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On the equation of the ρ-orthogonal additivity

Abstract. We solve a conditional functional equation of the form

$$x \perp^{\rho} y \implies f(x+y) = f(x) + f(y),$$

where f is a mapping from a real normed linear space $(X, \|\cdot\|)$ with $\dim X \geq 2$ into an abelian group $(G,+)$ and $\perp^{\rho}$ is a given orthogonality relation associated to the norm.

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1. Introduction. The Cauchy functional equation, i.e., the equation of additivity, has been widely investigated (see e.g. Aczél [1], Kuczma [10], Aczél & Dhombres [2]). Its conditional form described below deserves further studies.

A mapping f from a linear space X into a group $(G,+)$ is called *orthogonally additive* provided that for every $x, y \in X$ one has

$$x \perp y \quad \text{implies} \quad f(x+y) = f(x) + f(y), \tag{1}$$

where $\perp$ denotes an orthogonal relation defined on X.

For instance, in an inner product space $(X, \langle\cdot,\cdot\rangle)$ the functional

$$X \ni x \mapsto \langle x, x\rangle \in \mathbb{R}$$

is orthogonally additive (Pythagoras theorem). The notion of orthogonal additivity has intensively been studied by many authors; see e.g. Sundaresan [13], Drewnowski

& Orlicz [6], Gudder & Strawther [7], Rätz [12], Szabó [14, 15, 16, 17, 18, 19] and others.

Let $(X, \|\cdot\|)$ be a real normed linear space with $\dim X \geq 2$. Define the orthogonality relation $\perp^\rho$ on X as follows:

$$x \perp^\rho y \quad \text{if and only if} \quad \rho'_+(x,y) + \rho'_-(x,y) = 0, \tag{2}$$

where

$$\rho'_\pm(x,y) = \lim_{t\to 0^\pm} \frac{\|x+ty\|^2 - \|x\|^2}{2t}. \tag{3}$$

Our aim is to give the description of functions satisfying the following condition

$$x \perp^\rho y \quad \text{implies} \quad f(x+y) = f(x) + f(y). \tag{4}$$

2. Preliminaries. The functions ρ'_+ and ρ'_- given by (3) are well-defined and if $(X, \langle\cdot,\cdot\rangle)$ is a real inner product space, then both ρ'_+ and ρ'_- coincide with $\langle\cdot,\cdot\rangle$. Next results contain some of the properties of $\rho'_\pm$ (cf. e.g. Amir [3]).

Proposition 2.1 *Let $(X, \|\cdot\|)$ be a real normed space with $\dim X \geq 2$, and let $\rho'_+, \rho'_- : X \times X \to \mathbb{R}$ be given by (3). Then*

(a) $\rho'_\pm(0,y) = \rho'_\pm(x,0) = 0$ *for all* $x, y \in X$;

(b) $\rho'_\pm(x,x) = \|x\|^2$ *for all* $x \in X$;

(c) $\rho'_\pm(\alpha x, y) = \rho'_\pm(x, \alpha y) = \alpha\rho'_\pm(x,y)$ *for all* $x, y \in X$ *and* $\alpha \geq 0$;

(d) $\rho'_\pm(\alpha x, y) = \rho'_\pm(x, \alpha y) = \alpha\rho'_\mp(x,y)$ *for all* $x, y \in X$ *and* $\alpha \leq 0$;

(e) $\rho'_\pm(x, \alpha x + y) = \alpha\|x\|^2 + \rho'_\pm(x,y)$ *for all* $x, y \in X$ *and* $\alpha \in \mathbb{R}$;

(f) $\rho'_-(x,y) \leq \rho'_+(x,y)$ *for all* $x, y \in X$.

