ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXIX (1990)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXIX (1990)

ANDRZEJ DROZDOWICZ and JERZY POPENDA (Poznań)

Asymptotic behavior of the solutions of an n-th order difference equation

Introduction. The asymptotic behavior of the solutions of the n-th order differential equations have been considered by T. G. Hallam [3–4] and W. F. Trench [7]. Similar problems with regard to the second order difference equations were investigated by J. W. Hooker, W. T. Patula [5], A. Drozdowicz [1] and to the n-th order by J. Popenda [6]. This paper is a generalization of the result obtained by A. Drozdowicz and J. Popenda [2] for second order difference equation.

In the present paper the asymptotic behavior of the solutions of m-th order difference equation

$$\text{(E)} \qquad \Delta^m x_n + p_n f(x_n) = 0, \qquad m \text{ is a positive integer}$$

will be considered. A necessary and sufficient condition for the existence of a solution x of (E) which has the asymptotic behavior

$$\text{(AB)} \qquad \lim_{n\to\infty} x_n = C,$$

where C is a constant such that $f(C) \neq 0$ will be proved.

Let N denote the set of positive integers, R the set of real numbers. Throughout this paper it will be assumed that $f\colon R \to R$ is continuous and $p\colon N \to R_+ \cup \{0\}$. Next for a function $a\colon N \to R$ one introduces the difference operator Δ as follows:

$$\Delta a_n = a_{n+1} - a_n, \quad \Delta^m a_n = \Delta(\Delta^{m-1} a_n), \quad \text{where } a_n = a(n), \quad n \in N.$$

Moreover, let $\sum_{j=k}^{k-1} a_j = 0$.

One can observe that if f is defined and finite on R then there exists a solution of (E) for any initial values: $x_0, \ldots, x_m$.

1. A necessary condition

THEOREM 1. *A necessary condition for the existence of a solution x of* (E) *which has the asymptotic behavior* (AB) *is*

(NS) $$\sum_{j=1}^{\infty} j^{m-1} p_j < \infty.$$

Proof. Let x denote a solution of (E) having property (AB), i.e. $x_n \to C$ for $n \to \infty$. Then

(1.1) $$\Delta x_n \to 0, \quad \Delta^2 x_n \to 0, \ldots, \Delta^m x_n \to 0 \quad \text{as } n \to \infty.$$

Assume that $f(C) > 0$. (The case $f(C) < 0$ can be proved in a similar way.)

The continuity of f implies that there exists $\varepsilon > 0$ such that $f(t) > 0$ for $t \in I := [C-\varepsilon, C+\varepsilon]$. Since $x_n \to C$ as $n \to \infty$, there exists $n_1 = N(\varepsilon)$ such that $x_n \in I$ for all $n \geqslant n_1$. Therefore

(1.2) $$f(x_n) \geqslant C_0 := \min_{t \in I} f(t) > 0 \quad \text{for } n \geqslant n_1.$$

By summation of (E) over n and using (1.2), one obtains

$$\Delta^{m-1} x_n - \Delta^{m-1} x_k \leqslant -C_0 \sum_{j=k}^{n-1} p_j \quad \text{for } n \geqslant k \geqslant n_1.$$

In view of (1.1),

(1.3) $$\Delta^{m-1} x_k \geqslant C_0 \sum_{j=k}^{\infty} p_j \quad \text{for any } k \geqslant n_1.$$

Therefore, the series

(1.4) $$\sum_{j=k}^{\infty} p_j \quad \text{is convergent.}$$

By summation of (1.3) over k we obtain

$$\Delta^{m-2} x_n - \Delta^{m-2} x_s \geqslant C_0 \sum_{k=s}^{n-1} \sum_{j=k}^{\infty} p_j.$$

In view of (1.1) we get

$$C_0 \sum_{k=s}^{\infty} \sum_{j=k}^{\infty} p_j \leqslant -\Delta^{m-2} x_s < \infty.$$

Hence the series $\sum_{k=s}^{\infty} \sum_{j=k}^{\infty} p_j$ is convergent.

