|\overline{y_i}\|_{(L_1+L_2)(0,\infty)} = \|y_i\|_1.$$

Hence, taking into account that $\|y_n\|_1 \to 0$ and $u_i^* \leqslant g$, in the same manner as above (passing to a subsequence, if necessary), we get

$$\Big\|\sum_{i=1}^{n} u_i\Big\|_1 \leqslant C_2 \Big\|\sum_{i=1}^{n} \overline{u_i}\Big\|_{(L_1+L_2)(0,\infty)} \leqslant C_2 \Big\|\sum_{i=1}^{n} \overline{g_i}\Big\|_{(L_1+L_2)(0,\infty)}.$$

Thus, by (8), the equation

$$\Big\|\sum_{k=1}^{n} e_k\Big\|_{l_N} = \frac{1}{N^{-1}(1/n)}$$

and definition of the norm in $(L_1 + L_2)(0, \infty)$, we have

$$\frac{1}{N^{-1}(1/n)} \leqslant C_3 \bigg(n \int_0^{1/n} g(s)ds + \Big(n \int_{1/n}^{1} g(s)^2 ds \Big)^{1/2} \bigg), \quad n \in \mathbb{N},$$

or, by convexity of N,

$$\frac{1}{N^{-1}(t)} \leqslant C\left(\frac{1}{t}\int_0^t g(s)ds + \left(\frac{1}{t}\int_t^1 g(s)^2 ds\right)^{1/2}\right), \quad 0 < t \leqslant 1, \tag{9}$$

for some $C > 0$.

Let $\{g_k\}$ be a sequence of mean zero independent functions on $[0,1]$ such that $g_k^* = g$, $k = 1, 2, \ldots$ In $L_1[0,1]$ it is equivalent to the unit vector basis of the Orlicz space l_Q, where Q is the 2-concave Orlicz function satisfying the condition ([16, Theorem 1, p.X.8])

$$\lim_{t\to 0+} \frac{Q(t)}{t} = 0.$$

Therefore, by [8, Proposition 2.4] it follows that

$$\frac{1}{Q^{-1}(t)} \asymp \frac{1}{t}\int_0^t g(s)ds + \left(\frac{1}{t}\int_t^1 g(s)^2 ds\right)^{1/2}, \quad 0 < t \leqslant 1. \tag{10}$$

Moreover, since N is $(1+\varepsilon)$-convex and $(2-\varepsilon)$-concave at zero for some $\varepsilon > 0$, by [8, Theorem 3.3], we have

$$\frac{1}{N^{-1}(t)} \asymp \frac{1}{t}\int_0^t f(s)\,ds + \left(\frac{1}{t}\int_t^1 f(s)^2\,ds\right)^{1/2}, \quad 0 < t \leqslant 1, \tag{11}$$

where $f(s) := 1/N^{-1}(s)$. Let us show that the remaining conditions of Proposition 3.2 (ii) also hold.

First, (9) and (10) imply inequality (1). Moreover, $(X + L_2)'' = X'' + L_2$ [25, Theorem 3.1]. Therefore, by [26, Corollary 4.2] the fact that $X+L_2 \in I(L_1, L_2)$ implies $X''+L_2 \in I(L_1, L_2)$. Thus, since $g \in X''$, by Proposition 3.2 (ii) and (11), we obtain that $1/N^{-1} \in X''$. □

Recall that α_X is the lower Boyd index of a symmetric space X (see Preliminaries).

3.3. Corollary. *Let X be a separable symmetric space on $[0,1]$, $\alpha_X > 1/2$, and let N be an Orlicz function which is $(1+\varepsilon)$-convex and $(2-\varepsilon)$-concave at zero for some $\varepsilon > 0$. If X contains a strongly embedded subspace isomorphic to l_N, then $1/N^{-1} \in X''$.*

Proof. By [4, Theorem 1], $X \in I(L_1, L_2)$. Then, obviously, $X+L_2 \in I(L_1, L_2)$, and it remains to apply Theorem 3.1. □

3.4. Remark. In the case when $N(t) = t^{1/q}$, $1 < q < 2$, the result of Theorem 3.1 was proved by Raynaud in [29] for *every* separable symmetric space by a completely different approach based on the profound theorem of Dacuncha-Castelle and Krivine on the structure of l_q-subspaces of L_1 from [17]. So it is natural to ask whether Theorem 3.1 holds without the interpolation condition imposed on the space X.