Proposition 2.2 *The functions ρ'_+ and ρ'_- are continuous in the second variable.*

Proposition 2.3 (Precupanu [11]) *Let $X = \mathbb{R}^2$ and let u, v in $S_X := \{u \in X : \|u\| = 1\}$ be such that $u \neq \pm v$. Then*

$$\lim_{\tau\to 0^+} \rho'_+(u+\tau v, v) = \rho'_+(u,v).$$

Remark 2.4 Using the same argument to that used by Precupanu one can show that with the same assumptions

$$\lim_{\tau\to 0^-} \rho'_+(u+\tau v, v) = \rho'_+(u,v),$$

so, in fact, we have

$$\lim_{\tau\to 0} \rho'_+(u+\tau v, v) = \rho'_+(u,v). \tag{5}$$

Corollary 2.5 *Let $X = \mathbb{R}^2$ and let u, v in S_X be such that $u \neq \pm v$. Then*

$$\lim_{\tau\to 0} \rho'_-(u + \tau v, v) = \rho'_-(u, v). \tag{6}$$

Proof By the properties of $\rho'_\pm$ we have

$$\begin{aligned} \lim_{\tau\to 0} \rho'_-(u + \tau v, v) &= \lim_{\tau\to 0}(-\rho'_+(u + \tau v, -v)) = -\lim_{-\tau\to 0} \rho'_+(u + (-\tau)(-v), -v) \\ &= -\rho'_+(u, -v) = \rho'_-(u, v). \end{aligned}$$

■

Remark 2.6 Conditions (5) and (6) can be used in each two-dimensional linear space since such a space is isomorphic with $\mathbb{R}^2$.

As a consequence of the above results and by properties of $\rho'_\pm$, one has even more general result.

Corollary 2.7 *Let $(X, \|\cdot\|)$ be a real normed space with $\dim X \geq 2$. Functions $\mathbb{R} \ni t \mapsto \rho'_\pm(x + ty, y) \in \mathbb{R}$ are continuous at zero for every fixed x, y in X.*

For the next results we recall the definition of Birkhoff orthogonality: in a normed space *x is Birkhoff orthogonal to y* ($x \perp_B y$) if and only if for all real λ we have $\|x\| \leq \|x + \lambda y\|$ (for details the reader is referred to Birkhoff [5], James [9], Amir [3]).

Functions $\rho'_\pm$ characterize the Birkhoff orthogonality in the following sense (cf. James [9]; see also Amir [3]).

Proposition 2.8 *Let $(X, \|\cdot\|)$ be a real normed linear space with $\dim X \geq 2$. Then for all $x, y \in X$ and $\alpha \in \mathbb{R}$ the condition $x \perp_B y - \alpha x$ is satisfied if and only if $\rho'_-(x, y) \leq \alpha\|x\|^2 \leq \rho'_+(x, y)$.*

The orthogonality relation $\perp^\rho$ defined by (2) satisfies the following properties.

Proposition 2.9 *Let $(X, \|\cdot\|)$ be a real normed linear space with $\dim X \geq 2$. For all $x, y \in X$ and $\alpha \in \mathbb{R}$*

(a) *$x \perp^\rho y - \alpha x$ if and only if $2\alpha\|x\|^2 = \rho'_+(x, y) + \rho'_-(x, y)$;*

(b) *If $x \perp^\rho y - \alpha x$ then $\rho'_-(x, y) \leq \alpha\|x\|^2 \leq \rho'_+(x, y)$;*

(c) *If $x \perp^\rho y$ then $x \perp_B y$;*

(d) *If X is smooth, i.e., for all x, y in X one has $\rho'_-(x, y) = \rho'_+(x, y)$, then $x \perp^\rho y$ if and only if $x \perp_B y$.*

Let us state the notion of an abstract orthogonality space (see Rätz [12]).

Definition 2.10 An ordered pair $(X, \perp)$ is called an *orthogonality space* in the sense of Rätz whenever X is a real linear space with $\dim X \geq 2$ and $\perp$ is a binary relation on X such that

(i) $x \perp 0$ and $0 \perp x$ for all $x \in X$;

(ii) if $x, y \in X \setminus \{0\}$ and $x \perp y$, then x and y are linearly independent;

(iii) if $x, y \in X$ and $x \perp y$, then for all $\alpha, \beta \in \mathbb{R}$ we have $\alpha x \perp \beta y$;

(iv) for any two-dimensional subspace P of X and for every $x \in P$, $\lambda \in [0, \infty)$, there exists a $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

A normed linear space with Birkhoff orthogonality is a typical example of an orthogonality space (see Rätz [12], Szábo [15, 16]). James orthogonality, since it is not homogenous (see James [8]), cannot act as an example of a binary relation in such a space.