Changing the order of summation, we have

$$C_0 \sum_{k=s}^{\infty} \sum_{j=k}^{\infty} p_j = C_0 \sum_{j=s}^{\infty} (j+1-s) p_j \leqslant -\Delta^{m-2} x_s, \quad s \geqslant n_1.$$

The next summation (using (1.1)) gives

$$C_0 \sum_{j=s}^{\infty} \binom{j+2-s}{2} p_j \leqslant \Delta^{m-3} x_s, \quad s \geqslant n_1.$$

Repeating the above reasoning, we find

$$C_0 \sum_{k=n_1}^{n-1} \sum_{j=k}^{\infty} \binom{j+m-2-k}{m-2} p_j \leqslant (-1)^{m-2}(x_n - x_{n_1}).$$

Tending with n to infinity, we obtain

$$C_0 \sum_{k=n_1}^{\infty} \sum_{j=k}^{\infty} \binom{j+m-2-k}{m-2} p_j \leqslant (-1)^{m-2}(C - x_{n_1}).$$

Changing the order of summation, one can show that

$$C_0 \sum_{j=n_1}^{\infty} \binom{j+m-1-n_1}{m-1} p_j \leqslant (-1)^{m-2}(C - x_{n_1}).$$

Hence the series

$$\sum_{j=n_1}^{\infty} \binom{j+m-1-n_1}{m-1} p_j \quad \text{is convergent.} \tag{1.5}$$

It is easy to show that the condition

$$j^q = q!\binom{j+q-k}{q} + \sum_{r=0}^{q-1} (q-1-r)!\,(k-q+r)\binom{j+q-1-r-k}{q-1-r} j^r \tag{1.6}$$

holds for arbitrary $j, k, q \in N$.

One can observe that for $r = 0, \ldots, q-1$, $k \in N$, the inequality $q-1-r-k < 0$ holds, too. Hence

$$\binom{j-k+q-1-r}{q-1-r} j^r \leqslant \frac{1}{(q-1-r)!} j^{q-1}. \tag{1.7}$$

Let $q \geqslant 1$. For $q = 1$, the convergence of the series $\sum_{j=n_1}^{\infty} j^{q-1} p_j$ was shown in (1.4). Assume that the series $\sum_{j=n_1}^{\infty} j^{q-1} p_j$ is convergent for some $q \leqslant m-1$.

Then by virtue of (1.5), (1.6) and (1.7) it follows that

$$\begin{aligned}\sum_{j=k}^{\infty} j^q p_j &= \sum_{j=k}^{\infty} q!\binom{j+q-k}{q} p_j + \sum_{j=k}^{\infty} \sum_{r=0}^{q-1} (q-1-r)!\,(k-q+r)\binom{j+q-1-r-k}{q-1-r} j^r p_j \\ &\leqslant q! \sum_{j=k}^{\infty} \binom{j+q-k}{q} p_j + \sum_{r=0}^{q-1} \frac{(q-1-r)!\,(k-q+r)}{(q-1-r)!} \sum_{j=k}^{\infty} j^{q-1} p_j < \infty.\end{aligned}$$

Therefore the series $\sum_{j=k}^{\infty} j^q p_j$ is convergent. If $q = m-1$ then it follows that the series $\sum_{j=1}^{\infty} j^{m-1} p_j$ is convergent, too. ■

2. A sufficient condition

THEOREM 2. *Let for every* $k \in N$

$$\begin{cases} (i_R + p_k f)\colon R \to R & \textit{if } m \textit{ is even,} \quad \text{(S1)} \\ (i_R - p_k f)\colon R \to R & \textit{if } m \textit{ is odd} \quad \text{(S2)} \end{cases}$$

be surjections. (i_R *denotes here the identity function on* R.)

A sufficient condition for the existence of a solution x of (E) *which possesses the asymptotic behavior* (AB) *is* (NS).

Proof. The case $f(C) > 0$ will be considered. (The case $f(C) < 0$ can be shown in a similar way with some modifications.)

Let (NS) hold. Hence

$$\lim_{n\to\infty} \sum_{j=n}^{\infty} j^{m-1} p_j = 0. \tag{2.1}$$

One can observe that the sequence $\{\sum_{j=n}^{\infty} j^{m-1} p_j\}_{n=1}^{\infty}$ is nonincreasing. Analogously as in the proof of Theorem 1, there exists an interval $I := [C-\varepsilon, C+\varepsilon]$ ($\varepsilon > 0$) such that f is positive on I.

