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The problems of separability, duality, reflexivity and of comparison for generalized Orlicz–Sobolev spaces $W_M^k(\Omega)$

Abstract. We shall prove the theorem on representation of bounded linear functionals on generalized Orlicz–Sobolev space $W_M^k(\Omega)$ generated by a class of N-functions $M(t, u)$ with parameter t. We shall give some sufficient conditions in order $W_M^k(\Omega)$ be separable and some sufficient conditions in order that $W_M^k(\Omega)$ be a reflexive space. Moreover, we shall prove that if $W_{M_1}^k(\Omega)$ is a dense subspace of $W_{M_2}^k(\Omega)$, then $M_2 \prec M_1$.

0. Introduction. R^n denotes the n-dimensional real Euclidean space, μ = Lebesgue measure on R^n, Ω is an arbitrary non-void open set in $R^n, R_+ = [0, \infty)$. A function $M(\cdot,\cdot)$: $\Omega \times R_+ \to R_+$, which satisfies the conditions:

There exists a set A of measure zero such that:

(i) $M(t, u) = 0$ if and only if $u = 0$ for every $t \in \Omega \setminus A$;

(ii) $M(t, \alpha_1 u_1 + \alpha_2 u_2) \leqslant \alpha_1 M(t, u_1) + \alpha_2 M(t, u_2)$ for every $t \in \Omega \setminus A, u_1, u_2, \alpha_1, \alpha_2 \in R_+, \alpha_1 + \alpha_2 = 1$;

(iii) $M(t, u)$ is a measurable function of t for every fixed $u \geqslant 0$;

is called a *φ-function of the variable u with parameter t*.

A φ-function $M(t, u)$ with parameter t, which satisfies the condition:

(iv) there exists a set A of measure zero such that

$$\frac{M(t, u)}{u} \to 0 \quad \text{as } u \to 0 \quad \text{and} \quad \frac{M(t, u)}{u} \to \infty \quad \text{as } u \to \infty \quad \text{for every } t \in \Omega \setminus A,$$

is called an *N-function with parameter t*.

The following conditions for φ-functions $M(t, u)$ with parameter t will be used:

(v) there exists a number $u_0 > 0$ such that $\int_B M(t, u)\,dt < \infty$ for every compact set $B \subset \Omega$ and for every $0 \leqslant u \leqslant u_0$;

(vi) for every compact set $B \subset \Omega$ there exist a set $A_B, \mu(A_B) = 0$, a constant $C_B > 0$, and a function $h_B \geqslant 0$ belonging to $L_1(B)$ such that

$$u \leqslant C_B M(t, u) + h_B(t) \quad \text{for every } u \geqslant 0, t \in B \setminus A_B;$$

(vii) $\frac{M(t,u)}{u} \to \infty$ as $u \to \infty$ uniformly with respect to t on every compact set $B \subset \Omega$;

(Δ_2) there exist a constant $K > 0$, a set $A \subset \Omega$ of measure zero, and a non-negative function $h_1 \in L_1(\Omega)$ such that for every $t \in \Omega \backslash A$ and for every $u \geqslant 0$

$$M(t, 2u) \leqslant KM(t, u) + h_1(t).$$

Obviously, we may assume that the sets A in the conditions above are the same.

By F we denote the real space of all complex-valued and Lebesgue measurable functions defined on Ω with equality almost everywhere on Ω. For every φ-function $M(t, u)$ with parameter t, we define the functional (convex modular)

$$\varrho_M(f) = \int_\Omega M(t, |f(t)|)\,dt, \forall f \in F,$$

and the Orlicz–Musielak space (see [11])

$$L_M(\Omega) = \{f \in F\colon \exists\, \lambda > 0,\ \varrho_M(\lambda f) < \infty\}.$$

