ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXVI (1986)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXVI (1986)

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A note on generalized modular spaces. I

Abstract. This note is a continuation of note [4]. Here, concepts of locally s-convex $(0 < s \leqslant 1)$ and locally bounded premodular bases and semi-premodulars in a linear lattice are introduced. We study describing of premodular bases by semi-premodulars of adequate types.

This note is a continuation of note [4]. The results on modular bases contained in [2] and [3] are generalized here, with their adjustment to a real linear lattice S. Through the whole note by s we shall denote a real number such that $0 < s \leqslant 1$.

1.1. *Each filtrating base B in S composed of normal and s-convex sets is an s-premodular base.* We shall call such base a *locally s-convex premodular base.*

This immediately follows from the equality $\Gamma_s(N(U)) = \Gamma_s(U) = U$ which holds in this case for each $U \in B$ (cf. [4], 4.2).

1.2. *If B is a locally s-convex premodular base, then the pretopological bases $B^{\check{}}$ and $B^{\hat{}}$ are also locally s-convex.*

The local s-convexity of the base $B^{\check{}}$ immediately follows from the fact that the base $B^{\check{}}$ is a family of sets of the form $V = \alpha U$, where $U \in B$ and $\alpha \neq 0$. Now, let $\{U_n\}$ be any sequence of sets of B. Let us observe that

$$\Gamma_s(W(\{U_n\})) = \Gamma_s\Big(\bigcup_{n=1}^{\infty} (U_1 + \ldots + U_n)\Big) = \bigcup_{n=1}^{\infty} \Gamma_s(U_1 + \ldots + U_n)$$
$$\subset \bigcup_{n=1}^{\infty} \big(\Gamma_s(U_1) + \ldots + \Gamma_s(U_n)\big) = \bigcup_{n=1}^{\infty} (U_1 + \ldots + U_n) = W(\{U_n\})$$

and

$$N(W(\{U_n\})) = N\Big(\bigcup_{n=1}^{\infty} (U_1 + \ldots + U_n)\Big) = \bigcup_{n=1}^{\infty} N(U_1 + \ldots + U_n)$$
$$\subset \bigcup_{n=1}^{\infty} \big(N(U_1) + \ldots + N(U_n)\big) = \bigcup_{n=1}^{\infty} (U_1 + \ldots + U_n)$$
$$= W(\{U_n\}).$$

That means that each set of the base $B^{\hat{}}$ is normal and s-convex, so the base $B^{\hat{}}$ is also locally s-convex.

1.3. A filtrating functional ϱ defined on S is called a *locally s-convex semi-premodular* if it satisfies the conditions: (ii) (see [4], 3.1), and

(iii/lc^s) $\varrho(\alpha x+\beta y) \leqslant \sup\{\varrho(x), \varrho(y)\}$ for any numbers $\alpha, \beta \geqslant 0$ such that $\alpha^s+\beta^s \leqslant 1$ and for arbitrary $x, y \in S$.

A semi-prenorm which satisfies conditions (ii) and (iii/lc^s) is also called *locally s-convex*. Here, it is obvious that each locally s-convex semi-premodular is also a semi-s-premodular (cf. [4], 4.4). The condition of the local s-convexity (iii/lc^s) was introduced for the first time for the case $s = 1$ in [3].

If ϱ is a locally s-convex semi-premodular on S, then the base $B(\varrho)$, i.e., the family of sets of the form $U(\varrho, \varepsilon) = \{x \in S\colon\ \varrho(x) < \varepsilon\}$, where $\varepsilon > 0$, is locally s-convex.

It suffices only to show the s-convexity of $U(\varrho, \varepsilon)$. So, let $x, y \in U(\varrho, \varepsilon)$ and α, β be non-negative numbers such that $\alpha^s+\beta^s \leqslant 1$. Then there holds $\varrho(x) < \varepsilon$ and $\varrho(y) < \varepsilon$, and by virtue of condition (iii/lc^s) we get $\varrho(\alpha x+\beta y) < \varepsilon$. So $\alpha x+\beta y \in U(\varrho, \varepsilon)$, and it proves that the set $U(\varrho, \varepsilon)$ is s-convex.

