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The primary decomposition of differential modules

1. Introduction. In [8] A. Seidenberg proved the following theorem: *for any noetherian Ritt algebra each differential ideal A has an irredundant primary decomposition $A = A_1 \cap \ldots \cap A_s$, where $A_1, \ldots, A_s$ are differential ideals.*

A more general case is presented in [1]. In [7] the above theorem was proved by S. Sato for arbitrary noetherian differential rings.

In this paper, using methods similar to those of S. Sato, we prove that: *if R is a noetherian differential ring and M is a differential R-module finitely generated over R, then any differential submodule N of M has an irredundant primary decomposition $N = N_1 \cap \ldots \cap N_s$, where all N_i are differential submodules.*

From this fact a number of interesting conclusions follow concerning differential modules over a noetherian $d-MP$-ring.

In the last section we show an example of a differential ring for which the Differential Nakayama Lemma does not hold and a particular version of this lemma is given.

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2. Preliminary notions. A *differential ring* (shortly: a *d-ring*) is a pair (R, d), where R is a commutative ring with unit and $d\colon R \to R$ is a mapping, called *derivation*, which satisfies the conditions:

$$d(r+s) = d(r)+d(s), \quad d(rs) = rd(s)+sd(r) \quad \text{for arbitrary } r, s \in R.$$

A *differential module* (shortly: a *d-module*) over a d-ring (R, d) is a pair (M, δ), where M is a R-module and $\delta\colon M \to M$ is a mapping which satisfies the conditions: $\delta(m+n) = \delta(m)+\delta(n)$, $\delta(rm) = r\delta(m)+d(r)m$ for arbitrary $m, n \in M$, $r \in R$.

Let (R, d) be a d-ring and (M, δ) a d-module over (R, d). An ideal A in R is called a *d-ideal* if $d(A) \subset A$. Similarly a submodule N of M is called a *d-submodule* if $\delta(N) \subset N$.

If A is a d-ideal in R, then AM is a d-submodule of M. If N and P are d-submodules of M, then $(N: P) = \{r \in R;\ rP \subset N\}$ is a d-ideal in R. Similary, if A is a d-ideal and N a d-submodule, then $(N: A) = \{m \in M;\ Am \subset N\}$ is a d-submodule.

For an arbitrary subset T of $R(M)$ by $[T]$ we denote the smallest d-ideal (d-submodule) containing T.

We say that a d-module M is *d-finitely generated* if there is a finite number of elements $m_1, \ldots, m_n \in M$ such that $M = [m_1, \ldots, m_n]$. The d-ring (R, d) is called *$d-MP$ ring* if a radical of an arbitrary d-ideal in R is a d-ideal. Equivalent definitions of $d-MP$ ring may be found in [3]. If the d-ring (R, d) contains the field of rational numbers Q, then we call it a *Ritt algebra*. Every Ritt algebra is a $d-MP$ ring. A d-ideal A is d-maximal if it is maximal among all d-ideals in R different from R. If R is a $d-MP$ ring, then d-maximal ideals are prime (see [3]).

With every d-ring (R, d) we associate some ring (non-commutative in general) $D = D(R, d)$ (see [4], [5]) which is a left free R-module having basis $\{1, t, t^2, \ldots\}$, with the multiplication defined by: $r \cdot t = rt$, $t^n \cdot t^m = t^{n+m}$, $t \cdot r = d(r) + rt$. If (M, δ) is a d-module over (R, d), then M together with the multiplication $(r_n t^n + \ldots + r_0)m = r_n \delta^n(m) + \ldots + r_0 \cdot m$ is a left D-module. If M is a D-module, then the mapping $\delta\colon M \to M$, $\delta(m) = tm$, makes (M, δ) a d-module over (R, d). Any d-module over (R, d) is d-finitely generated iff it is finitely generated as $D(R, d)$-module.

For a R-module M by $\mathrm{Ass}_R(M)$ we denote the set of all prime ideals in R associated with M (see [5]).

3. Primary decomposition. Let (R, d) be a noetherian d-ring, (M, δ) a d-module finitely generated over R, and N a d-submodule of M.