For every φ-function $M(t, u)$, $L_M(\Omega)$ is a real vector space with usual scalar multiplication and addition of functions. The functional $\|\cdot\|_{L_M}$ defined by

$$\|f\|_{L_M} = \inf\{\varepsilon > 0\colon \varrho_M(f/\varepsilon) \leqslant 1\}$$

is a norm on $L_M(\Omega)$. $\|\cdot\|_{L_M}$ is called the *Luxemburg norm* (see [8]). Further, for any fixed non-negative integer k we define

$$W_M^k(\Omega) = \{f \in L_M(\Omega)\colon \forall\, |\alpha| \leqslant k\ \exists\, D^\alpha f \in L_M(\Omega)\},$$

where $D^\alpha f = \partial^{|\alpha|} f/\partial t_1^{\alpha_1} \ldots \partial t_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex with $\alpha_i \geqslant 0$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, denote the distributional derivatives of the function f of order $|\alpha|$. The space $W_M^k(\Omega)$ is called the *generalized Orlicz–Sobolev space* (see [2]). Let

$$\bar{\varrho}_M(f) = \sum_{|\alpha| \leqslant k} \varrho_M(D^\alpha f) \quad \text{and} \quad \|f\|_{W_M^k} = \inf\{\varepsilon > 0\colon \bar{\varrho}_M(f/\varepsilon) \leqslant 1\}$$

for $f \in W_M^k(\Omega)$. These functionals are a convex modular and a norm on $W_M^k(\Omega)$, respectively. If M is a φ-function with parameter satisfying condition (vi), then the pair $\langle W_M^k(\Omega), \|\cdot\|_{W_M^k}\rangle$ is a Banach space (see [2]).

If a φ-function $M(t, u)$ satisfies condition (v), then the inclusion

$$C_0^\infty(\Omega) \subset W_M^k(\Omega) \tag{0.1}$$

holds for every non-negative integer k, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω with compact support in Ω.

If a φ-function $M(t, u)$ with parameter t satisfies condition (vi), then there holds the inclusion $L_M(\Omega) \subset L_1^{\mathrm{loc}}(\Omega)$. Condition (vi) is also necessary in order that the inclusion $L_M(\Omega) \subset L_1^{\mathrm{loc}}(\Omega)$ hold (see [8]). Thus, if condition (vi) is satisfied, then for every function $f \in L_M(\Omega)$ the functional T_f defined by

$$T_f(\varphi) = \langle f, \varphi \rangle = \int_\Omega f(t)\varphi(t)\,dt, \qquad \forall\, \varphi \in C_0^\infty(\Omega),$$

is a regular distribution and so $W_M^k(\Omega) = L_M(\Omega)$ if $k = 0$.

If $M(t, u)$ is an N-function with parameter t, then we define the complementary N-function $N(t, u)$ to $M(t, u)$ by

$$N(t, u) = \sup_{v>0} \{uv - M(t, v)\}; \qquad \forall\, t \in \Omega,\ u \geqslant 0.$$

Henceforth, $M(t, u)$ and $N(t, u)$ denote a pair of complementary N-functions. There holds the following Young's inequality

$$uv \leqslant M(t, u) + N(t, v); \qquad \forall\, t \in \Omega \backslash A,\, u,\ v \in R_+. \tag{0.2}$$

Moreover, if $p(t, u)$ and $q(t, u)$ denote the right-hand derivatives of N-functions $M(t, u)$ and $N(t, u)$ with respect to the variable u for fixed t, respectively, then there hold the following Young's equalities:

$$\begin{aligned} up(t, u) &= M(t, u) + N\big(t, p(t, u)\big), & \forall\, t \in \Omega \backslash A,\ u \geqslant 0, \\ vq(t, v) &= M\big(t, q(t, v)\big) + N(t, v), & \forall\, t \in \Omega \backslash A,\ v \geqslant 0. \end{aligned} \tag{0.3}$$

Now, let $l = \sum_{|\alpha| \leqslant k} 1$ and let $L_M^l(\Omega) = \prod_{i=1}^{l} L_M(\Omega)$, i.e. $L_M^l(\Omega)$ is the l-tuple Cartesian product of $L_M(\Omega)$. Then every element $f \in L_M^l(\Omega)$ is of the form $f = (f_1, \ldots, f_l)$, where $f_i \in L_M(\Omega)$, $i = 1, \ldots, l$. We define

$$\tilde{\varrho}_M(f) = \sum_{i=1}^{l} \varrho_M(f_i) \quad \text{and} \quad \|f\|_{L_M^l} = \inf\{\varepsilon > 0\colon \tilde{\varrho}_M(f/\varepsilon) \leqslant 1\}$$

for every $f \in L_M^l(\Omega)$. Obviously these functionals are a convex modular and a norm on $L_M^l(\Omega)$, respectively, and the pair $\langle L_M^l(\Omega), \|\cdot\|_{L_M^l}\rangle$ is a Banach space. We define also on $L_M^l(\Omega)$ the Orlicz norm ${}^1\|\cdot\|_{L_M^l}$ by

$${}^1\|f\|_{L_M^l} = \sup\Big\{\Big|\sum_{i=1}^{l} \int_\Omega f_i(t) g_i(t)\,dt\Big|\colon \|g\|_{L_N^l} \leqslant 1\Big\}.$$

