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An approximation theorem in Musielak-Orlicz-Sobolev spaces

Abstract. In this paper we prove the uniform boundedness of the operators of convolution in the Musielak-Orlicz spaces and the density of $C_0^\infty(R^n)$ in the Musielak-Orlicz-Sobolev spaces by assuming a condition of Log-Hölder type of continuity.

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1. Introduction. Let Ω be an open set in R^n and let φ be a real-valued function defined in $\Omega \times R_+$ and satisfying the following conditions :
a) $\varphi(x,.)$ is an N-function [convex, increasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0, \frac{\varphi(x,t)}{t} \to 0$ as $t \to 0$, $\frac{\varphi(x,t)}{t} \to \infty$ as $t \to \infty$]
b) $\varphi(.,t)$ is a measurable function for any $t \in R_+$.
A function $\varphi(x,t)$ which satisfies the conditions a) and b) is called a Musielak-Orlicz function.

We define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)dx,$$

where $u : \Omega \mapsto R$ is a Lebesgue measurable function. In the following the measurability of a function $u : \Omega \mapsto R$ means the Lebesgue measurability.

The set

$$K_\varphi(\Omega) = \{u : \Omega \to R \text{ mesurable } / \varrho_{\varphi,\Omega}(u) < +\infty\}$$

is called the generalized Orlicz class.

The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing

the set $K_\varphi(\Omega)$.
Equivelently:

$$L_\varphi(\Omega) = \left\{ u : \Omega \to R \text{ mesurable } / \varrho_{\varphi,\Omega}(\frac{|u(x)|}{\lambda}) < +\infty, \text{ for some } \lambda > 0 \right\}$$

Let

$$\psi(x,s) = \sup_{t \geqslant 0}\{st - \varphi(x,t)\},$$

that is, ψ is the Musielak-Orlicz function complementary to $\varphi(x,t)$ in the sense of Young with respect to the variable s.

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\{\lambda > 0 / \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx, \leqslant 1\}.$$

which is called the Luxemburg norm and the so-called Orlicz norm by :

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_\psi \leqslant 1} \int_\Omega |u(x)v(x)|dx.$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [19].

We say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall |\alpha| \leqslant m \; D^\alpha u \in L_\varphi(\Omega)\}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with nonnegative integers α_i $|\alpha| = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.

Let

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leqslant m} \varrho_{\varphi,\Omega}(D^\alpha u) \text{ and } ||u||^m_{\varphi,\Omega} = \inf\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leqslant 1\}$$

for $u \in W^m L_\varphi(\Omega)$. These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $< W^m L_\varphi(\Omega), ||u||^m_{\varphi,\Omega} >$ is a Banach space if φ satisfies the following condition [19]:

(1) there exist a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) \geqslant c$.

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

For two Musielak-Orlicz functions φ and ψ the following inequality is called the young inequality [19]:

$$t.s \leqslant \varphi(x,t) + \psi(x,s) \text{ for } t,s \geqslant 0,\ x \in \Omega \tag{2}$$

This inequality implies the inequality

$$|||u|||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}(u) + 1. \tag{3}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular :

$$||u||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}(u) \text{ if } ||u||_{\varphi,\Omega} > 1 \tag{4}$$

$$||u||_{\varphi,\Omega} \geqslant \varrho_{\varphi,\Omega}(u) \text{ if } ||u||_{\varphi,\Omega} \leqslant 1 \tag{5}$$

For two complementary Musielak-Orlicz functions φ and ψ let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$ we have the Hölder inequality [19]:

$$|\int_\Omega u(x)v(x)\ dx| \leqslant ||u||_{\varphi,\Omega}|||v|||_{\psi,\Omega}. \tag{6}$$

In this paper we assume that there exists a constant $A > 0$ such that for all $x, y \in \Omega : |x - y| \leqslant \frac{1}{2}$ we have :

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leqslant t^{\frac{A}{\log(\frac{1}{|x-y|})}} \tag{7}$$

for all $t \geqslant 1$.

For examples of Musielak-Orlicz functions which verify (7) see §Examples.

The aim of this paper is to prove the density of space of class C^∞ functions with compact supports in R^n $C_0^\infty(R^n)$ in the space $W^m L_\varphi(R^n)$ for the modular convergence, under the assumption (7).