Reversely, *if the base $B(\varrho)$ of the filtrating functional ϱ is locally s-convex, then ϱ is a locally s-convex semi-premodular.*

Let us take arbitrary $x, y \in S$ and any numbers $\alpha, \beta \geqslant 0$ such that $\alpha^s+\beta^s \leqslant 1$. If $\varrho(x) = \infty$ or $\varrho(y) = \infty$, then condition (iii/lc^s) for ϱ is obvious. So let $\varrho(x) < \infty$ and $\varrho(y) < \infty$. We select any ε such that $\sup\{\varrho(x), \varrho(y)\} < \varepsilon$. Then $x, y \in U(\varrho, \varepsilon)$. Since the set $U(\varrho, \varepsilon)$ is by the assumption s-convex, so we get $\alpha x+\beta y \in U(\varrho, \varepsilon)$. Hence $\varrho(\alpha x+\beta y) < \varepsilon$. This way we get condition (iii/lc^s) for ϱ. Moreover, let x and y be such that $|x| \leqslant |y|$. If $\varrho(y) = \infty$, then condition (ii) for ϱ is obvious. So let $\varrho(y) < \infty$. We take any number ε such that $\varrho(y) < \varepsilon$. Then there holds $y \in U(\varrho, \varepsilon)$. Since by the assumption the set $U(\varrho, \varepsilon)$ is normal, we have $x \in U(\varrho, \varepsilon)$. Then $\varrho(x) < \varepsilon$, and hence we get condition (ii) for ϱ.

1.4. *A premodular base B can be described by a locally s-convex semi-premodular if and only if it is equivalent to some at most countable locally s-convex premodular base.*

If the premodular base B is describable by the locally s-convex semi-premodular ϱ, then we can observe that the family $B'(\varrho)$ of sets of the form $U(\varrho, 2^{-n})$, where $n = 1, 2, \ldots$, is at most countable locally s-convex premodular base equivalent to the base $B(\varrho)$, so it is also equivalent to B. Conversely, let B be a premodular base equivalent to some at most countable locally s-convex premodular base $B_1 = \{U_n\}$. We may assume that the sequence of sets $\{U_n\}$ is descending; in the other case by virtue of condition (a) we could select a subsequence $\{U_{n_k}\}$ from the sequence $\{U_n\}$ such that $U_{n_1} = U_1$ and $U_{n_{k+1}} \subset U_{n_k} \cap U_k$ for $k = 1, 2, \ldots$, and then we could get a base equivalent to B_1 which had already this property. We define the functional ϱ on S by the formula

$$\varrho(x) = \begin{cases} 2^{-n} & \text{if } x \in U_n \setminus U_{n+1} \ (n = 0, 1, \ldots), \\ 0 & \text{if } x \in \bigcap_{n=1}^{\infty} U_n, \end{cases}$$

where $U_0 = S$. From 3.3 in [4] we know that ϱ is a semi-premodular on S that describes the base B. So it suffices only to show that ϱ is locally s-convex. We take any elements $x, y \in S$ and any numbers $\alpha, \beta \geqslant 0$ such that $\alpha^s + \beta^s \leqslant 1$. Let us observe that $\sup\{\varrho(x), \varrho(y)\} = 0$ or is equal to 2^{-n}, where n is some non-negative integer. In the first case there holds $x, y \in \bigcap_{n=1}^{\infty} U_n$, in the other one $x, y \in U_n$. Since the sets U_n and $\bigcap_{n=1}^{\infty} U_n$ are s-convex, in the first case we get $\alpha x + \beta y \in \bigcap_{n=1}^{\infty} U_n$ and next $\varrho(\alpha x + \beta y) = 0$, while in the second one $\alpha x + \beta y \in U_n$ which gives $\varrho(\alpha x + \beta y) \leqslant 2^{-n}$. Hence we conclude that ϱ satisfies the condition of the local s-convexity (iii/lc^s).