There hold the following inequalities (see [6])

$$\|f\|_{L_M^l} \leqslant {}^1\|f\|_{L_M^l} \leqslant 2\,\|f\|_{L_M^l}, \qquad \forall f \in L_M^l(\Omega).$$

Let us suppose that the l multiindices α satisfying $|\alpha| \leqslant k$ are linearly ordered in a convenient fashion so that with each $f \in W_M^k(\Omega)$ we may associate a well-defined vector Pf in $L_M^l(\Omega)$ given by

$$Pf = (D^\alpha f)_{|\alpha| \leqslant k}. \tag{0.4}$$

We have $\|f\|_{W^k_M} = \|Pf\|_{L^l_M}$, so P is an isometric isomorphism of $W^k_M(\Omega)$ onto a subspace $PW^k_M(\Omega) = W$ of the $L^l_M(\Omega)$. If $k > 0$, then $PW^k_M(\Omega)$ is a closed proper subspace of $L^l_M(\Omega)$.

Let X' denote the dual space of X for any Banach space. It is easily seen that if $M(t, u)$ satisfies condition (v), then the Lebesgue measure and the open set $\Omega \subset R^n$ satisfy conditions A and B from [8] with the sequence $\{T_n\}$ of compact subsets of Ω. Thus, there holds the following (see [8]):

0.1. LEMMA. *If $f^* \in [L_M(\Omega)]^1$, where $M(t, u)$ is an N-function with parameter t satisfying condition* (v) *and* (Δ_2), *then there exists a unique function $f \in L_N(\Omega)$ such that for every $g \in L_M(\Omega)$*

$$f^*(g) = \langle f, g\rangle = \int_\Omega f(t)g(t)dt \quad \text{and} \quad \|f^*\| = {}^1\|f\|_{L_N}.$$

1. The separability of $W^k_M(\Omega)$. We shall prove the following

1.1. THEOREM. *If $M(t, u)$ is a φ-function with parameter t continuous with respect to t for every fixed $u \geqslant 0$ and satisfying conditions* (vi) *and* (Δ_2), *then the space $W^k_M(\Omega)$ is separable.*

Proof. The space $L_M(\Omega)$ is separable. This follows from density of $C_0^\infty(\Omega)$ in $L_M(\Omega)$ (see [3]). Hence also $L^l_M(\Omega)$ is separable. Since the operator P defined by (0.4) is an isometry between $W^k_M(\Omega)$ and $PW^k_M(\Omega) \subset L^l_M(\Omega)$ and $W^k_M(\Omega)$ is complete, $W = PW^k_M(\Omega)$ is a closed subspace of $L^l_M(\Omega)$. Thus, W and $W^k_M(\Omega) = P^{-1}W$ are separable spaces.

2. Duality, the space $W^{-k}_M(\Omega)$. First, we shall write the following lemma which immediately follows from Lemma 0.1.

2.1. LEMMA. *If an N-function $M(t, u)$ with parameter t satisfies conditions* (v) *and* (Δ_2), *then to every $f^* \in [L^l_M(\Omega)]'$ there corresponds a unique $f \in L^l_N(\Omega)$ such that*

$$f^*(g) = \sum_{i=1}^{l} \langle f_i, g_i\rangle, \quad \forall g \in L^l_M(\Omega). \tag{2.0}$$

Moreover, $\|f^*\| = {}^1\|f\|_{L_N}$.

There holds the following Hölder's inequality

$$\Big|\sum_{i=1}^{l} \langle f_i, g_i\rangle\Big| \leqslant \|f\|_{L^l_M}{}^1\|g\|_{L^l_N}, \quad \forall f \in L^l_M(\Omega),\ g \in L^l_N(\Omega).$$

Thus each element $f \in L^l_N(\Omega)$ defines a bounded linear functional f^* on $W^k_M(\Omega)$ by

$$f^*(g) = \sum_{|\alpha| \leqslant k} \langle D^\alpha g, f_\alpha\rangle, \tag{2.1}$$

where the element $f \in L^l_N(\Omega)$ is rewritten in the form $f = (f_\alpha)_{|\alpha| \leqslant k}$.