We will show that a similar result is valid, without any extra interpolation condition, if a separable symmetric space X is contained in the Marcinkiewicz space $M(\varphi)$ with $\varphi(t) := t/N^{-1}(t)$. More precisely, we have then $X'' = M(\varphi)$.

3.5. Theorem. *Suppose the Orlicz function N is $(1+\varepsilon)$-convex and $(2-\varepsilon)$-concave at zero for some $\varepsilon > 0$, and X is a separable symmetric space on $[0,1]$ such that $X \subset M(\varphi)$, $\varphi(t) := t/N^{-1}(t)$. Then if X contains a strongly embedded subspace isomorphic to the Orlicz space l_N, we have $X'' = M(\varphi)$.*

Let us begin with the following simple lemma.

3.6. Lemma. *If an Orlicz function N is $(1+\varepsilon)$-convex at zero for some $\varepsilon > 0$, and $\varphi(t) = t/N^{-1}(t)$, then*

$$\|x\|_{M(\varphi)} \asymp \sup_{0<t\leqslant 1} x^*(t)N^{-1}(t).$$

Proof. Let us estimate from above the dilation function $\mathcal{M}_\varphi(t)$ defined by

$$\mathcal{M}_\varphi(t) := \sup_{0<s\leqslant 1} \frac{\varphi(st)}{\varphi(s)}$$

for $0 < t \leqslant 1$.

Since the function $N(t^{1/(1+\varepsilon)})$, $0 < t \leqslant 1$, is convex, we have

$$N\big((st)^{1/(1+\varepsilon)}\big) \leqslant tN\big(s^{1/(1+\varepsilon)}\big), \quad 0 < s, t \leqslant 1,$$

or

$$N(uv) \leqslant v^{1+\varepsilon}N(u), \quad 0 < u, v \leqslant 1,$$

Hence

$$N^{-1}(s) \cdot t^{1/(1+\varepsilon)} \leqslant N^{-1}(st),$$

and so

$$\varphi(st) = \frac{st}{N^{-1}(st)} \leqslant \frac{st^{\varepsilon/(1+\varepsilon)}}{N^{-1}(s)}.$$

As a result, we obtain

$$\mathcal{M}_\varphi(t) \leqslant t^{\varepsilon/(1+\varepsilon)}, \quad 0 < t \leqslant 1,$$

whence $\mathcal{M}_\varphi(t) \to 0$ as $t \to 0+$. Thus, applying [23, Theorem II.5.3], we have

$$\|x\|_{M(\varphi)} \asymp \sup_{0<t\leqslant 1} \frac{1}{\varphi(t)} x^*(t) = \sup_{0<t\leqslant 1} x^*(t)N^{-1}(t).$$

□

Proof of Theorem 3.5. Since $X \subset M(\varphi)$, we have $X'' \subset M(\varphi)'' = M(\varphi)$. Combining this with Lemma 3.6, we obtain that for all $x \in X''$

$$\sup_{0<t\leqslant 1} x^*(t)N^{-1}(t) < \infty. \tag{12}$$

Further, in the same way as in the proof of Theorem 3.1, we can find a function $g = g^* \in X''$ and an Orlicz function Q satisfying relations (9) and (10). Then, from (12) it follows that

$$g(t) \leqslant \frac{C'}{N^{-1}(t)}, \quad 0 < t \leqslant 1,$$

and therefore, by (10),

$$\frac{1}{Q^{-1}(t)} \asymp \frac{1}{t}\int_0^t g(s)\,ds + \left(\frac{1}{t}\int_t^1 g(s)^2\,ds\right)^{1/2}$$
$$\leqslant C''\left(\frac{1}{t}\int_0^t \frac{ds}{N^{-1}(s)} + \left(\frac{1}{t}\int_t^1 \frac{ds}{(N^{-1}(s))^2}\right)^{1/2}\right).$$