1.5. *If ϱ is a locally s-convex semi-premodular, then the semi-prenorm defined by the formula* (cf. 5.6 in [4])

$$\varrho^{\vee}(x) = \begin{cases} \inf\{\varepsilon > 0\colon \varrho(x/\varepsilon^{1/s}) \leqslant \varepsilon\} & \textit{if } \varrho(\lambda x) < \infty \textit{ for some } \lambda > 0, \\ \infty & \textit{otherwise } (x \in S), \end{cases}$$

is also locally s-convex.

We take any elements $x, y \in S$ and any numbers $\alpha, \beta \geqslant 0$ such that $\alpha^s + \beta^s \leqslant 1$. If $\varrho^{\vee}(x) = \infty$ or $\varrho^{\vee}(y) = \infty$, then condition (iii/lc^s) for $\varrho^{\vee}$ is obvious. In the other case we take any ε such that $\sup\{\varrho^{\vee}(x), \varrho^{\vee}(y)\} < \varepsilon$. Then we have $\varrho(x/\varepsilon^{1/s}) \leqslant \varepsilon$ and $\varrho(y/\varepsilon^{1/s}) \leqslant \varepsilon$. Hence, by virtue of condition (iii/lc^s) for ϱ, we get

$$\begin{aligned} \varrho\big((\alpha x + \beta y)/\varepsilon^{1/s}\big) &= \varrho\big(\alpha(x/\varepsilon^{1/s}) + \beta(y/\varepsilon^{1/s})\big) \\ &\leqslant \sup\{\varrho(x/\varepsilon^{1/s}), \varrho(y/\varepsilon^{1/s})\} \leqslant \varepsilon. \end{aligned}$$

Therefore $\varrho^{\vee}(\alpha x + \beta y) \leqslant \varepsilon$. Hence we conclude that in this case $\varrho^{\vee}$ also satisfies condition (iii/lc^s).

2.1. Let U be any set in S. We shall denote by $Nc^s(U)$ the smallest normal and s-convex set in S containing the set U.

Now let B be a filtrating base in S. We shall denote by $Nc^s(B)$ the family of all sets $Nc^s(U)$, where $U \in B$. From the properties of the filtrating bases we conclude that the family $Nc^s(B)$ is also a filtrating base. Since, in addition to this, the family $Nc^s(B)$ is composed of normal and s-convex sets, so it is a locally s-convex premodular base. For this reason we shall call the family $Nc^s(B)$ the *locally s-convex premodular base generated by the filtrating base B.*

2.2. *The locally s-convex premodular base $Nc^s(B)$ generated by the filtrating base B has the following properties:*

1° $Nc^s(B) \dashv B$,

2° *if B_1 is a locally s-convex premodular base such that $B_1 \dashv B$, then $B_1 \dashv Nc^s(B)$.*

Property 1° follows from the obvious inclusion $U \subset Nc^s(U)$ for arbitrary $U \subset S$. The predecessor of implication 2° while written in the split form looks as follows: there exists a number $\alpha \neq 0$ such that for every $U_1 \in B_1$ there exists $U \in B$ such that $\alpha U \subset U_1$. Since U_1 is a normal and s-convex set, so it must also hold $\alpha Nc^s(U) \subset U_1$. Hence we get the consequent of implication 2°.

2.3. *If B_1 and B_2 are filtrating bases such that $B_1 \dashv B_2$, then also $Nc^s(B_1) \dashv Nc^s(B_2)$.*

One can easy conclude it from 2.2.

2.4. *If B is a pretopological base, then $Nc^s(B)$ is also such a base.*

Let us observe that if the base B satisfies condition (Δ_2): "*for every $U \in B$ there exists $U' \in B$ such that $2U' \subset U$*", then also $2Nc^s(U') \subset Nc^s(U)$, so the base $Nc^s(B)$ satisfies this condition as well. The result from 2.2 in paper [4] completes the rest of the proof.