Denoting $C_1 := \max_{t\in I} f(t)$, from (2.1) it follows that

$$C_1 \sum_{j=n}^{\infty} j^{m-1} p_j \leqslant \varepsilon \quad \text{for all } n \geqslant N(\varepsilon, C_1).$$

Let

$$n_2 = \max\{\min\{n\in N\colon\ C_1 \sum_{j=n}^{\infty} j^{m-1} p_j \leqslant \varepsilon\},\ m-1\}.$$

Next let l_∞ denote the Banach space of bounded sequences $x = \{h_i\}_{i=1}^{\infty}$ with the norm $\|x\| = \sup_{i\geqslant 1} |h_i|$.

Moreover, let us define the set $T \subset l_\infty$ as follows:

$$x = \{h_i\}_{i=1}^{\infty} \in T \quad \text{if } \begin{cases} h_k = C & \text{for } k = 1, 2, \ldots, n_2-1, \\ h_k \in I_k & \text{for } k \geqslant n_2 \end{cases} \tag{2.2}$$

where

$$I_k := \Big[C - C_1 \sum_{j=k}^{\infty} j^{m-1} p_j,\ C + C_1 \sum_{j=k}^{\infty} j^{m-1} p_j\Big], \qquad k \geqslant n_2.$$

It is easy to show that T is bounded, convex and closed in l_∞. The fact that T is compact will be shown.

Let us write $\operatorname{diam}[a, b] = b-a$, $a, b \in R$. By virtue of (NS) it follows that $\operatorname{diam} I_n \to 0$ as $n \to \infty$. Let us choose any $\varepsilon_1 > 0$. If ε_1 is such that $\operatorname{diam} I_{n_2} < \varepsilon_1$ then $v = \{C, C, C, \ldots\} \in l_\infty$ is an ε_1-net.

The case $\operatorname{diam} I_{n_2} \geqslant \varepsilon_1$ will be considered. Let $n_3 \geqslant n_2$ be such that $\operatorname{diam} I_{n_3} \geqslant \varepsilon_1$ and $\operatorname{diam} I_{n_3+1} < \varepsilon_1$. (In any case one can find n_3 because $\operatorname{diam} I_n \to 0$ as $n \to \infty$.) Then it is easy to show that the set of elements of the space l_∞ of the form

$$v^{1,2,\ldots,n_3-n_2+1}_{s_1,s_2,\ldots,s_{n_3-n_2+1}} = \{C, \ldots, C, C+s_1\varepsilon_1, \ldots, C+s_{n_3-n_2+1}\varepsilon_1, C, \ldots\}$$

where

$$s_i = 0, \pm 1, \pm 2, \ldots, \pm r_i := \mathrm{En}\left[\frac{\operatorname{diam} I_{n_2+i-1}}{2\varepsilon_1}\right]+1, \quad i = 1, 2, \ldots, n_3-n_2+1,$$

forms an ε_1-net. (En denotes the entier function.)

One can observe that there are n_3-n_2+1 intervals I_k for which $\operatorname{diam} I_k \geqslant \varepsilon_1$. In every interval I_k we take $2r_k+1$ values which differ from C by an integral multiple of ε_1 and do not exceed the borders of I_k. Next we take all permutations of these values such that the first element belongs to I_{n_2}, the second to I_{n_2+1} and so on, and the last one to I_{n_3}. One can observe that the number of these permutations is equal to $\prod_{i=1}^{n_3-n_2+1}(1+2r_i)$. Therefore

$$\operatorname{card}\{v^{1,2,\ldots,n_3-n_2+1}_{s_1,s_2,\ldots,s_{n_3-n_2+1}}\} = \prod_{i=1}^{n_3-n_2+1}(1+2r_i) < \infty.$$

Hence the ε_1-net is finite. By the Hausdorff theorem, T is compact.

Let us define the operator A on T in the following way. For any $x \in T$ (defined by (2.2))

$$Ax = y = \{b_1, b_2, \ldots, b_{n_2-1}, b_{n_2}, \ldots, b_k, \ldots\},$$

where

$$b_n = \begin{cases} C & \text{for } n = 1, 2, \ldots, n_2-1, \\ C - \sum_{j=n}^{\infty} \binom{j+m-1-n}{m-1} p_j f(h_j) & \text{for } n \geqslant n_2, \text{ if the order of the} \\ & \text{equation (E) is even,} \\ [C + \sum_{j=n}^{\infty} \binom{j+m-1-n}{m-1} p_j f(h_j), & \text{for } n \geqslant n_2 \\ & \text{if the order } m \text{ is odd].} \end{cases}$$

The case where m is even. We shall show that A is a function from T to T. By observing that $I_k \subset I$ it follows that $0 < f(h_k) \leqslant C_1$ for $k \geqslant n_2$.