Our result generalizes that of Gossez in [14] in the case of classical Orlicz spaces and that of Samko [21] in the case of variable exponent Sobolev spaces $W^{m,p(x)}(R^n)$.

A similar result has been proved by Hudzik in [16] and [17] by assuming the following condition:

$$\int M(x, |f_\varepsilon(x)|)dx \leqslant K \int M(x, |f(x)|)dx \tag{8}$$

for all function $f \in L_M(R^n)$,where f_ε is a regularized function of f. In our paper we don't assume any condition of type(8).

For others approximations results in Musielak-Orlicz-Sobolev spaces and some applications to nonlinear partial differential equations see [9].

And for nonlinear equations in classical Orlicz spaces see [1], [2], [3], [5], [6], [8], [10], [13], [11], [15], [12] and references within.

2. Main results. Let $K(x)$ be a measurable function with support in the ball $B_R = B(0, R)$ and let

$$K_\varepsilon(x) = \frac{1}{\varepsilon^n} K(\frac{x}{\varepsilon}).$$

We consider the family of operators

$$K_\varepsilon f(x) = \int_\Omega K_\varepsilon(x-y) f(y)\ dy. \tag{9}$$

We define

$$\Omega_R = \{x \in R^n : dist(x, \Omega) \leqslant R\} \supseteq \Omega, 0 < R < \infty.$$

THEOREM 2.1 *Let $K(x) \in L^\infty(B_R)$ and let φ and ψ be two complementary Musielak-Orlicz functions such that φ satisfies the conditions (1), (7) and*

$$\text{if } D \subset \Omega \text{ is a bounded measurable set, then } \int_D \varphi(x,1)dx < \infty. \tag{10}$$

And ψ satisfies the following condition:

$$\psi(x,1) \leqslant C \ \ a.e \ in \ \Omega. \tag{11}$$

Then the operators K_ε are uniformly bounded from $L_\varphi(\Omega)$ into $L_\varphi(\Omega_R)$, namely

$$||K_\varepsilon f||_{\varphi,\Omega_R} \leqslant C||f||_{\varphi,\Omega} \ \forall f \in L_\varphi(\Omega), \tag{12}$$

where $C > 0$ does not depend on ε.

REMARK 2.2 For any Musielak-Orlicz function φ we can replace it by a Musielak-Orlicz function $\overline{\varphi}$ which is globally equivalent to φ such that $\overline{\varphi}(x,1) + \overline{\psi}(x,1) = 1$, where $\overline{\psi}$ is the Musielak-Orlicz function complementary to $\overline{\varphi}$ (see [20], §2.4). Hence by (1) we may assume without loss of generality that the condition (11) is always satisfied.

THEOREM 2.3 *Let φ and $K(x)$ satisfy the assumptions of theorem 1 and*

$$\int_{B_R} K(y)\, dy = 1. \tag{13}$$

Then (9) is an identity approximation in $L_\varphi(\Omega)$, that is,

$$\exists\lambda > 0 : \lim_{\varepsilon\to 0} \varrho_{\varphi,\Omega_R}(\frac{K_\varepsilon f - f}{\lambda}) = 0,\ \ f \in L_\varphi(\Omega). \tag{14}$$

Let

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n |B(0,1)|} \int_{y\in\Omega, |y-x|<\varepsilon} f(y)\, dy \tag{15}$$

COROLLARY 2.4 *Under the assumptions of Theorem 2.1 ,*

$$\lim_{\varepsilon\to 0} \varrho_{\varphi,\Omega}(\frac{f_\varepsilon - f}{\lambda}) = 0 \text{ for some } \lambda > 0. \tag{16}$$

REMARK 2.5 . The statement (16) is an analogue of mean continuity property for Musielak-Orlicz spaces, but with respect to the averaged śhiftóperator (15).