2.5. *If B is a premodular base, then*

$$(Nc^s(B))^\vee = Nc^s(B^\vee) \quad \textit{and} \quad (Nc^s(B))^\wedge \sim Nc^s(B^\wedge).$$

For any set $U \in B$ and for any number $\alpha \neq 0$ there holds the obvious equality $Nc^s(\alpha U) = \alpha Nc^s(U)$. So $Nc^s(B^\vee) = (Nc^s(B))^\vee$. By virtue of results 1.2, 2.2, 2.3, 2.4 and results 5.2 and 5.3 in paper [4] on the one hand we have $B^\wedge \dashv B$, further $Nc^s(B^\wedge) \dashv Nc^s(B)$, and finally $Nc^s(B^\wedge) \dashv (Nc^s(B))^\wedge$, while on the other hand: $Nc^s(B) \dashv B$, further $(Nc^s(B))^\wedge \dashv B^\wedge$, and finally $(Nc^s(B))^\wedge \dashv Nc^s(B^\wedge)$. So $(Nc^s(B))^\wedge \sim Nc^s(B^\wedge)$.

2.6. Let ϱ be a filtrating functional on S. We define a functional $Nc^s\varrho$ on S by the formula

$$(Nc^s\varrho)(x) = \inf\Big\{\sup_{1 \leqslant k \leqslant m} \varrho(x_k)\colon |x| \leqslant \sum_{k=1}^{m} \alpha_k |x_k|,\ \alpha_k \geqslant 0,\ \sum_{k=1}^{m} \alpha_k^s \leqslant 1,\ x_k \in S\Big\}.$$

The functional $Nc^s\varrho$ is a locally s-convex semi-premodular such that $B(Nc^s\varrho) = Nc^s(B(\varrho))$ for any filtrating functional ϱ.

By virtue of 1.3 and 2.1 it suffices only to show that $B(Nc^s\varrho) = Nc^s(B(\varrho))$. To this end we take $x \in U(Nc^s\varrho, \varepsilon)$, where $\varepsilon > 0$, i.e., $(Nc^s\varrho)(x)$

$< \varepsilon$. Then there exist elements $x_1, \ldots, x_m$ in S and numbers $\alpha_1, \ldots, \alpha_m \geqslant 0$ such that

$$\sup_{1 \leqslant k \leqslant m} \varrho(x_k) < \varepsilon, \quad |x| \leqslant \sum_{k=1}^{m} \alpha_k |x_k|, \quad \text{and} \quad \sum_{k=1}^{m} \alpha_k^s \leqslant 1.$$

Hence we get $x_1, \ldots, x_m \in U(\varrho, \varepsilon)$, and further $x \in Nc^s(U(\varrho, \varepsilon))$. So we obtain $U(Nc^s \varrho, \varepsilon) \subset Nc^s(U(\varrho, \varepsilon))$. Taking $x \in Nc^s(U(\varrho, \varepsilon))$ let us observe that then there exist elements $x_1, \ldots, x_m \in U(\varrho, \varepsilon)$ and numbers $\alpha_1, \ldots, \alpha_m \geqslant 0$ such that

$$|x| \leqslant \sum_{k=1}^{m} \alpha_k |x_k| \quad \text{and} \quad \sum_{k=1}^{m} \alpha_k^s \leqslant 1.$$

Since $\sup_{1 \leqslant k \leqslant m} \varrho(x_k) < \varepsilon$, we get $(Nc^s \varrho)(x) < \varepsilon$, and hence $x \in U(Nc^s \varrho, \varepsilon)$. This proves that the inclusion $U(Nc^s \varrho, \varepsilon) \supset Nc^s(U(\varrho, \varepsilon))$ holds, too. So, for any $\varepsilon > 0$ the equality $U(Nc^s \varrho, \varepsilon) = Nc^s(U(\varrho, \varepsilon))$ holds.