For $k \geqslant n_2$ and $j \geqslant k$, the inequality

$$0 < \binom{j+m-1-k}{m-1} p_j f(h_j) \leqslant \frac{j^{m-1} C_1 p_j}{(m-1)!} \leqslant j^{m-1} C_1 p_j \tag{2.3}$$

holds. Hence

$$C \geqslant C - \sum_{j=k}^{\infty} \binom{j+m-1-k}{m-1} p_j f(h_j) \geqslant C - C_1 \sum_{j=k}^{\infty} j^{m-1} p_j.$$

This means that $b_k \in I_k$ for $k \geqslant n_2$. Therefore, $y \in T$.

Next we shall show that A is continuous. Since f is continuous on R, it is uniformly continuous on I. In view of this fact, for each $\varepsilon_2 > 0$ there exists $\delta_1 > 0$ such that the condition $|t_1-t_2| < \delta_1$ implies $|f(t_1)-f(t_2)| < \varepsilon_2$. Consider the sequence $\{x^\alpha\}_{\alpha=1}^{\infty}$, $x^\alpha \in T$ such that

$$\|x^\alpha - x^0\| \to 0, \quad \text{i.e.} \quad \sup_{n \geqslant 1} |h_n^\alpha - h_n^0| \to 0 \text{ as } \alpha \to \infty. \tag{2.4}$$

From (2.4) it follows that there exists $n_4 = N(\delta_1)$ such that

$$\|x^\alpha - x^0\| < \delta_1, \quad \text{i.e.} \ \sup_{n\geq 1} |h_n^\alpha - h_n^0| < \delta_1 \ \text{for} \ \alpha \geq n_4.$$

Hence, for every $\alpha \geq n_4$ and for each $i \in N$, $|h_i^\alpha - h_i^0| < \delta_1$. Then for $\alpha \geq n_4$

$$\begin{aligned} \|Ax^\alpha - Ax^0\| &= \sup_{n\geq 1} |b_n^\alpha - b_n^0| \\ &= \sup_{n\geq n_2} \Big| \sum_{j=n}^{\infty} \tbinom{j+m-1-n}{m-1} p_j f(h_j^\alpha) - \sum_{j=n}^{\infty} \tbinom{j+m-1-n}{m-1} p_j f(h_j^0) \Big|, \end{aligned}$$

where $b^0 = Ax^0$, $b^\alpha = Ax^\alpha$.

Since the series $\sum_{j=n}^{\infty} \binom{j+m-1-n}{m-1} p_j f(h_j^\alpha)$, $\sum_{j=n}^{\infty} \binom{j+m-1-n}{m-1} p_j f(h_j^0)$ are convergent (which follows from (2.3)), we have

$$\|Ax^\alpha - Ax^0\| \leq \varepsilon_2 \sum_{j=n_2}^{\infty} \tbinom{j+m-1-n_2}{m-1} p_j, \quad \alpha \geq n_4.$$

Hence A is continuous.

By the Schauder fixed point theorem, there exists in T a solution of the equation $x = Ax$. Let $z = \{d_1, d_2, \ldots, d_{n_2-1}, d_{n_2}, \ldots\}$ denote such a solution. Since $z \in T$, it can be written as follows:

$$z = \{C, C, \ldots, C, d_{n_2}, d_{n_2+1}, \ldots\},$$

and

$$\begin{aligned} Az = \{C, C, \ldots, C, C &- \sum_{j=n_2}^{\infty} \tbinom{j+m-1-n_2}{m-1} p_j f(d_j), \\ & C - \sum_{j=n_2+1}^{\infty} \tbinom{j+m-n_2-2}{m-1} p_j f(d_j), \ldots\}. \end{aligned}$$

This means that

$$d_{n_2+k} = C - \sum_{j=n_2+k}^{\infty} \tbinom{j+m-1-n_2-k}{m-1} p_j f(d_j) \quad \text{for } k \geq 0. \tag{2.5}$$

Using the operator Δ to (2.5) m times, one obtains

$$\Delta^m d_{n_2+k} = (-1)^m p_{n_2+k} f(d_{n_2+k}), \quad k \geq 0.$$

This means that the sequence $\{d_n\}_{n=1}^{\infty}$ fulfils equation (E) but for $n \geq n_2$ only.

