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On a special case of asymptotic stability in a uniform manner with respect to the initial conditions

Assume that $y = y_0$ is a solution of the differential equation

$$\frac{dy}{dx} = f(x, y). \tag{1}$$

In [1] (Theorem 4, p. 236) sufficient conditions for asymptotic stability of this solution in a uniform manner with respect to the initial conditions have been given.

In the present paper we give a generalization of Theorem 4 from [1] (cf. Remark II in [1], p. 238). The proof of our theorem is more simple and much shorter than that of Theorem 4 in [1].

Theorem. *Let us suppose that*

1° *the function $f(x, y)$ is defined and continuous in the plane set*

$$D = \{(x, y):\ x \in \langle a, +\infty),\ y \in (y_0 - \alpha,\ y_0 + \alpha)\},$$

where $\alpha > 0$ (in particular, it may be $\alpha = +\infty$),

2° *$f(x, y) \leqslant 0$ for $(x, y) \in D_1$ and $f(x, y) \geqslant 0$ for $(x, y) \in D_2$, where*

$$D_1 = \{(x, y):\ x \in \langle a, +\infty),\ y \in (y_0, y_0 + \alpha)\},$$
$$D_2 = \{(x, y):\ x \in \langle a, +\infty),\ y \in (y_0 - \alpha, y_0)\}$$

(in Theorem 4 of [1] we suppose that $f(x, y) < 0$ inside D_1 and $f(x, y) > 0$ inside D_2),

3° *there exist the limits*

$$\lim_{\substack{x \to +\infty \\ y \searrow \bar{y}}} f(x, y) = \delta, \qquad \delta < 0 \qquad \textit{for every } \bar{y} \in (y_0, y_0 + \alpha),$$

$$\lim_{\substack{x \to +\infty \\ y \nearrow \bar{y}}} f(x, y) = \gamma, \qquad \gamma > 0 \qquad \textit{for every } \bar{y} \in (y_0 - \alpha, y_0),$$

4° *for every $\bar{y} \in (y_0 - \alpha, y_0 + \alpha)$ the real function u defined by the formula $u(x) = |f(x, \bar{y})|$ is non-decreasing for $x \in \Delta_a = \langle a, +\infty)$.*

Under these assumptions the solution $y = y_0$ of equation (1) *is asymptotically stable in a uniform manner with respect to the arbitrary initial conditions, which means that*

1. *the integral $y = y_0$ is asymptotic stable in Δ_a,*

2. *for every $\varepsilon > 0$ there exists a number $A > 0$ (depending only on ε) such that for every $x_0 \geqslant a$ and a fixed value $\beta \in (0, a)$ an arbitrary solution $y = \varphi(x)$ of equation* (1) *starting at a segment*

$$K = \{(x, y): x = x_0, y \in \langle y_0 - \beta, y_0 + \beta \rangle\}$$

satisfies the inequality $|\varphi(x) - y_0| < \varepsilon$ for $x > x_0 + A$.

Proof. It follows from assumptions 1°–3° and from Theorem 3 of [1] (p. 235) that the solution $y = y_0$ of equation (1) is asymptotic stable in Δ_a. Therefore condition 1 is satisfied.

We shall now show that condition 2 holds. It follows from Theorem 3 of [1] that for a fixed $x_0 \in \Delta_a$ and $\beta \in (0, a)$ there exists a number $A > 0$ such that the inequality

$$|\varphi(x_0) - y_0| \leqslant \beta \tag{2}$$

implies the inequality

$$|\varphi(x) - y_0| < \varepsilon \tag{3}$$

for $x > x_0 + A$ and for an arbitrary solution $y = \varphi(x)$ satisfying the initial inequality (2).

We shall now show that for the same values ε and A the inequality

$$|\varphi(x) - y_0| < \varepsilon$$

is satisfied for every $x > x_1 + A$, $x_1 > x_0$ and for an arbitrary solution $y = \varphi(x)$ of equation (1) satisfying the initial inequality

$$|\varphi(x_1) - y_0| \leqslant \beta. \tag{4}$$

Let $y = \psi(x)$ be a maximum solution of equation (1) satisfying the initial condition

$$\psi(x_1) = y_0 + \beta, \tag{5}$$

where $x_1 > x_0$. We shall prove that

$$|\psi(x) - y_0| < \varepsilon \tag{6}$$

for $x > x_1 + A$.

For this purpose we denote by $y = \Phi(x)$ the maximum solution of equation (1) satisfying the initial condition

$$\Phi(x_0) = y_0 + \beta. \tag{7}$$

Let $\Psi(x)$ be the function defined by the formula

$$\Psi(x) = \psi(x+h), \tag{8}$$

where

$$h = x_1 - x_0. \tag{9}$$

It follows from (5), (7) and (8) that

$$\Psi(x_0) = \Phi(x_0) = y_0 + \beta. \tag{10}$$

Assumption 4° and condition (8) imply

$$\frac{d\Psi(x)}{dx} = f\big(x+h, \psi(x+h)\big) \leqslant f\big(x, \psi(x+h)\big) = f\big(x, \Psi(x)\big) \tag{11}$$

for $x \geqslant x_0$.

Since

$$\frac{d\Phi(x)}{dx} = f\big(x, \Phi(x)\big)$$

or $x \geqslant x_0$, it follows from (10), (11) and from Theorem 9.5 of [2] (p. 27) that

$$\Psi(x) \leqslant \Phi(x)$$

for $x \geqslant x_0$. Therefore

$$\psi(x+h) \leqslant \Phi(x)$$

for $x \geqslant x_0$. As the function $y = \Phi(x)$ satisfies inequality (3) for $x > x_0 + A$ it follows that

$$|\psi(x+h) - y_0| = \psi(x+h) - y_0 \leqslant \Phi(x) - y_0 < \varepsilon$$

for $x > x_0 + A$. By this condition and by (9) we have (6) for $x > x_1 + A$.

In a similar way we can prove that the inequality

$$|\tilde{\psi}(x) - y_0| < \varepsilon \tag{12}$$

is satisfied for $x > x_1 + A$, where $y = \tilde{\psi}(x)$ is a minimal solution of equation (1) such that

$$\tilde{\psi}(x_1) = y_0 - \beta.$$

From inequalities (6), (12) and from Lemma 1 of [1] (p. 234) it follows that the inequality

$$|\varphi(x) - y_0| < \varepsilon$$

is satisfied for $x > x_1 + A$ and for arbitrary solution $y = \varphi(x)$ of equation (1) starting at a segment

$$K_1 = \{(x, y):\ x = x_1,\ y \in \langle y_0 - \beta, y_0 + \beta \rangle\}.$$

The both conditions 1° and 2° are satisfied, therefore the proof of theorem is completed.

References

[1] W. Pawelski, *On a simple case of asymptotic stability*, Comm. Math. 13 (1970), p. 233–239.

[2] J. Szarski, *Differential inequalities*, Warszawa 1965.