LEMMA 1. *For any $x \in R$ there is a natural number k such that $(N: x^k)$ is a d-submodule of M and $(N: x^n) = (N: x^k)$ for any $n \geqslant k$.*

Proof. For any $m \in U = \bigcup_{s=0}^{\infty} (N: x^s)$ we have $x^s m \in N$ for some s and then the element $\delta(x^s m) = x^s \delta(m) + s x^{s-1} d(x) m$ is in N, thus $x^{s+1} \delta(m) \in N$, i.e. $\delta(m) \in U$. It means that U is a d-submodule of M. It suffices now to consider the sequence $(N: x^1) \subset (N: x^2) \subset \ldots$

DEFINITION 2. A d-submodule N of M is *d-primary* if for any d-ideal A and any d-submodule P of M, from $AP \subset N$ it follows that either $P \subset N$ or $A^n \subset (N: M)$ for some natural number n.

DEFINITION 3. A d-submodule N of M is *d-irreducible* if it is not an intersection of two d-submodules different from N.

LEMMA 4. *If N is a d-primary d-submodule, then it is a primary submodule.*

Proof. Let for given $r \in R, m \in M$, the element rm be in N. We must show that either $m \in N$ or $r \in \sqrt{(N\colon M)}$. By Lemma 1 there is a natural number k such that $(N\colon r^k)$ is a d-submodule of M. $rm \in N$ implies $m \in (N\colon r^k)$ since $r^k m \in N$; hence $[m] \subset (N\colon r^k)$ and therefore $r^k \in (N\colon [m])$.

Now, $(N\colon [m])$ is a d-ideal in R, thus $[r^k] \subset (N\colon [m])$, i.e. $[r^k][m] \subset N$. Since N is d-primary, we have either $[m] \subset N$ or $[r^k]^n \subset (N\colon M)$, i.e. $m \in N$ or $r \in \sqrt{(N\colon M)}$.

LEMMA 5. *If N is d-irreducible d-submodule of M, then N is a d-primary d-submodule.*

Proof. Assume that for a d-ideal A and a d-submodule P we have $AP \subset N$ and $A \not\subset \sqrt{(N\colon M)}$. Let $N = N_1 \cap \ldots \cap N_k$ be a primary decomposition of N. Since $A \not\subset \sqrt{(N\colon M)} = \bigcap_{i=1}^{k} \sqrt{(N_i\colon M)}$, we have $A \not\subset \sqrt{(N_i\colon M)}$ for some i. Assume that $A \not\subset \sqrt{(N_i\colon M)}$ for $i = 1, 2, \ldots, s$ and $A \subset \sqrt{(N_j\colon M)}$ for $j = s+1, \ldots, k$.

If $s = k$, then, for any $i = 1, 2, \ldots, k$, $(N_i\colon A) = N_i$ and therefore $P \subset (N\colon A) = \bigcap_{i=1}^{k} (N_i\colon A) = \bigcap_{i=1}^{k} N_i = N$. Assume that $s < k$. Since R is noetherian, there is a natural number n such that $A^n \subset (N_j\colon M)$ for $j = s+1, \ldots, k$. In this case $(N_i\colon A^n) = N_i$ for $i = 1, 2, \ldots, s$ and $(N_j\colon A^n) = M$ for $j = s+1, \ldots, k$. Thus we have

$$N \subset (N\colon A^n) \cap (N + A^n M) \subset \bigcap_{i=1}^{s} N_i \cap \bigcap_{j=s+1}^{k} N_j = N,$$

$$\text{i.e.} \quad N = (N\colon A^n) \cap (N + A^n\colon M).$$

Since $AP \subset N$, we have $A^n P \subset N$ and $N + P \subset (N\colon A^n)$. Therefore

$$N \subset (N+P) \cap (N + A^n M) \subset (N\colon A^n) \cap (N + A^n M) = N,$$

$$\text{i.e.} \quad N = (N+P) \cap (N + A^n M).$$

By d-irreducibility of $N \neq N + A^n M$ we have that $N = N+P$, i.e. $P \subset N$.