We define on the space $L^l_N(\Omega)$ a relation of equivalence R by: $f_1 R f_2$ for $f_1, f_2 \in L^l_N(\Omega)$ if and only if f_1 and f_2 define by formula (2.1) the same bounded linear functional on $W^k_M(\Omega)$.

2.2. THEOREM. *Let $M(t, u)$ be an N-function with parameter t satisfying conditions* (v), (vi), (Δ_2) *and let $f^* \in [W^k_M(\Omega)]'$. Then there exists an element $f \in L^l_N(\Omega)$ such that, writing the element f in the form $(f_\alpha)_{|\alpha| \leq k}$, we have*

$$f^*(g) = \sum_{|\alpha| \leq k} \langle D^\alpha g, f_\alpha \rangle \tag{2.2}$$

for all $g \in W^k_M(\Omega)$. Moreover, $\|f^*\| = \inf\{{}^1\|f\|_{L^l_N}\} = \min\{{}^1\|f\|_{L^l_N}\}$, the infimum being taken over (and attained), on the set of all $f \in L^l_N(\Omega)$, which define the functional f^*, i.e. (2.2) *holds for every $g \in W^k_M(\Omega)$. Thus, the space $[W^k_M(\Omega)]'$ is isometrically isomorphic to the quotient space $L^l_N(\Omega)/R$ with the norm*

$$\|[f]\|_{L^l_N/R} = \inf\{{}^1\|f\|_{L^l_N} \colon f \in [f]\}.$$

Proof. Using Lemma 2.1, the proof is analogous to the proof of the respective theorem for the space $W^k_p(\Omega)$ (Theorem 3.8 in [1]).

2.3. Some remarks. Let us denote by $\mathring{W}^k_M(\Omega)$ the closure in $W^k_M(\Omega)$ of the set $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{W^k_M}$, and by R_0 a relation of equivalence on $L^l_N(\Omega)$ defined as follows: $f_1 R_0 f_2$ for $f_1, f_2 \in L^l_N(\Omega)$ iff f_1 and f_2 define the same linear bounded functional on $\mathring{W}^k_M(\Omega)$. Then the dual space $[\mathring{W}^k_M(\Omega)]'$ is isometrically isomorphic to $L^l_N(\Omega)/R_0$ with norm

$$\|[f]_0\| = \inf\{{}^1\|f\|_{L^l_N} \colon f \in [f]_0\},$$

where $[f]_0$ denotes the equivalence class of the element f with respect to the relation R_0.

If $M(t, u)$ is an N-function with parameter t satisfying conditions (v), (vi) and (Δ_2), then every element $f^* \in [\mathring{W}^k_M(\Omega)]'$ is an extension of a distribution $T \in \mathscr{D}'(\Omega)$ to $\mathring{W}^k_M(\Omega)$, with T of the form

$$T(\varphi) = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha T_{f_\alpha}(\varphi), \quad \forall \varphi \in C^\infty_0(\Omega), \tag{2.3}$$

where $f = (f_\alpha)_{|\alpha| \leq k}$ is an element of $L^l_N(\Omega)$ determining the functional f^*. Obviously, for every $f \in [f]$ the distribution T defined by (2.3) is the same, but if T is any element of $\mathscr{D}'(\Omega)$ having the form (2.3) for some $f \in L^l_N(\Omega)$, then the continuous extension of T to $W^k_M(\Omega)$ may be not unique. However, T possesses a unique extension to $\mathring{W}^k_M(\Omega)$.

We denote by $W^{-k}_M(\Omega)$ the Banach space consisting of distributions $T \in \mathscr{D}(\Omega)$ satisfying (2.3) for some $f \in L^l_N(\Omega)$, normed by

$$\|T\| = \inf\{{}^1\|f\|_{L^l_N} \colon f \text{ satisfies (2.3)}\}.$$

By the above remarks, $W^{-k}_M(\Omega)$ is isometrically isomorphic to $[\mathring{W}^k_M(\Omega)]'$

(so also to $L_N^l(\Omega)/R_0$), provided the N-function $M(t, u)$ with parameter t satisfies conditions (v), (vi) and (Δ_2).