2.7. *If ϱ is a locally s-convex semi-premodular, then $Nc^s \varrho = \varrho$.*

In this case we have $Nc^s(U(\varrho, \varepsilon)) = U(\varrho, \varepsilon)$ for every $\varepsilon > 0$. By this and the proof of 2.6 we get $U(Nc^s \varrho, \varepsilon) = U(\varrho, \varepsilon)$ for every $\varepsilon > 0$. The above implies that $(Nc^s \varrho)(x) = \varrho(x)$ for every $x \in S$.

2.8. *The equality $Nc^s(\varrho^\vee) = (Nc^s \varrho)^\vee$ holds for any semi-s-premodular.*

Let $x \in S$ be such that $(Nc^s(\varrho^\vee))(x) < \infty$. We take any number ε such that $(Nc^s(\varrho^\vee))(x) < \varepsilon$. Then there exist elements $x_1, \ldots, x_m \in S$ and numbers $\alpha_1, \ldots, \alpha_m \geqslant 0$ such that

$$\sup_{1 \leqslant k \leqslant m} \varrho^\vee(x_k) < \varepsilon, \quad |x| \leqslant \sum_{k=1}^{m} \alpha_k |x_k|, \quad \text{and} \quad \sum_{k=1}^{m} \alpha_k^s \leqslant 1.$$

Hence we get

$$(*) \qquad \sup_{1 \leqslant k \leqslant m} \varrho(x_k/\varepsilon^{1/s}) < \varepsilon, \quad |(x/\varepsilon^{1/s})| \leqslant \sum_{k=1}^{m} \alpha_k |(x_k/\varepsilon^{1/s})|, \quad \sum_{k=1}^{m} \alpha_k^s \leqslant 1.$$

So $(Nc^s \varrho)(x/\varepsilon^{1/s}) < \varepsilon$, and further $(Nc^s \varrho)^\vee(x) \leqslant \varepsilon$. That way we get the inequality $(Nc^s \varrho)^\vee(x) \leqslant (Nc^s(\varrho^\vee))(x)$. Now, let $x \in S$ be such that $(Nc^s \varrho)^\vee(x) < \infty$. We take any number ε with the property $(Nc^s \varrho)^\vee(x) < \varepsilon$. Then there holds $(Nc^s \varrho)(x/\varepsilon^{1/s}) < \varepsilon$. Further, there exist elements $x_1, \ldots, x_m \in S$ and numbers $\alpha_1, \ldots, \alpha_m \geqslant 0$ such that $(*)$ holds. Hence we get $\sup_{1 \leqslant k \leqslant m} \varrho^\vee(x_k) \leqslant \varepsilon$ and next $(Nc^s(\varrho^\vee))(x) \leqslant \varepsilon$. So the inequality $(Nc^s(\varrho^\vee))(x) \leqslant (Nc^s \varrho)^\vee(x)$ is also satisfied.

3.1. We shall call a premodular base B in S *almost locally bounded* if it satisfies the following condition:

(1b*) there exists a set $U_0 \in B$ such that for every set $U \in B$ there exists a number $\alpha \neq 0$ such that $\alpha U_0 \subset U$, while we shall call it *locally bounded* if it satisfies the condition:

(1b) for any sets $U_1, U_2 \in B$ there exists a number $\alpha \neq 0$ such that $\alpha U_1 \subset U_2$.

We shall call a semi-premodular ϱ defined on S *almost locally bounded* if the base $B(\varrho)$ is almost locally bounded and we shall call it *locally bounded* if its base $B(\varrho)$ is locally bounded. Conditions (1b*) and (1b) for a semi-premodular have the form:

(1b*) there exists $\delta > 0$ such that for every $\varepsilon > 0$ there exists $\alpha > 0$ such that for any $x \in S$ there holds the implication: if $\varrho(x) < \delta$, then $\varrho(\alpha x) < \varepsilon$,

(1b) for any numbers $\varepsilon, \delta > 0$ there exists a number $\alpha > 0$ such that there holds the implication: if $\varrho(x) < \delta$, then $\varrho(\alpha x) < \varepsilon$.