Based on the results from the papers by Rätz [12] and Baron and Volkmann [4] we have the following theorem concerning the orthogonal additivity for a function defined on an orthogonality space.

Theorem 2.11 *Let $(X, \perp)$ be an orthogonality space and let $(G, +)$ be an abelian group. A mapping $f : X \to G$ satisfies condition* (1) *if and only if there exist an additive mapping $a : X \to G$ and a biadditive and symmetric mapping $b : X \times X \to G$ such that*

$$f(x) = a(x) + b(x, x) \ \textit{for all} \ \ x \in X \tag{7}$$

and

$$b(x, y) = 0 \ \textit{for all} \ \ x, y \in X \ \textit{with} \ x \perp y. \tag{8}$$

As an immediate consequence of the above result, we deduce that each Birkhoff orthogonally additive mapping has the form (7). And finally, on account of Proposition 2.9 (d), since in smooth spaces the relations $\perp^\rho$ and $\perp_B$ are equivalent, as a corollary we get the following result.

Corollary 2.12 *Let $(X, \|\cdot\|)$ be a smooth normed linear space with* $\dim X \geq 2$, *and let $(G, +)$ be an abelian group. A mapping $f : X \to G$ satisfies condition* (4) *if and only if there exist an additive mapping $a : X \to G$ and a biadditive and symmetric mapping $b : X \times X \to G$ such that f has the form* (7) *and condition* (8) *with $\perp := \perp^\rho$ is satisfied.*

The question is: What about spaces which are not smooth?

3. Main results. Assume that $(X, \|\cdot\|)$ is a normed linear space with $\dim X \geq 2$. We will show that the relation $\perp^\rho$ satisfies the four properties of the orthogonality space. The first three are easy to be checked. In order to check the fourth one we need some auxiliary results.

LEMMA 3.1 *For any two vectors x and w in X we have*

$$\lim_{t\to t_0} \rho'_{\pm}(x+tw,w) = \rho'_{\pm}(x+t_0w,w).$$

PROOF By Corollary 2.7 we can write

$$\lim_{t\to t_0} \rho'_{\pm}(x+tw,w) = \lim_{s\to 0} \rho'_{\pm}(x+t_0w+sw,w) = \rho'_{\pm}(x+t_0w,w).$$

LEMMA 3.2 *For any two vectors x and w in X we have*

$$\lim_{t\to t_0} \rho'_{\pm}(x+tw,x) = \rho'_{\pm}(x+t_0w,x).$$

PROOF By Proposition 2.1 (c)-(e) for $t \neq 0$ we have

$$\rho'_{\pm}(x+tw,x) = \|x+tw\|^2 - t\rho'_{\mp\operatorname{sgn} t}(x+tw,w),$$

where $\operatorname{sgn} t := \frac{t}{|t|}$, and by Lemma 3.1 we obtain: if $t_0 = 0$, immediately

$$\lim_{t\to 0} \rho'_{\pm}(x+tw,x) = \|x\|^2,$$

and if $t_0 \neq 0$ then

$$\lim_{t\to t_0} \rho'_{\pm}(x+tw,x) = \|x+t_0w\|^2 - t_0\rho'_{\mp\operatorname{sgn} t_0}(x+t_0w,w) = \rho'_{\pm}(x+t_0w,x).$$

LEMMA 3.3 *For any $x \in X \setminus \{0\}$ and $\lambda \in [0,\infty)$ there exists $z \in X \setminus \{0\}$ such that*

$$\Big[\rho'_+(x,z) + \rho'_-(x,z)\Big]\Big[\rho'_+(z,x) + \rho'_-(z,x)\Big] = \frac{4\|x\|^2\|z\|^2}{\lambda+1}. \tag{9}$$

PROOF If $\lambda = 0$, it is enough to take $z := x$. Assume that $\lambda > 0$. Let $w \in X$ be linearly independent of x and such that $\rho'_+(x,w) + \rho'_-(x,w) \neq 0$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) := \frac{4\|x\|^2\|x+tw\|^2}{\lambda+1} - \Big[\rho'_+(x,x+tw)+\rho'_-(x,x+tw)\Big]\Big[\rho'_+(x+tw,x)+\rho'_-(x+tw,x)\Big].$$