THEOREM 6. *Let (R, d) be a noetherian d-ring and (M, δ) a d-module finitely generated over R. Then any d-submodule N of M has an irredundant primary decomposition $N = N_1 \cap \ldots \cap N_n$ such that N_i are d-submodules of M.*

Proof. Using Lemmas 4 and 5 the argument is standard.

4. Conclusions from Theorem 6 for noetherian $d-MP$ rings. We assume now that R is a noetherian $d-MP$ ring and M is a d-module finitely generated over R.

From Theorem 6 we have an immediate

COROLLARY 7. *Any prime ideal associated with a d-module M is a d-ideal.*

LEMMA 8. *For any $m \in M$ if $(o\colon m)$ is a d-ideal, then $(o\colon m) = (o\colon [m])$.*

Proof. See [2], Lemma 2.

Lemma 9. *If $M \neq 0$, then there exists a d-submodule $N \neq 0$ and a prime d-ideal P such that N is a torsion-free d-submodule over R/P.*

Proof. Let P be a maximal ideal in the family $\{(o: m);\ o \neq m \in M\}$. It is known that $(o: x) = P$ is a prime ideal. By Corollary 7, P is a d-ideal.

Put $N = [x]$. Clearly, N is a non-zero d-submodule and, by Lemma 8, $P = (o: x) = (o: [x]) = (O: N)$, thus $PN = 0$, i.e. N is a d-module over the d-ring R/P.

Now assume that $rn = o$, $r \in R \setminus P$, $o \neq n \in N$. Then $r \in (o: n)$, $r \notin (o: x) = P$, which gives $(o: x) \subsetneq (o: n)$, contrary to the maximality of $(o: x)$.

Corollary 10. *If $M \neq 0$, then there exist a sequence of d-submodules $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_k = M$ and a sequence of prime d-ideals $P_1, \ldots, P_k$ in R such that M_i/M_{i-1} is a torsion-free d-module over the d-ring R/P_i, $i = 1, \ldots, k$.*

Proof. Let N and P be as in Lemma 9. We put $M_1 = N$ and $P_1 = P$. Then $M_1/M_0 = N$ is a torsion-free d-module over R/P_1. If $M_1 = M$, then there is nothing more to do. If $M_1 \subsetneq M$, then we apply Lemma 9 to the d-module $M/M_1 \neq 0$. Thus there exist a d-submodule $N_1 \neq 0$ of M/M_1 and a prime d-ideal P_2 such that N_1 is a torsion-free d-module over R/P_2. We take $M_2 = \varphi^{-1}(N_1)$, where $\varphi\colon M \to M/M_1$ is canonical. So we have $0 \subsetneq M_1 \subsetneq M_2$ and $M_2/M_1 = N_1$ is torsion-free d-module over R/P_2. Since M is noetherian, this procedure ends.

Corollary 11. *Assume that $M \neq 0$ is a d-simple d-module (i.e. M is without any proper d-submodels). Then*

(1) *$(O: M)$ is a prime d-ideal,*

(2) *M is a torsion-free d-module over $R/(O: M)$,*

(3) *for any $o \neq m \in M$, we have $(o: m) = (O: M)$.*

Proof. (1) Since $M \neq 0$, the set $\mathrm{Ass}_R(M)$ is non-empty. Let $P = (o: m) \in \mathrm{Ass}_R(M)$. By Corollary 7, P is a d-ideal. Thus Lemma 8 implies that $(O: M) = (O: [m]) = (o: m) = P$ is a prime d-ideal;

(2) Follows from (1) and from the proof of Lemma 9;

(3) For such m, since $[m] = M$, we have $(O: M) = (O: [m]) \subset (o: m)$. Assume that $(O: M) \subsetneq (o: m)$ and take $x \in (o: m)$ such that $x \notin (O: M)$. Since M is d-simple, O is a d-primary d-submodule, and by Lemma 4 it is a primary submodule of M. But $xm = o$; hence $m = o$ or $x \in \sqrt{(O: M)} = (O: M)$, a contradiction.