On the other hand, since the function N is $(1+\varepsilon)$-convex and $(2-\varepsilon)$-concave at zero for some $\varepsilon > 0$, by [8, Theorem 3.3], we have (11) with $f(s) = 1/N^{-1}(s)$. Therefore, from the preceding inequality it follows that

$$\frac{1}{Q^{-1}(t)} \leqslant \frac{C}{N^{-1}(t)}, \quad 0 < t \leqslant 1.$$

This inequality combined with (9) and (10) yields

$$\frac{1}{N^{-1}(t)} \asymp \frac{1}{t}\int_0^t g(s)ds + \left(\frac{1}{t}\int_t^1 g(s)^2 ds\right)^{1/2}, \quad 0 < t \leqslant 1.$$

Hence, again taking into account the properties of N, by [8, Theorem 1.1 and Proposition 2.4], we infer that

$$g(t) \asymp \frac{1}{N^{-1}(t)}$$

for all sufficiently small $t > 0$. As a result, the function $1/N^{-1}$ belongs to X'', which, in view of Lemma 3.6, is equivalent to the inclusion $M(\varphi) \subset X''$. Since the reverse embedding also holds, the proof is complete. □

If a symmetric space is situated very "close" to L_2, it may be a non-interpolation space with respect to the couple (L_1, L_2). However, for some such spaces we have the following result.

3.7. Corollary. *Let an Orlicz function N satisfy the conditions of Theorem 3.5. If a symmetric space X is such that $X \subset L_{r,\infty}$ for every $r < 2$, then X does not contain a strongly embedded subspace isomorphic to the Orlicz space l_N. In particular, this holds for the Lorentz spaces $L_{2,q}$, $1 \leqslant q \leqslant \infty$.*

Proof. As above, $\varphi(t) := t/N^{-1}(t)$, $0 < t \leqslant 1$.

Since the function $N(t^{1/(2-\varepsilon)})$, $0 < t \leqslant 1$, is concave, we have

$$N\big((st)^{1/(2-\varepsilon)}\big) \geqslant tN\big(s^{1/(2-\varepsilon)}\big), \quad 0 < s, t \leqslant 1,$$

or

$$N(uv) \geqslant v^{2-\varepsilon}N(u), \quad 0 < u, v \leqslant 1,$$

Therefore,

$$N^{-1}(s) \cdot t^{1/(2-\varepsilon)} \geqslant N^{-1}(st),$$

and since $N(1) = 1$, we obtain

$$N^{-1}(t) \leqslant t^{1/(2-\varepsilon)}.$$

Thus, by Lemma 3.6 it follows that $M(\varphi) \supset L_{r_1,\infty}$, where $r_1 := 2 - \varepsilon < 2$. Now, choosing any r_2 from the interval $(r_1, 2)$ and taking into account the conditions of the corollary, we infer that $M(\varphi) \supsetneqq L_{r_2,\infty} \supset X$. Therefore, passing twice to dual spaces, we obtain $M(\varphi) \supsetneqq L_{r_2,\infty} \supset X''$, and the result follows from Theorem 3.5. □

Let $1 < p < 2$ and let $\{g_n^p\}$ be a sequence of mean zero independent functions on $[0, 1]$ equimeasurable with the function $g(u) = u^{-1/p}$, $0 < u \leqslant 1$. Then if X is a symmetric space such that $X \supset L_{p,\infty}$, we have

$$\Big\| \sum_{n=1}^{\infty} a_n g_n^p \Big\|_X \asymp \|(a_n)\|_{l_p}$$

with constants independent of $(a_n) \in l_p$ [13, Theorem III.3]. The following theorem, being in a sense converse to Theorem 3.1, gives necessary and sufficient conditions under which an analogous result holds for the arbitrary Orlicz function $N(t)$ situated sufficiently "far" from the extreme functions t and t^2. As above, $M(\varphi)$ is the Marcinkiewicz space with $\varphi(t) := t/N^{-1}(t)$.