3.2. *Let B_1 and B_2 be two premodular bases in S. If $B_1 \sim B_2$ and the base B_1 is almost locally bounded, then the base B_2 has the same property.*

3.3. *A premodular base B in S is almost locally bounded if and only if the base $B^\vee$ is almost locally bounded and B is locally bounded if and only if $B^\vee$ is locally bounded.*

3.4. *For every almost locally bounded premodular base B in S there is an equivalent locally bounded premodular base B_1 in S. It can be assumed that $B_1 \subset B$.*

Proofs of 3.2, 3.3, and 3.4 are analogous to the proofs of results 8.2, 8.3, and 8.4 in paper [2], so are omitted here.

3.5. *For every almost locally bounded semi-premodular ϱ there exists a locally bounded semi-premodular ϱ' equivalent to it.*

Proof is similar to the proof of result 4.2 in paper [3], so is omitted here.

3.6. *If a semi-s-premodular ϱ is locally bounded, then a semi-prenorm $\varrho^\vee$ is also locally bounded.*

We take numbers $\varepsilon, \delta > 0$. From the assumption it follows the existence of $\alpha > 0$ such that $\varrho(x) < \delta$ implies $\varrho(\alpha x) < \frac{1}{2}\varepsilon$ for every $x \in S$. Since $\varrho^\vee(x) < \delta$ implies $\varrho(x/\delta^{1/s}) < \delta$, so $\varrho^\vee(x) < \delta$ implies $\varrho(\alpha x/\delta^{1/s}) < \frac{1}{2}\varepsilon$, and further $\varrho^\vee((\frac{1}{2}\varepsilon)^{1/s}\alpha x/\delta^{1/s}) \leqslant \frac{1}{2}\varepsilon < \varepsilon$. That proves the semi-prenorm $\varrho^\vee$ is also locally bounded.

4.1. We call a filtrating functional ϱ defined on S an *s-convex semi-premodular* if it satisfies the conditions: (ii) and

(iii/c^s) $\varrho(\alpha x + \beta y) \leqslant \alpha^s \varrho(x) + \beta^s \varrho(y)$ for any numbers $\alpha, \beta \geqslant 0$ such that $\alpha^s + \beta^s \leqslant 1$ and for any $x, y \in S$. Under this condition let us assume $0 \cdot \infty = 0$.

In particular, we call also a semi-prenorm that satisfies condition (iii/c^s) an *s-convex semi-prenorm*. The condition of s-convexity (iii/c^s) was introduced by W. Orlicz (cf. [1], [6]).

4.2. *Each s-convex semi-premodular ϱ is locally s-convex and locally bounded.*

Let us observe that from condition (iii/c^s) for ϱ it follows condition (iii/lc^s) for ϱ. For any numbers $\varepsilon, \delta > 0$ we take $\alpha = \inf\{1, (\varepsilon/\delta)^{1/s}\}$ and from condition (iii/c^s) we get the implication: if $\varrho(x) < \delta$, then $\varrho(\alpha x) \leqslant \alpha^s \varrho(x) < \varepsilon$.

4.3. *If ϱ is an s-convex semi-premodular, then the functional*

$$\varrho^0(x) = \begin{cases} \inf\{\varepsilon > 0\colon \ \varrho(x/\varepsilon^{1/s}) \leqslant 1\} & \text{if } \varrho(\lambda x) < \infty \text{ for some } \lambda > 0, \\ \infty & \text{otherwise } (x \in S), \end{cases}$$

is an s-homogeneous semi-prenorm equivalent to the semi-prenorm $\varrho^{\check{}}$ defined as in 1.5.