COROLLARY 2.6 *Under the assumptions of Theorem 2.1 with $\Omega = R^n$, $C_0^\infty(R^n)$ is dense in $L_\varphi(R^n)$ with respect to the modular topology.*

THEOREM 2.7 *Let φ be a Musielak-Orlicz function which satisfies the assumptions of Theorem 2.1 with $\Omega = R^n$. Then $C_0^\infty(R^n)$ is dense in $W^m L_\varphi(R^n)$ with respect to the modular topology.*

Examples. Let $p : \Omega \mapsto [1,\infty)$ be a measurable function such that there exist a constant $c > 0$ such that for all points $x, y \in \Omega$ with $|x-y| < \frac{1}{2}$, we have the inequality

$$|p(x) - p(y)| \leqslant \frac{c}{\log(\frac{1}{|x-y|})}.$$

Then the following Musielak-Orlicz functions satisfy the conditions of Theorem 2.1 :

1. $\varphi(x,t) = t^{p(x)}$ such that $\sup_{x\in\Omega} p(x) < \infty$,
2. $\varphi(x,t) = t^{p(x)} \log(1+t)$,
3. $\varphi(x,t) = t(\log(t+1))^{p(x)}$,

4. $\varphi(x,t) = (e^t)^{p(x)} - 1.$

3. Proofs.

Proof (of Theorem 2.1) .
We assume that

$$||f||_{\varphi,\Omega} \leqslant 1. \tag{17}$$

It suffices to show that

$$\varrho_{\varphi,\Omega_R}(K_\varepsilon f) = \int_{\Omega_R} \varphi(x, |K_\varepsilon f(x)|)dx \leqslant c \tag{18}$$

for some ε such that $0 < \varepsilon \leqslant \varepsilon^0 \leqslant 1$, and $c > 0$ independent of f.

Let

$$\Omega_R = \cup_{k=1}^N \omega_R^k$$

be any partition of Ω_R into small parts ω_R^k comparable with the given ε:

$$diam\ \omega_R^k \leqslant \varepsilon,\ k = 1, 2, 3..., N = N(\varepsilon).$$

We represent the integral in (18) as

$$\varrho_{\varphi,\Omega_R}(K_\varepsilon f) = \sum_{k=1}^N \int_{\omega_R^k} \varphi(x, |\int_\Omega K_\varepsilon(x-y)f(y)\ dy|)dx. \tag{19}$$

We put

$$\varphi_k(t) = \inf\{\varphi(x,t), x \in \Omega_R^k\} \leqslant \inf\{\varphi(x,t), x \in \omega_R^k\} \tag{20}$$

where some larger partition $\Omega_R^k \supset \omega_R^k$ will be chosen later comparable with ε :

$$diam\ \Omega_R^k \leqslant m\varepsilon\ , m > 1. \tag{21}$$

Hence :

$$\varrho_{\varphi,\Omega_R}(K_\varepsilon f) = \sum_{k=1}^N \int_{\omega_R^k} A_k(x,\varepsilon)\ \varphi_k(|\int_\Omega K_\varepsilon(x-y)f(y)\ dy|)\ dx\ . \tag{22}$$

where

$$A_k(x,\varepsilon) := \frac{\varphi(x, |\int_\Omega K_\varepsilon(x-y)f(y)\ dy|)}{\varphi_k(|\int_\Omega K_\varepsilon(x-y)f(y)\ dy|)}$$

We shall prove the uniform estimate

$$A_k(x,\varepsilon) \leqslant c,\ x \in \omega_R^k \tag{23}$$

where $c > 0$ does not depend on $x \in \omega_R^k,\ k$ and $\varepsilon \in (0, \varepsilon^0)$ with some $\varepsilon^0 > 0$.

By (6) we have

$$\begin{aligned}\alpha(x,\varepsilon) := |\int_\Omega K_\varepsilon(x-y)f(y)\,dy| &\leqslant \frac{M}{\varepsilon^n}\int_\Omega |\chi_{B_{\varepsilon R}}(y)f(y)|dy \\ &\leqslant \frac{M}{\varepsilon^n}||f||_\varphi\ |||\chi_{B_{\varepsilon R}}|||_\psi\end{aligned}$$

where $M = \sup_{B_R}|K(y)|$.

By (3) and the condition (11) we obtain

$$|||\chi_{B_{\varepsilon R}}|||_\psi \leqslant c_2|B_{\varepsilon R}| + 1 \leqslant c_2 + 1 \tag{24}$$

for $0 < \varepsilon \leqslant |B(0,1)|^{-\frac{1}{n}} := \varepsilon_1^0$.