3. Reflexivity of $W_M^k(\Omega)$. First, we shall prove two lemmas.

3.1. LEMMA. *If $M(t, u)$ and $N(t, u)$ are complementary N-functions with parameter t and M satisfies conditions* (v) *for every $u \geqslant 0$ and* (vii), *then N satisfies condition* (v) *for every $u \geqslant 0$.*

Proof. We have (see (0.3))

$$N(t, u) = uq(t, u) - M\big(t, q(t, u)\big).$$

Obviously, $M\big(t, q(t, u)\big) \leqslant uq(t, u)$, because $N(t, u)$ is non-negative. Hence $M\big(t, q(t, u)\big)/q(t, u) \leqslant u$. Let $B \subset \Omega$ be a compact set and $u \geqslant 0$ an arbitrary fixed number. Then by condition (vii) there exists a v_u such that

$$M(t, v)/v \geqslant u, \quad \forall\, v \geqslant v_u,\ \forall\, t \in B.$$

Hence we obtain that $q(t, u) \leqslant v_u$ for every $t \in B$. Thus we have

$$\int_B N(t, u)\,dt \leqslant \int_B uv_u\,dt + \int_B M(t, v_u)\,dt < \infty.$$

3.2. LEMMA. *If $M(t, u)$ and $N(t, u)$ satisfy condition (Δ_2) and if M satisfies conditions* (v) *and* (vii), *then the space $L_M^l(\Omega)$ is reflexive for every positive integer l.*

Proof. The N-functions $M(t, u)$ and $N(t, u)$ satisfy the assumptions of Lemma 2.1. It suffices to prove that (see [1]) the natural isometry $\mathcal{K}: L_M^l(\Omega) \ni f \mapsto f^{**} \in [L_M^l(\Omega)]''$ given by

$$f^{**}(f^*) = f^*(f), \quad \forall f^* \in [L_M^l(\Omega)]'$$

maps $L_M^l(\Omega)$ onto $[L_M^l(\Omega)]''$ uniquely. From Lemma 2.1 it follows that exists an isometry $T\colon L_N^l(\Omega) \xrightarrow{\text{onto}} [L_M^l(\Omega)]'$ given by (2.0). Thus, if $f^{**} \in [L_M^l(\Omega)]''$, then there exists a unique $z^* \in [L_N^l(\Omega)]'$ such that

$$f^{**}(f^*) = z^*(T^{-1}f^*), \quad \forall\, f^* \in [L_M^l(\Omega)]'.$$

It suffices to put $z^* = f^{**} \circ T$. Let us assume that the functionals $z^* \in [L_N^l(\Omega)]'$ and $f^* \in [L_M^l(\Omega)]'$ are determined by the elements $z \in L_M^l(\Omega)$ and $f \in L_N^l(\Omega)$, respectively. So we obtain

$$f^{**}(f^*) = \sum_{i=1}^{l} \langle z_i, f_i \rangle = f^*(z), \quad \forall\, f^* \in [L_M^l(\Omega)]'.$$

Thus, the element $f^{**} \in [L_M^l(\Omega)]''$ is uniquely determined by the element $z \in L_M^l(\Omega)$ and $\|f^{**}\| = {}^1\|z\|_{L_M}$.

3.3. THEOREM. *If $M(t, u)$ and $N(t, u)$ satisfy conditions* (vi), (Δ_2) *and if $M(t, u)$ satisfies conditions* (v) *and* (vii), *then for every non-negative integer k the space $W_M^k(\Omega)$ is reflexive.*

Proof. Let $l = \sum_{|\alpha| \leqslant k} 1$. Since, by Lemma 3.2, the space $L_M^l(\Omega)$ is reflexive and $W_M^k(\Omega)$ is isometric to the closed subspace $W = PW_M^k(\Omega)$ of $L_M^l(\Omega)$, so $W_M^k(\Omega)$ is reflexive.