The functional ϱ^0 is an s-homogeneous semi-prenorm, because it is the proper Minkowski functional of the normal and s-convex set $\{x \in S\colon \ \varrho(x) \leqslant 1\}$. Now let us observe that if $\varrho^{\check{}}(x) < \varepsilon < 1$, then $\varrho(x/\varepsilon^{1/s}) \leqslant \varepsilon < 1$, and further $\varrho^0(x) \leqslant \varepsilon$. It follows from the above that $\varrho^0(x) \leqslant \varrho^{\check{}}(x)$ for any $x \in S$ such that $\varrho^{\check{}}(x) < 1$. Let us also observe that if $\varrho^0(x) < \varepsilon^2 < 1$ then by virtue of condition (iii/c^s) there holds $\varrho(x/\varepsilon^{1/s}) \leqslant \varepsilon\varrho(x/\varepsilon^{2/s}) < \varepsilon$, and further $\varrho^{\check{}}(x) \leqslant \varepsilon$. Hence we get $\varrho^{\check{}}(x) \leqslant (\varrho^0(x))^{1/2}$ for every $x \in S$ such that $\varrho^0(x) < 1$. We conclude from these two inequalities that the semi-prenorms ϱ^0 and $\varrho^{\check{}}$ are equivalent.

4.4. *A locally s-convex and locally bounded base B can be described by an s-convex semi-premodular if and only if it is equivalent to some at most countable premodular base.*

If a locally s-convex and locally bounded base B is describable by a semi-premodular (in particular, by an s-convex semi-premodular), then by virtue of result 3.3 in paper [4] this base is equivalent to some premodular at most countable base. Now assume that the locally s-convex and locally bounded premodular base B is equivalent to some at most countable premodular base. Then, as one can easy observe, there exists at most countable premodular base $B_1 = \{U_n\}$, equivalent to the base B, composed of the sets of the base B and such that $U_{n+1} \subset U_n$ for $n = 1, 2, \ldots$ Since the base B_1 is composed of the sets which belong to the base B it is also locally s-convex and locally bounded. Now we take the proper Minkowski functionals of the normal and s-convex sets U_n, i.e., functionals

$$p_n(x) = \begin{cases} \inf\{\varepsilon > 0\colon \ (x/\varepsilon^{1/s}) \in U_n\} & \text{if } \lambda x \in U_n \text{ for some } \lambda \neq 0, \\ \infty & \text{otherwise } (x \in S), \end{cases}$$

for $n = 1, 2, \ldots$ It is obvious that these functionals are s-convex semi-prenorms on S. Further we take the functionals

$$\varrho_n(x) = \sup\{0, p_n(x) - 1\} \quad (x \in S, \ n = 1, 2, \ldots).$$

The functionals ϱ_n are s-convex semi-premodulars on S. In fact for any $x, y \in S$ and any numbers $\alpha, \beta \geqslant 0$ such that $\alpha^s + \beta^s \leqslant 1$ we have

$$p_n(\alpha x+\beta y)-1 \leqslant \alpha^s p_n(x)+\beta^s p_n(y)-(\alpha^s+\beta^s)$$
$$= \alpha^s(p_n(x)-1)+\beta^s(p_n(y)-1),$$

and hence we get condition (iii/c^s) for ϱ_n. Condition (ii) for ϱ_n is obvious. Since the base $B_1 = \{U_n\}$ is locally bounded, for each positive integer n there exists a number $\alpha_n > 0$ such that $U_n \subset \alpha_n U_{n+1}$, the inequality $p_{n+1}(x) \leqslant \alpha_n^s p_n(x)$ so is satisfied. Here we can assume $\alpha_n \geqslant 1$ for $n = 1, 2, \ldots$ and take $\alpha_0 = 1$. Now we define the functional

$$\varrho(x) = \sum_{n=1}^{\infty} 2^{-n}(\alpha_0\alpha_1\ldots\alpha_{n-1})^{-s}\varrho_n(x) \qquad (x \in S).$$

Since the functionals ϱ_n are s-convex semi-premodulars one can easily conclude that the functional ϱ is also s-convex semi-premodular on S. It suffices to show that $B(\varrho) \sim B_1$. For any number $\varepsilon > 0$ we select a positive integer m such that $2^{-m} < \varepsilon$. Let $x \in U_m$. Let us observe that $\varrho_n(x) = 0$ for $n = 1, \ldots, m$ and $p_m(x) \leqslant 1$. The above and the inequalities $p_{n+1}(x) \leqslant \alpha_n^s p_n(x)$ for $n = 1, 2, \ldots$ imply

$$\varrho(x) \leqslant \sum_{n=m+1}^{\infty} 2^{-n}(\alpha_0\alpha_1\ldots\alpha_{n-1})^{-s}p_n(x) \leqslant 2^{-m}p_m(x)\sum_{n=1}^{\infty} 2^{-n} \leqslant 2^{-m} < \varepsilon.$$