Corollary 12. *If for all d-maximal d-ideals $\mathfrak{M}$ in R, $M_{\mathfrak{M}} = 0$, then $M = 0$.*

Proof. Assume that $M \neq 0$. Then there is a prime d-ideal P of the form $P = (o: x)$, for some $x \in M$, $x \neq o$, since $\mathrm{Ass}_R(M) \neq \emptyset$. Let $\mathfrak{M}$ be a d-maximal d-ideal containing P. Then $M_{\mathfrak{M}} = 0$, thus $x/1 = o$ in $M_{\mathfrak{M}}$. It follows now that, for some $a \in R \backslash \mathfrak{M}$, $ax = 0$, hence $s \in (o\colon x) = P \subset \mathfrak{M}$, a contadiction.

5. The Differential Nakayama Lemma. Let $Jd(R)$ denote the intersection of all d-maximal d-ideals of the d-ring (R, d). We call $Jd(R)$ the *Jacobson d-radical.*

DEFINITION 13. We say that d-ring (R, d) *satisfies the Differential Nakayama Lemma* if for any d-ideal $A \subset Jd(R)$ and any d-finitely generated d-module M the condition $AM = M$ implies $M = 0$.

Now we give an example of a d-ring which does not satisfy the Differential Nakayama Lemma.

EXAMPLE 14. Let k be a field of characteristics zero, $R = k[x]$ a ring of polynomials in one variable x over k and let $d(x) = x$, $d(k) = 0$. Since the only d-maximal d-ideal in the d-ring (R, d) is (x), we have $Jd(R) = (x)$.

Note that for any $w \in D = D(R, d)$ there is $w' \in D$ such that $wx = xw'$. Indeed,

(a) if $w \in R$, then $wx = xw$,

(b) since $tx = d(x) + xt = x + xt = x(1+t)$, we have $t^n \cdot x = x(1+t)^n$,

(c) if $w = r_0 + r_1 t + \ldots + r_n t^n$ is an arbitrary element of D, then

$$wx = \Big(\sum_{i=0}^{n} r_i t^i\Big)x = \sum_{i=0}^{n} r_i x(1+t)^i = x\Big(\sum_{i=0}^{n} r_i (1+t)^i\Big).$$

Let $f\colon k[x] \to k$ be such a homomorphism of rings that $f(x) = 1$ and $f(k) = k$ for any $k \in k$. The homomorphism f induces on k a structure of R-module given $wk = f(w)k$.

Put $M = D \otimes_R k$. Since M is a left D-module generated by the element $1 \otimes_R 1$, M is a d-module d-finitely generated over (R, d). We show now that $(x)M = M$. Take $m \in M$. Then $m = w(1 \otimes 1)$ for some $w \in D$. Thus we have:

$$\begin{aligned} m &= w(1 \otimes 1) = w(1 \otimes 1 \cdot 1) = w\big(1 \otimes f(x) \cdot 1\big) = w(1 \otimes x \cdot 1) \\ &= w(1 \cdot x \otimes 1) = wx(1 \otimes 1) = xw'(1 \otimes 1), \quad \text{i.e.} \quad m \in (x)M. \end{aligned}$$

This proves that the d-ring (R, d) does not satisfy the Differential Nakayama Lemma.

With some limitations on d-ring R and d-module M one may prove the following version of the Differential Nakayama Lemma, different from previous one.

PROPOSITION 15. *Let (R, d) be a noetherian d-MP ring and (M, δ) a d-module finitely generated over R. If A is d-ideal such that $A \subset Jd(R)$ and $AM = M$, then $M = 0$.*

Proof. *If* $\mathfrak{M}$ is an arbitrary d-maximal d-ideal in R, then $A \subset Jd(R) \subset \mathfrak{M}$, $A_{\mathfrak{M}} M_{\mathfrak{M}} = M_{\mathfrak{M}}$ and $A_{\mathfrak{M}} \subset \mathfrak{M}R_{\mathfrak{M}}$. From the Nakayama Lemma, $M_{\mathfrak{M}} = 0$; hence by Corollary 12, $M = 0$.

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