4. The space $H_M^{-k}(\Omega)$ and duality. Let $M(t, u)$ and $N(t, u)$ satisfy the assumptions of Theorem 3.3. Then the space $W_M^k(\Omega)$ is reflexive. Each element $g \in L_N(\Omega)$ determines an element T_g of $[W_M^k(\Omega)]'$ by means of $T_g(f) = \langle f, g\rangle$. Moreover,

$$|T_g(f)| = |\langle f, g\rangle| \leqslant \|f\|_{W_M^k}\, {}^1\|g\|_{L_N}, \quad \text{where } {}^1\|g\|_{L_N} = {}^1\|g\|_{L_N^1}. \tag{4.1}$$

We define the $(-k, N)$-norm of $g \in L_N(\Omega)$ to be the norm of T_g, that is,

$$\|g\|_{-k,N} = \sup\{|\langle f, g\rangle|\colon \|f\|_{W_M^k} \leqslant 1\}.$$

Obviously,

$$\|g\|_{-k,N} \leqslant {}^1\|g\|_{L_N} \quad \text{and equality holds for } k = 0.$$

Moreover, there holds the following Hölder's inequality

$$|\langle f, g\rangle| = \|f\|_{W_M^k} \left|\left\langle \frac{f}{\|f\|_{W_M^k}}, g\right\rangle\right| \leqslant \|f_{W_M^k}\|g\|_{-k,N}.$$

Let

$$V = \{T_g\colon g \in L_N(\Omega)\}.$$

Obviously, V is a linear subspace of the space $[W_M^k(\Omega)]'$. We shall show that V is dense in $[W_M^k(\Omega)]'$. This is easily seen by showing that if $F \in [W_M^k(\Omega)]''$ satisfies $F(T_g) = 0$ for every $T_g \in V$, then $F(T) = 0$ for every $T \in [W_M^k(\Omega)]'$. Since $W_M^k(\Omega)$ is reflexive, there exists $f \in W_M^k(\Omega)$ such that

$$\langle f, g\rangle = T_g(f) = F(T_g) = 0, \quad \forall\, g \in L_N(\Omega). \tag{4.2}$$

Since $N(t, u)$ satisfies condition (v), $C_0^\infty(\Omega)$ is contained in $L_N(\Omega)$; hence from (4.2) it follows that $f = 0$ in $W_M^k(\Omega)$. Hence $\|F\| = {}^1\|f_{W_M^k} = {}^1\|Pf\|_{L_M^l} = 0$. Thus $F = 0$.

Let $H_N^{-k}(\Omega)$ denote the completion of $L_N(\Omega)$ with respect to the norm $\|\cdot\|_{-k,N}$. Then we have

4.1. THEOREM. *If $M(t, u)$ and $N(t, u)$ satisfy conditions* (vi), (Δ_2) *and if $M(t, u)$ satisfies conditions* (v) *and* (vii), *then the space $[W_M^k(\Omega)]'$ is isomorphic to the space $H_N^{-k}(\Omega)$.*

Proof. We denote by H the closure of $L_N(\Omega)$ with respect to the norm $\|\cdot\|_{-k,N}$. Obviously, the spaces $V_1 = \{T_g\colon g \in H\}$ and $H_N^{-k}(\Omega)$ are isomorphic. From the density of V in $[W_M^k(\Omega)]'$ it follows that $V_1 = [W_M^k(\Omega)]'$ and thus $H_N^{-k}(\Omega)$ and $[W_M^k(\Omega)]'$ are isomorphic.

5. Comparison of the generalized Orlicz–Sobolev spaces.

5.1. LEMMA. *If $M_1(t,u)$ and $M_2(t,u)$ are φ-functions with parameter t and $W^k_{M1}(\Omega) \subset W^k_{M2}(\Omega)$, where k is a non-negative integer number, then there exists a positive constant $K > 0$ such that*

$$\|f\|_{W^k_{M_2}} \leqslant K\|f\|_{W^k_{M_1}}, \quad \forall f \in W^k_{M_1}(\Omega). \tag{5.1}$$

Proof. It is sufficient to prove that the embedding operation from $W^k_{M1}(\Omega)$ into $W^k_{M2}(\Omega)$ is closed, i.e. the conditions $\|f_n - f\|_{W^k_{M_1}} \to 0$ and $\|f_n - g\|_{W^k_{M_2}} \to 0$ as $n \to \infty$ imply $f = g$ almost everywhere on Ω.

