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On pseudocompact extensions

The aim of this paper is to investigate the set of pseudocompact extensions and to prove in particular that there does not exist the greatest element in this set.

All spaces in this paper are Tychonoff and all maps are continuous. We shall usually assume that the embeddings are inclusions.

A space pX will be said to be a *pseudocompact extension of* X if pX is pseudocompact and X is embedded in pX as a dense subset. There exists a set of pseudocompact extensions such that each pseudocompact extension of X is equivalent to the one from this set.

The following theorem will be intensively used in this paper.

THEOREM 1 (Gillman and Jerison [2], p. 95). *A Tychonoff space X is pseudocompact iff every non-empty zero-set in βX meets X.*

By this theorem we can produce many pseudocompact extensions which are not compact. To do this it suffices to drop out from $\beta X \setminus X$ a set which does not contain any zero-set in βX (in particular we may drop out a one-point set). Good examples of pseudocompact extensions were given by Fine and Gillman [1].

Let νX denote the Hewitt realcompactification of X (see [2]). It is easy to see that νX can be characterized as βX with all zero-sets contained in the remainder removed. We infer the following

THEOREM 2. *If X is a Tychonoff space, then $\pi X = X \cup (\beta X \setminus \nu X)$ is a pseudocompact extension of X.*

Proof. According to the above remark and Theorem 1 it suffices to notice that the remainder of the Čech–Stone compactification of πX equals $\nu X \setminus X$.

A map $f: X \to Y$ will be said to be *π-extendable* provided there exists a map $\pi f: \pi X \to \pi Y$ such that $\pi f | X = f$. Clearly, the extension πX leads to a functor of the category of Tychonoff spaces and π-extendable maps into the subcategory of pseudocompact spaces and continuous maps. Let us call this functor π.

THEOREM 3. *The functor π is adjoint to the embedding of the category of pseudocompact spaces into the category of Tychonoff spaces and π-extendable maps.*

Proof. To prove this it suffices to notice that $\pi X = X$ iff X is pseudocompact (see [2], p. 125).

Remark. The above theorem shows that among other pseudocompact extensions πX is a natural one. Clearly, πX is the greatest in the family of π-extendable pseudocompact extensions of X. However, it will be shown later that it is not the greatest pseudocompact extension in the family of all pseudocompact extensions, in contrast to the other known extensions which lead to the functors adjoint to the appropriate embeddings as: the Čech–Stone compactification, the Hewitt realcompactification, the Katětov H-closed extension.

A filter is said to be a *z-ultrafilter* provided it is a maximal one in the family of filters which consists of zero-sets. We say that a z-ultrafilter has a *countable intersection property* (c.i.p.) if it is closed with respect to the countable intersections.

The following theorem gives a topological characterization of π-extendable maps.

THEOREM 4. *A map $f\colon X \to Y$ is π-extendable iff for each z-ultrafilter $\mathscr{F}$ with the c.i.p. and with the empty intersection in Y, every z-ultrafilter in X containing the family $\{f^{-1}(Z)\colon Z \in \mathscr{F}\}$ has the c.i.p.*

Proof. It is known (see [2], p. 118) that each point $y \in \nu Y \setminus Y$ is appointed by a unique z-ultrafilter in Y with the c.i.p. and with the empty intersection, i.e., there exists $\mathscr{F}$ being a z-ultrafilter in Y with the c.i.p. and with the empty intersection such that $\{y\} = \bigcap \{\mathrm{cl}_{\beta Y} Z\colon Z \in \mathscr{F}\}$. Hence the points of $\nu X \setminus X$ are appointed by z-ultrafilters with c.i.p. But f is π-extendable iff $(\beta f)^{-1}(y) \subset (\nu X \setminus X)$ for every $y \in \nu Y \setminus Y$, and theorem follows in virtue of preceding remarks.

Note. The above theorem may be reformulated as follows: a map $f\colon X \to Y$ is π-extendable iff for each decreasing sequence $\{Z_n\colon n = 1, 2, \ldots\}$ of zero-sets in X such that $\bigcap_{n=1}^{\infty} Z_n = \emptyset$ and for each z-ultrafilter $\mathscr{F}$ with the c.i.p. in Y and with the empty intersection, there exist $Z \in \mathscr{F}$ and an integer n such that $Z \cap f(Z_n) = \emptyset$.

We shall show an additional motivation of π-extendable maps: the following theorem together with Theorem 4, gives a criterion in topological terms for a closed subspace of pseudocompact space to be pseudocompact. Namely

THEOREM 5. *If an embedding $i\colon A \subset X$, where A is closed and X is pseudocompact, is π-extendable, then A is pseudocompact.*

Note. The converse implication is always true, i.e., an inclusion $A \subset X$ is π-extendable whenever A is pseudocompact, so we have a characterization of closed pseudocompact subspaces of pseudocompact spaces.

Proof. At first, let us notice, that the extension map $\pi i\colon \pi A \to \pi X$ maps the remainder into the remainder, because $\beta i\colon \beta A \to \beta X$ maps the remainder into the remainder. Since the remainder of πX is empty, the remainder of πA is empty. Thus A is pseudocompact.

The following lemma will be used in the proof of the next theorem.

LEMMA. *Let p_1X and p_2X be pseudocompact extensions of a given space X. If p_1X is not smaller than p_2X and p_2X is not smaller than βX, then $p_1X \subset p_2X \subset \beta X$.*

Proof. Observe that if a pseudocompact extension pX is greater than βX, then $pX \subset \beta X$. If p_1X is greater than p_2X and p_2X is greater than βX, then both p_1X and p_2X are embedded in βX. There exists a map $\varphi\colon p_1X \to p_2X$ such that $\varphi | X = \mathrm{id}_X$. Clearly, the extension $\beta\varphi\colon \beta X \to \beta X$ has to be the identity. Thus p_1X is embedded in p_2X.

Now we shall show that

THEOREM 6. *If X is not pseudocompact, then there does not exist the greatest pseudocompact extension of X.*

Proof. Suppose, on the contrary, that pX is the greatest pseudocompact extension of X. It suffices to show, by the lemma, that $\tilde{X} = X \cup \cup (\beta X \setminus pX)$ is a pseudocompact extension of X.

Clearly, $\beta\tilde{X} = \beta X$. Let us suppose that X is not pseudocompact. Then, by Theorem 1, there exists a point $x_0 \in \beta X \setminus \tilde{X}$ and a zero-set $Z \subset \beta X$ such that $x_0 \in Z \subset \beta X \setminus \tilde{X}$. On the other hand, since pX is not majorizable, the space $pX \setminus \{x_0\}$ is not pseudocompact. Clearly, βX is the Čech–Stone compactification of $pX \setminus \{x_0\}$. Then, by Theorem 1, there exists a zero-set $R \subset \beta X$ such that $pX \cap R = \{x_0\}$. Since $\beta X \setminus \tilde{X} = pX \setminus X$, hence $Z \subset pX \setminus X$ and $Z \cap R = \{x_0\}$ is a zero-set in βX and $x_0 \in \beta X \setminus X$. It is known (see [2], p. 132) that every zero-set in βX contained in the remainder is at least of power 2^c. Hence we get a contradiction. Thus, the space $\tilde{X}$ is the pseudocompact extension, which completes the proof.

References

[1] N. J. Fine and L. Gillmăn, *Remote points in βR*, Proc. Amer. Math. Soc. 13 (1962), p. 29–36.

[2] L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton 1960.

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