3.8. Theorem. *Suppose the Orlicz function N is $(1 + \varepsilon)$-convex and $(2 - \varepsilon)$-concave at zero for some $\varepsilon > 0$, and let $\{g_n^N\}$ be a sequence of mean zero independent functions on $[0, 1]$ equimeasurable with the function $1/N^{-1}(t)$, $0 < t \leqslant 1$. The following conditions are equivalent.*

(i) *For every symmetric space X such that $X \supset M(\varphi)$, we have, with constants independent of $(a_n) \in l_N$,*

$$\Big\| \sum_{n=1}^{\infty} a_n g_n^N \Big\|_X \asymp \|(a_n)\|_{l_N}.$$

(ii) *For every symmetric space X such that $X \supset M(\varphi)$, with constants independent of $n \in \mathbb{N}$,*

$$\Big\| \sum_{k=1}^{n} g_n^N \Big\|_X \asymp \frac{1}{N^{-1}(1/n)}.$$

(iii) *There exists a constant* $K > 0$ *such that for all* $0 < u, v \leqslant 1$ *we have*

$$N(uv) \leqslant KN(u)N(v). \tag{13}$$

Proof. As above, without loss of generality, we may (and will) assume that N is strictly increasing and $N(1) = 1$. Let us begin by proving the implication (iii) ⇒ (i). Since, by [8, Proposition 2.4 and Theorem 3.3],

$$\Big\|\sum_{n=1}^{\infty} a_n g_n^N\Big\|_X \geqslant c\Big\|\sum_{n=1}^{\infty} a_n g_n^N\Big\|_1 \asymp \big\|(a_n)\big\|_{l_N},$$

it is sufficient to prove that for some $C > 0$ and every $(a_n) \in l_N$

$$\Big\|\sum_{n=1}^{\infty} a_n g_n^N\Big\|_X \leqslant C\big\|(a_n)\big\|_{l_N}. \tag{14}$$

First, from the embedding $X \supset M(\varphi)$ and [18, Theorem 1] it follows that

$$\Big\|\sum_{n=1}^{\infty} a_n g_n^N\Big\|_X \leqslant \Big\|\sum_{n=1}^{\infty} a_n g_n^N\Big\|_{M(\varphi)} \asymp \|G_a\|_{(M(\varphi)+L_2)(0,\infty)}, \tag{15}$$

where $G_a(u) := \sum_{n=1}^{\infty} a_n \bar{g}_n^N(u)$, $u > 0$. Since the function N is $(2-\varepsilon)$-concave at zero, then by the definition of the norm in the space $(M(\varphi) + L_2)(0,\infty)$

$$\|G_a\|_{(M(\varphi)+L_2)(0,\infty)} \asymp \|G_a^* \chi_{[0,1]}\|_{M(\varphi)} + \|G_a^* \chi_{(1,\infty)}\|_2.$$

Noting that in view of [18, Theorem 1] and [8, Proposition 2.4 and Theorem 3.3],

$$\|G_a^* \chi_{(1,\infty)}\|_2 \leqslant C'\|G_a\|_{(L_1+L_2)(0,\infty)} \asymp \Big\|\sum_{n=1}^{\infty} a_n g_n^N\Big\|_1 \asymp \big\|(a_n)\big\|_{l_N},$$

we see that to prove (14) it is sufficient to verify the estimate

$$\|G_a^* \chi_{[0,1]}\|_{M(\varphi)} \leqslant C\big\|(a_n)\big\|_{l_N} \tag{16}$$

with some $C > 0$.