Hence

$$\alpha(x,\varepsilon) \leqslant \frac{c_1}{\varepsilon^n}. \tag{25}$$

We observe now that by (7) and (20) we have

$$\frac{\varphi(x,t)}{\varphi_k(t)} = \frac{\varphi(x,t)}{\varphi(\xi_k,t)} \leqslant t^{\frac{A}{\log(\frac{1}{|x-\xi_k|})}} \tag{26}$$

where $x \in \omega_R^k,\ \xi_k \in \Omega_R^k$. Evidently $|x-\xi_k| \leqslant diam\ \Omega_R^k \leqslant m\varepsilon$.
Therefore,

$$\begin{aligned}A_k(x,\varepsilon) &= \frac{\varphi(x,\alpha(x,\varepsilon))}{\varphi(\xi_k,\alpha(x,\varepsilon))} \leqslant (\alpha(x,\varepsilon))^{\frac{A}{\log(\frac{1}{m\varepsilon})}} \\ &\leqslant (c_1\varepsilon^{-n})^{\frac{A}{\log(\frac{1}{m\varepsilon})}} \leqslant (c_1)^{\frac{A}{\log(\frac{1}{m})}}(\varepsilon^{-n})^{\frac{A}{\log(\frac{1}{m\varepsilon})}}\end{aligned} \tag{27}$$

under the assumption that $0 < \varepsilon \leqslant \frac{1}{2m} := \varepsilon_2^0$.

Then from (27)

$$A_k(x,\varepsilon) \leqslant c_4 := c_3 e^{2nA},\ c_3 = (c_1)^{\frac{A}{\log(\frac{1}{m})}} \tag{28}$$

for $x \in \omega_R^k$ and

$$0 < \varepsilon \leqslant \frac{1}{m^2} := \varepsilon_3^0. \tag{29}$$

Therefore, we have the uniform estimate (23) with $c = c_3 e^{2nA}$ and $0 < \varepsilon \leqslant \varepsilon^0,\ \varepsilon^0 = \min_{1\leqslant k\leqslant 3}\varepsilon_k^0,\ \varepsilon_k^0$ being given above.

Using the estimate (23) we obtain from (22)

$$\varrho_{\varphi,\Omega_R}(K_\varepsilon f) = c\sum_{k=1}^{N}\int_{\omega_R^k}\varphi_k(|\int_\Omega K_\varepsilon(x-y)f(y)\ dy|)dx\ . \tag{30}$$

So by the Jensen integral inequality we obtain

$$\begin{aligned}\varrho_{\varphi,\Omega_R}(K_\varepsilon f) &\leqslant c\textstyle\sum_{k=1}^{N}\int_{|y|<\varepsilon R}|K_\varepsilon(y)|dy\int_{\omega_R^k}\varphi_k(f(x-y))dx\\ &= c\textstyle\sum_{k=1}^{N}\int_{|y|<R}|K(y)|dy\int_{x+\varepsilon y\in\omega_R^k}\varphi_k(f(x))dx\end{aligned} \tag{31}$$

Obviously, the domain of integration in x in the last integral is embedded into the domain

$$\bigcup_{y\in B_{\varepsilon R}}\{x : x+y\in\omega_R^k\} \tag{32}$$

which does not depend on y. Now, we choose the sets Ω_R^k in (20), which were not determined until now, as the sets (32). Then, evidently, $\Omega_R^k\supset\omega_R^k$, and it is easily seen that

$$diam\ \Omega_R^k\leqslant(1+2R)\varepsilon \tag{33}$$

so the requirement (21) is satisfied with $m=1+2R$.

From (32) we have

$$\begin{aligned}\varrho_{\varphi,\Omega_R}(K_\varepsilon f) &\leqslant c\textstyle\sum_{k=1}^{N}\int_{|y|<R}|K(y)|dy\int_{\Omega_R^k}\varphi_k(f(x))dx\\ &\leqslant c\textstyle\int_{|y|<R}|K(y)|dy\sum_{k=1}^{N}\int_{\Omega_R^k\cap\Omega}\varphi_k(f(x))dx\end{aligned} \tag{34}$$