So $U_m \subset U(\varrho, \varepsilon)$. Thus we have $B(\varrho) \dashv B_1$. On the other hand, for any positive integer m we take $\delta = 2^{-m}(\alpha_0\alpha_1\ldots\alpha_{m-1})^{-s}$, and let $x \in U(\varrho, \delta)$, so let $\varrho(x) < \delta$. Then we have $\varrho_m(x) < 1$ and next $p_m(x) < 2$. Hence we get $U(\varrho, \delta) \subset 2^{1/s}U_m$, and this proves that $B_1 \dashv B(\varrho)$.

5.1. Let $\{\psi_n\}$ be a sequence of non-negative and convex functions for $u \geqslant 0$ and equal to 0 for $u = 0$, and let S denote the linear lattice of real functions measurable on an interval (a, b). Let us define

$$\varrho_n(x) = \int_a^b \psi_n(|x(t)|^s)\,dt, \qquad n = 1, 2, \ldots$$

and further

$$\varrho(x) = \sup_n \frac{\varrho_n(x)}{n(1+\varrho_n(x))} \qquad \text{for } x \in S.$$

If $\varrho_n(x) = \infty$, then we assume here $\varrho_n(x)/(1+\varrho_n(x)) = 1$.

The functional ϱ is a locally s-convex semi-premodular on S. It is clear that ϱ satisfies conditions (i) and (ii). So it suffices only to show that ϱ satisfies condition (iii/lc^s). We take any elements $x, y \in S$ and any numbers

$\alpha, \beta \geqslant 0$ such that $\alpha^s + \beta^s \leqslant 1$. Since the functions ψ_n are non-decreasing and convex for $u \geqslant 0$, we have

$$\begin{aligned}\varrho_n(\alpha x+\beta y) &\leqslant \int_a^b \psi_n(\alpha^s|x(t)|^s+\beta^s|y(t)|^s)\,dt \\ &\leqslant \alpha^s \int_a^b \psi_n(|x(t)|^s)\,dt + \beta^s \int_a^b \psi_n(|y(t)|^s)\,dt \\ &= \alpha^s \varrho_n(x) + \beta^s \varrho_n(y) \leqslant \sup\{\varrho_n(x), \varrho_n(y)\}.\end{aligned}$$

Further, since the function $\varphi(u) = u/(1+u)$ and $\varphi(\infty) = 1$ is non-decreasing for $0 \leqslant u \leqslant \infty$, we get

$$\begin{aligned}\varrho(\alpha x+\beta y) &= \sup_n \frac{1}{n}\varphi\big(\varrho_n(\alpha x+\beta y)\big) \\ &\leqslant \sup_n \frac{1}{n}\varphi\big(\sup\{\varrho_n(x), \varrho_n(y)\}\big) \\ &= \sup_n \sup\left\{\frac{1}{n}\varphi\big(\varrho_n(x)\big), \frac{1}{n}\varphi\big(\varrho_n(y)\big)\right\} \\ &= \sup\left\{\sup_n \frac{1}{n}\varphi\big(\varrho_n(x)\big), \sup_n \frac{1}{n}\varphi\big(\varrho_n(y)\big)\right\} = \sup\{\varrho(x), \varrho(y)\}.\end{aligned}$$

5.2. The functionals ϱ_n given in 5.1 constitute an example of s-convex semi-premodulars on the lattice S of real measurable functions on an interval (a, b).

5.3. There exist locally convex and locally bounded bases in linear lattices, which are not describable by a semi-premodular. An example is the base given in 6.3 in paper [4].

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