From the first condition follows that $f_n\chi_A \to 0$ with respect to the measure for every measurable set A of finite measure (see [5]). Since the Lebesgue measure on R^n is σ-finite, we may find a subsequence $\{z_n\}$ of the sequence $\{f_n\}$ such that $z_n(t) \underset{n\to\infty}{\longrightarrow} f(t)$ for a.e. $t \in \Omega$. We have also $\|z_n - g\|_{W^k_{M_2}} \to 0$ as $n \to \infty$. Now, we may find a subsequence $\{h_n\}$ of the sequence $\{z_n\}$ such that $h_n(t) \underset{n\to\infty}{\longrightarrow} g(t)$ for a.e. $t \in \Omega$. Thus we have $h_n(t) \underset{n\to\infty}{\longrightarrow} f(t)$ and $h_n(t) \underset{n\to\infty}{\longrightarrow} g(t)$ for a.e. $t \in \Omega$. Hence $f = g$ almost everywhere on Ω.

5.2. COROLLARY. *If $W^k_{M_1}(\Omega) \subset W^k_{M2}(\Omega)$, then $[W^k_{M2}(\Omega)]' \subset [W^k_{M_1}(\Omega)]'$.*

Proof. If $f^* \in [W^k_{M2}(\Omega)]'$, $f_n \in W^k_{M1}(\Omega)$, $n = 1, 2, \ldots$ and $\|f_n\|_{W^k_{M_1}} \to 0$ as $n \to \infty$, then $\|f_n\|_{W^k_{M_2}} \to 0$ as $n \to \infty$, and thus $f^*(f_n) \to 0$ as $n \to \infty$.

5.3. THEOREM. *If $M_1(t,u)$, $M_2(t,u)$ are N-functions with parameter satisfying conditions* (v), (vi), (Δ_2) *and if $W^k_{M_1}(\Omega) \subset W^k_{M_2}(\Omega)$, where $W^k_{M_1}(\Omega)$ is dense in $W^k_{M2}(\Omega)$ with respect to the norm $\|\cdot\|_{W^k_{M_2}}$, then $M_2 \prec M_1$, i.e. there exist a set A of measure zero, a constant $K > 0$, and a non-negative function $h \in L_1(\Omega)$ such that*

$$M_2(t,u) \leqslant M_1(t, Ku) + h(t) \quad \textit{for every } t \in \Omega \backslash A,\ u \geqslant 0.$$

Proof. We define the relations $\sim_1$ and $\sim_2$ on spaces $L^l_{N_1}(\Omega)$ and $L^l_{N_2}(\Omega)$, where $l = \sum_{|\alpha| \leqslant k} 1$ and $N_1(t,u)$, $N_2(t,u)$ are complementary N-functions to $M_1(t,u)$ and $M_2(t,u)$, respectively, as follows: $f_1 \sim_1 f_2$ for f_1, $f_2 \in L^l_{N_1}(\Omega)$ iff these elements determine the same bounded linear functionals on $W^k_{M_1}(\Omega)$; $f_1 \sim_2 f_2$ for $f_1, f_2 \in L^l_{N_2}(\Omega)$ iff these elements determine the same bounded linear functionals on $W^k_{M2}(\Omega)$. Now we shall prove that under assumptions of the theorem the relations $\sim_1$ and $\sim_2$ coincide. Obviously, if $f^* \in [W^k_{M2}(\Omega)]'$, then $f^* \in [W^k_{M_1}(\Omega)]'$. Moreover, if $f^*(f) = 0$ for every $f \in W^k_{M2}(\Omega)$, then also $f^*(f) = 0$ for every $f \in W^k_{M1}(\Omega)$. Conversely, if $f^* \in [W^k_{M2}(\Omega)]'$ and $f^*(f) = 0$ for every $f \in W^k_{M1}(\Omega)$, then by density of $W^k_{M1}(\Omega)$ in $W^k_{M2}(\Omega)$ we have $f^*(f) = 0$ for every $f \in W^k_{M2}(\Omega)$.