In view of (33), the covering $\{\omega_k=\Omega_R^k\cap\Omega\}_{k=1}^N$ has a finite multiplicity (that is, each point $x\in\Omega$ belongs simultaneously to not more than a finite number n_0 of the sets w_k, $n_0\leqslant 1+(1+2R)^n$ in this case) .
Therefore,

$$\varrho_{\varphi,\Omega_R}(K_\varepsilon f)\leqslant c_5\int_\Omega\tilde{\varphi}(x,f(x))\ dx, \tag{35}$$

where

$$\tilde{\varphi}(x,t)=\max_i\varphi_i(t),$$

the maximum being taken with respect to all the sets ω_k containing x. Evidently, $\tilde{\varphi}(x,t)\leqslant\varphi(x,t)\ \ \forall x\in\Omega$.
Then from (35) and (17) we arrive at the final estimate

$$\varrho_{\varphi,\Omega_R}(K_\varepsilon f)\leqslant c_5\int_\Omega\varphi(x,f(x))\ dx\leqslant c_5. \tag{36}$$

■

Proof (of Theorem 2.3) .
To prove (14), we use the Theorem 2.1, which provides the uniform boundedness of the operators K_ε from $L_\varphi(\Omega)$ into $L_\varphi(\Omega_R)$. Then by the Banach-Steinhaus theorem it suffices to verify that (14) holds for some dense set in $L_\varphi(\Omega)$. So, it is sufficient to prove (14) for the characteristic function $\chi_E(x)$ of any bounded measurable set $E \subset \Omega$ [19]. We have

$$K_\varepsilon(\chi_E)(x) - \chi_E(x) = \int_{B_R} k(y)[\chi_E(x-\varepsilon y) - \chi_E(x)]dy,$$

Hence for $\lambda > 0$

$$\varrho_{\varphi,\Omega_R}(\frac{K_\varepsilon(\chi_E) - \chi_E}{\lambda}) = \int_{\Omega_R} \varphi(x, \frac{1}{\lambda}\int_{B_R} k(y)[\chi_E(x-\varepsilon y) - \chi_E(x)]dy)dx$$

$$\leqslant \int_{B_R} k(y)(\int_{\Omega_R} \varphi(x, \frac{1}{\lambda}[\chi_E(x-\varepsilon y) - \chi_E(x)])dx)dy \quad \blacksquare$$

by the Fubini theorem and the Jensen inequality. Hence by the condition (10) and the Lebesgue dominated convergence theorem we obtain (14) for some $\lambda > 0$.

Proof (of Corollary 2.4) .
To obtain Corollary 2.4 from Theorem 2.1, it suffices to choose $K(y) = \frac{1}{|B(0,1)|}\chi_{B(0,1)}(y)$.

Proof (of Corollary 2.6) .
Let $\chi_N(x) = \chi_{B(0,N)}(x)$. Then the functions $f^N(x) = \chi_N(x)f(x)$ have compact supports and for the approximate of $f(x) \in L_\varphi(R^n)$ by f^N, we have :

$$\varrho_{\varphi,R^n}(\frac{f-f^N}{\lambda}) = \int_{R^n} \varphi(x, (\frac{f-f^N}{\lambda})(x))dx = \int_{|x|>N} \varphi(x, \frac{f(x)}{\lambda})dx \to 0$$

as $N \to \infty$.

Therefore, we may consider $f(x)$ with a compact support in the ball B_N from the very beginning. To approximate the $f(x)$ by C_0^∞, we use the identity approximation

$$f_\varepsilon(x) = \int_{R^n} K_\varepsilon(x-t)f(t)dt = \int_{|y|<1} K(y)f(x-\varepsilon y)dy \tag{37}$$

where $k(y) \in C_0^\infty(R^n)$ has its support in the ball B_1 and satisfies

$$\int_{|y|<1} K(y)dy = 1.$$

Then, evidently, $f_\varepsilon(x) \in C^\infty(R^n)$ and has a compact support because $f_\varepsilon \equiv 0$ if $|x| > N + \varepsilon$.

Therefore, for $\varepsilon < 1$, there exist some $\lambda > 0$ such that

$$\varrho_{\varphi,R^n}(\frac{f_\varepsilon - f}{\lambda}) = \varrho_{\varphi,B_{N+1}}(\frac{K_\varepsilon f - f}{\lambda}) \to 0 \tag{38}$$

as $\varepsilon \to 0$, by Theorem 2.3. ■

PROOF (OF THEOREM 2.7) .
The proof follows from Theorem 2.3 and Corollary 2.6 in two steps.