We have

$$\varphi(0) = 4\|x\|^4\left(\frac{1}{\lambda+1} - 1\right) < 0.$$

If $t_1 := -\dfrac{2\|x\|^2}{\rho'_+(x,w) + \rho'_-(x,w)}$, then

$$\rho'_+(x,x+t_1w) + \rho'_-(x,x+t_1w) = 2\|x\|^2 + t_1\left(\rho'_+(x,w) + \rho'_-(x,w)\right) = 0$$

and

$$\varphi(t_1) = \frac{4\|x\|^2\|x+t_1w\|^2}{\lambda+1} > 0.$$

On account of Proposition 2.2 and Lemma 3.2 function φ is continuous and, consequently, between t_1 and 0 there exists t_0 such that $\varphi(t_0) = 0$, i.e., condition (9) is satisfied with $z := x + t_0w$. ∎

Now we are able to prove

Proposition 3.4 *For any two-dimensional subspace P of X and for every $x \in P$, $\lambda \in [0, \infty)$, there exists a $y \in P$ such that $x \perp^\rho y$ and $x + y \perp^\rho \lambda x - y$.*

Proof Fix $x \in X$. If $x = 0$ then take $y := 0$. For $x \neq 0$ take nonzero $z \in X$ such that (9) is satisfied. Define

$$y := -x + \frac{\lambda + 1}{2\|z\|^2}\Big[\rho'_+(z, x) + \rho'_-(z, x)\Big] z.$$

We have

$$\begin{aligned}
\rho'_+(x, y) + \rho'_-(x, y) &= \rho'_+\Big(x, -x + \frac{\lambda+1}{2\|z\|^2}\big[\rho'_+(z,x) + \rho'_-(z,x)\big]z\Big) \\
&\quad + \rho'_-\Big(x, -x + \frac{\lambda+1}{2\|z\|^2}\big[\rho'_+(z,x) + \rho'_-(z,x)\big]z\Big) \\
&= -2\|x\|^2 + \frac{\lambda+1}{2\|z\|^2}\Big[\rho'_+(z,x) + \rho'_-(z,x)\Big]\Big[\rho'_+(x,z) + \rho'_-(x,z)\Big] \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
&\rho'_+(x + y, \lambda x - y) + \rho'_-(x + y, \lambda x - y) \\
= \;& \rho'_+\Big(\frac{\lambda+1}{2\|z\|^2}\big[\rho'_+(z,x) + \rho'_-(z,x)\big]z,\; (\lambda+1)x - \frac{\lambda+1}{2\|z\|^2}\big[\rho'_+(z,x) + \rho'_-(z,x)\big]z\Big) \\
& + \rho'_-\Big(\frac{\lambda+1}{2\|z\|^2}\big[\rho'_+(z,x) + \rho'_-(z,x)\big]z,\; (\lambda+1)x - \frac{\lambda+1}{2\|z\|^2}\big[\rho'_+(z,x) + \rho'_-(z,x)\big]z\Big) \\
= \;& -2\frac{(\lambda+1)^2}{4\|z\|^4}\Big[\rho'_+(z,x) + \rho'_-(z,x)\Big]^2 \|z\|^2 \\
& + \frac{(\lambda+1)^2}{2\|z\|^2}\Big[\rho'_+(z,x) + \rho'_-(z,x)\Big]\Big[\rho'_+(z,x) + \rho'_-(z,x)\Big] = 0,
\end{aligned}$$

which concludes the proof. ■

Relation $\perp^\rho$ satisfies (iv), so our main result follows immediately.

Theorem 3.5 *Let $(X, \|\cdot\|)$ be a real normed linear space with $\dim X \geq 2$, and let $(G, +)$ be an abelian group. A mapping $f : X \to G$ satisfies condition (4) if and only if there exist an additive mapping $a : X \to G$ and a biadditive and symmetric mapping $b : X \times X \to G$ such that f has the form (7) and condition (8) with $\perp := \perp^\rho$ holds true.*

Remark 3.6 As a corollary from Proposition 3.4 we infer also that the Birkhoff orthogonality satisfies condition (iv) in the definition of the orthogonality space (one may compare this with quite sophisticated considerations from the papers by Szabó [15, 16]).

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