Thus, we may take $L^l_{N1}/\sim$ and $L^l_{N2}/\sim$ instead of $L^l_{N1}/\sim_1$ and $L^l_{N2}/\sim_2$, respectively. Hence, by Corollary 5.2, we have $L^l_{N2}/\sim \subset L^l_{N1}/\sim$. Thus $L^l_{N2}(\Omega) \subset L^l_{N_1}(\Omega)$ and further $L_{N2}(\Omega) \subset L_{N_1}(\Omega)$. By [8], Theorem 1.8, we obtain that there exist a set A of measure zero, a constant $K > 0$, and a non-negative function $h \in L_1(\Omega)$ such that

$$N_1(t, u) \leqslant N_2(t, Ku) + h(t) \quad \text{for every } t \in \Omega \backslash A,\ u \geqslant 0.$$

Hence, we obtain

$$M_2(t, u) = \sup_{v>0} \{uv - N_2(t, v)\} = \sup_{v>0} \{uv - N_1(t, v/K) + h(t)\}$$

$$\leqslant \sup_{v>0} \left\{Ku \frac{v}{K} - N_1(t, v/K)\right\} + h(t) = M_1(t, Ku) + h(t).$$

Thus the proof is complete.

5.4. Remark. If the assumptions of Theorem 5.3 are satisfied, $\Omega = R^n$. Now let f_δ denote the modification of f, $M_2(\cdot, u)$ is a continuous function on Ω for every fixed $u \geqslant 0$ and $\|f_\delta\|_{L_{M_2}} \leqslant K_1 \|f\|_{L_{M_2}}$ for $0 \leqslant \delta \leqslant \delta_0$, $f \in L_{M_2}(\Omega)$, then $W^k_{M_1}(R^n)$ is dense in $W^k_{M_2}(R^n)$.

Proof. By assumptions, $C_0^\infty(R^n)$ is dense in $W^k_{M_2}(R^n)$ and $C_0^\infty(R^n) \subset W^k_{M_1}(R^n)$ (see [4]).

5.5. Remark. If condition (v) is satisfied, then for every compact set $B \subset \Omega$, $L^\infty(B) \subset L_M(B)$. Conversely, if $L^\infty(B) \subset L_M(B)$ for every compact set $B \subset \Omega$, then the N-function $M(t, u)$ with parameter t satisfies condition (v).

Proof. $L^\infty(B)$ is the Orlicz space generated by the φ-function without parameter

$$\varphi(u) = \begin{cases} 0 & \text{if } |u| \leqslant 1, \\ \infty & \text{if } |u| > 1. \end{cases}$$

The function $\varphi(u)$ has infinite values for $|u| > 1$ and does not satisfy condition: $\varphi(u) = 0$ implies $u = 0$. But such φ-functions were also considered by A. Kozek in [7] and [8]. By Theorem 1.8 from [8], we have

$$M(t, u) \leqslant \varphi(Ku) + h(t), \quad \forall\, u \geqslant 0,\ \forall\, t \in B \backslash A,\ \mu(A) = 0 \tag{5.2}$$

with an $K > 0$ and a non-negative function $h \in L_1(B)$. If $Ku \leqslant 1$, i.e. $u \leqslant 1/K$, then by (5.2), we obtain

$$M(t, u) \leqslant h(t), \quad \forall\, t \in B \backslash A,\ \mu(A) = 0, \tag{5.3}$$

i.e. condition (v) holds with $u_0 = 1/K$. The first part of the remark is obvious.

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References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York–San Francisco–London 1975.

[2] H. Hudzik, *A generalization of Sobolev space* (I), Functiones et Approximatio 2 (1976), p. 67–73.

[3] —, *A generalization of Sobolev space* (II), ibidem 3 (1976), p. 77–85.

[4] —, *Density of* $C_0'(R^n)$ *in generalized Orlicz–Sobolev space* $W_M^k(R^n)$, ibidem 7, p. 15–21.

[5] — and A. Kamińska, *Some remarks on convergence in Orlicz spaces*, Comm. Math. 21 (1979), p. 81–88.

[6] —, —, *Equivalence of the Orlicz and Luxemburg norms in generalized Orlicz spaces* $L_M^\mu(T)$, Functiones et Approximatio 9, in print.

[7] A. Kozek, *Orlicz spaces of functions with values in Banach spaces*, Comm. Math. 19 (1977), p. 259–288.

[8] —, *Convex integral functionals on Orlicz spaces*, ibidem 21 (1979), p. 109–135.

[9] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, *Convex functions and Orlicz spaces* (translated from the first Russian edition), P. Noordhoff–Groningen–The Netherlands, 1961.

[10] W. A. Luxemburg, *Banach function spaces*, Thesis, Technische Hogeschool te Delft, 1955.

[11] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. 18 (1959), p. 49–65.

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