1. Let $f(x) \in W^m L_\varphi(R^n)$ and let $\mu(r),\ 0 \leqslant r \leqslant \infty$, be a smooth step-function : $\mu(x) \equiv 1$ for $0 \leqslant |x| \leqslant 1, \mu(x) \equiv 0$ for $|x| \geqslant 2,\ \mu(x) \in C_0^\infty(R^n)$ and $0 \leqslant \mu(x) \leqslant 1$. Then

$$f^N(x) := \mu(\frac{x}{N}) f(x) \in W^m L_\varphi(R^n) \tag{39}$$

for every $N \in R_+$ and f^N has compact support in B_{2N}.

The function (39) approximate $f(x)$ in $W^m L_\varphi(R^n)$. Indeed, denoting $\nu_N(x) = 1 - \mu(\frac{x}{N})$, we know that $\nu_N(x) \equiv 0$ for $|x| < N$, so using the Leibnitz formula for differentiation, we have for $\lambda > 0$

$$\begin{aligned}
\overline{\varrho}_{\varphi,R^n}(\frac{f - f^N}{\lambda}) &= \sum_{|j|\leqslant m} \varrho_{\varphi,R^n}(\frac{D^j(\nu_N f)}{\lambda}) \\
&= \sum_{|j|\leqslant m} \varrho_{\varphi,R^n}(\sum_{0\leqslant k\leqslant j} c_j^k \frac{D^k(\nu_N) D^{j-k} f}{\lambda}) \\
&\leqslant \sum_{|j|\leqslant m} \sum_{0\leqslant k\leqslant j} c_j^k \varrho_{\varphi,R^n}(\frac{D^k(\nu_N) D^{j-k} f}{\lambda}) \\
&\leqslant \sum_{|j|\leqslant m} \varrho_{\varphi,R^n}(\frac{\nu_N D^j(f)}{\lambda}) \\
&+ c \sum_{|j|\leqslant m} \sum_{1\leqslant k\leqslant j} \varrho_{\varphi,R^n}(\frac{D^k(\nu_N) D^{j-k} f}{\lambda}) \\
&\leqslant \sum_{|j|\leqslant m} \varrho_{\varphi,R^n}(\frac{\nu_N D^j(f)}{\lambda}) \\
&+ c \sum_{|j|\leqslant m} \sum_{1\leqslant k\leqslant j} \frac{1}{N^{|k|}} \varrho_{\varphi,R^n}(\frac{D^{j-k} f}{\lambda}).
\end{aligned}$$

Hence there exist a $\lambda > 0$ which depend to m, n such that each term on the last hand side goes to Zero as $N \to \infty$, the first one by the Lebesgue dominated convergence theorem and the second one by direct examination.

2. By step 1 we may consider $f(x) \in W^m L_\varphi(R^n)$ with compact support. Then we arrange the approximation (37), evidently, $f_\varepsilon \in C_0^\infty(R^n)$. Indeed, for any j we have

$$D^j f_\varepsilon(x) = \frac{1}{\varepsilon^{n+|j|}} \int_{|y|<1} (D^j K)(\frac{x-t}{\varepsilon}) f(t) dt \in C^\infty(R^n) \tag{40}$$

and $f_\varepsilon(x)$ has compact support because $f_\varepsilon(x) \equiv 0$ if $|x| > 1 + \beta$, where $\beta = \sup_{x \in suppf} |x|$.

We have

$$\begin{aligned} \overline{\varrho}_{\varphi,R^n}(\frac{f-f_\varepsilon}{\lambda}) &\leqslant \sum_{|j|\leqslant m} \varrho_{\varphi,R^n}(\frac{D^j f - K_\varepsilon(D^j f)}{\lambda}) \\ &= \sum_{|j|\leqslant m} \varrho_{\varphi,\Omega_1}(\frac{D^j f - K_\varepsilon(D^j f)}{\lambda}) \end{aligned}$$

where $\Omega_1 = \{x : dist(x, \Omega) \leqslant 1\}$, $\Omega = suppf(x)$. It suffices to apply Theorem 2.3. ∎

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