Let $\|(a_n)\|_{l_N} = 1$. Then $\sum_{n=1}^{\infty} N(|a_n|) = 1$, and since N increases and $N(1) = 1$, we have $|a_n| \leqslant 1$ for all $n \in \mathbb{N}$. Moreover, for every $\tau > 0$,

$$\begin{aligned} m\{u > 0 : |G_a(u)| > \tau\} &= \sum_{n=1}^{\infty} m\{t \in [0,1] : |a_n g_n^N(t)| > \tau\} \\ &= \sum_{n=1}^{\infty} m\left\{t \in [0,1] : \frac{1}{N^{-1}(t)} > \frac{\tau}{|a_n|}\right\} \\ &= \sum_{n=1}^{\infty} N\left(\frac{|a_n|}{\tau}\right). \end{aligned}$$

In particular, since N strictly increases, this implies that

$$m\{u>0 : |G_a(u)|>\tau\}>1 \quad \text{if } \tau<1.$$

Therefore, from Lemma 3.6, inequality (13), and concavity of the function N^{-1} it follows that

$$\begin{aligned}\|G_a^*\chi_{[0,1]}\|_{M(\varphi)} &\asymp \sup_{0<t\leqslant 1} G_a^*(t)N^{-1}(t) = \sup_{\tau\geqslant 1}\tau N^{-1}\left(m\{u>0 : |G_a(u)|>\tau\}\right)\\ &= \sup_{\tau\geqslant 1}\tau N^{-1}\left(\sum_{n=1}^{\infty} N\left(\frac{|a_n|}{\tau}\right)\right)\\ &\leqslant \sup_{\tau\geqslant 1}\tau N^{-1}\left(K\sum_{n=1}^{\infty} N(|a_n|)N\left(\frac{1}{\tau}\right)\right)\\ &= \sup_{\tau\geqslant 1}\tau N^{-1}\left(KN\left(\frac{1}{\tau}\right)\right)\leqslant K.\end{aligned}$$

Thus inequality (16) is proved.

Since implication (i) ⇒ (ii) is obvious, it remains to show that (ii) implies (iii).

Combining the hypothesis with the equivalence from (15), we obtain

$$\Big\|\sum_{n=1}^{\infty} \bar{g}_n^N\Big\|_{(M(\varphi)+L_2)(0,\infty)} \leqslant \frac{C'}{N^{-1}(1/n)}, \quad n\in\mathbb{N}.$$

Hence, again by the definition of the norm in $(M(\varphi)+L_2)(0,\infty)$,

$$\Big\|\frac{1}{N^{-1}(\cdot/n)}\Big\|_{M(\varphi)} \leqslant \frac{C''}{N^{-1}(1/n)}, \quad n\in\mathbb{N}.$$

Since, by Lemma 3.6,

$$\Big\|\frac{1}{N^{-1}(\cdot/n)}\Big\|_{M(\varphi)} \asymp \sup_{0<t\leqslant 1}\frac{N^{-1}(t)}{N^{-1}(t/n)},$$

we infer that

$$\frac{N^{-1}(t)}{N^{-1}(t/n)} \leqslant \frac{C}{N^{-1}(1/n)}, \quad n\in\mathbb{N},$$

or

$$N^{-1}(1/n)N^{-1}(t)\leqslant CN^{-1}(t/n)$$

for all $t\in(0,1]$ and $n\in\mathbb{N}$. Therefore, in view of $2-\varepsilon$-concavity of N we obtain

$$N(N^{-1}(1/n)N^{-1}(t))\leqslant N(CN^{-1}(t/n))\leqslant C^{2-\varepsilon}t/n,$$

which combined with convexity of N implies (13). □

3.9. Remark. Up to this point, we did not consider the case when $N(t) = t^2$. As follows from [29, Proposition 1], if a separable symmetric space X contains l_2 as a strongly embedded subspace, then its second dual X'' contains the standard Gaussian random variable. It turns out that the same assertion holds even under a weaker condition, that does not specify the strongly embedded subspace of X. Indeed, if X contains a strongly embedded infinite dimensional subspace B, then the norms on X and L_1 are equivalent on B. Hence the canonical inclusion $I: X \to L_1$ is not strictly singular and by [19, Theorem 1] (see also [2]) $X \supset G$, where G is the closure of L_∞ in the Orlicz space generated by the function $e^{t^2} - 1$, $t > 0$. It remains to note that the latter embedding is equivalent to the fact that the Gaussian random variable belongs to X''.

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