Now the existence of the solution $\{x_n\}_{n=1}^{\infty}$ of (E) such that $x_n = d_n$ for $n \geq n_2$ will be proved. One can observe that (E) can be rewritten in the form

$$x_n + p_n f(x_n) = -x_{m+n} + \tbinom{m}{1} x_{m+n-1} - \tbinom{m}{2} x_{m+n-2} + \ldots - (-1)^{m-1} \tbinom{m}{m-1} x_{n+1}.$$

If $n = n_2 - 1$ then one gets

$$\begin{aligned} x_{n_2-1} &+ p_{n_2-1} f(x_{n_2-1}) \\ &= -x_{m+n_2-1} + \tbinom{m}{1} x_{m+n_2-2} + \ldots - (-1)^{m-1} \tbinom{m}{m-1} x_{n_2}. \end{aligned} \tag{2.6}$$

But we require x_n to be equal to d_n for $n \geqslant n_2$. From (2.6) one obtains

$$x_{n_2-1}+p_{n_2-1}f(x_{n_2-1}) = -d_{m+n_2-1}+\binom{m}{1}d_{m+n_2-2}+\ldots-(-1)^{m-1}\binom{m}{m-1}d_{n_2}.$$

By virtue of (S1) it follows that the equation

$$x+p_{n_2-1}f(x) = -d_{m+n_2-1}+\binom{m}{1}d_{m+n_2-2}+\ldots-(-1)^{m-1}\binom{m}{m-1}d_{n_2}$$

has solutions. Let us denote one of them by x_{n_2-1}.

Analogously, one can calculate $x_{n_2-2}, x_{n_2-3}, \ldots, x_2, x_1$ one after the other. In consequence one gets a sequence which fulfils (2.6), i.e. which fulfils (E), too. Moreover, this sequence is identical with $\{z_n\}_{n=1}^{\infty}$ for $n \geqslant n_2$ and it has the asymptotic behavior (AB) because $\lim_{n\to\infty} d_n = C$.

The case where m is odd. In a similar way as in the proof of the case where m is even but with some modifications one can show that A is a function from T to T and A is continuous. Using Schauder's fixed point theorem, analogously as above one finds that

$$z = Az = \Big\{C, \ldots, C, C+\sum_{j=n_2}^{\infty}\binom{j+m-1-n_2}{m-1}p_j f(h_j),$$
$$C+\sum_{j=n_2+1}^{\infty}\binom{j+m-n_2-2}{m-1}p_j f(h_j), \ldots\Big\}.$$

This means that

$$d_{n_2+k} = C+\sum_{j=n_2+k}^{\infty}\binom{j+m-1-n_2-k}{m-1}p_j f(d_j), \quad k \geqslant 0. \tag{2.7}$$

Using the operator Δ to (2.7) m times yields

$$\Delta^m d_{n_2+k} = -p_{n_2+k}f(d_{n_2+k}), \quad k \geqslant 0.$$

Hence the sequence $\{z_n\}_{n=1}^{\infty}$ fulfils (E) but for $n \geqslant n_2$ only. Recalling that m is odd, one can observe that (E) can be rewritten as

$$x_n - p_n f(x_n) = x_{m+n}-\binom{m}{1}x_{m+n-1}+\binom{m}{2}x_{m+n-2}-\ldots+(-1)^{m-1}\binom{m}{m-1}x_{n+1}.$$

Reasoning similarly to the case where m is even but using (S2), one can calculate $x_{n_2-1}, x_{n_2-2}, \ldots, x_2, x_1$ one after the other.

In consequence one gets a sequence which fulfils (E) and which is identical with $\{z_n\}_{n=1}^{\infty}$ for $n \geqslant n_2$. It has the asymptotic behavior (AB), too. ■

References

[1] A. Drozdowicz, *On the asymptotic behavior of solutions of the second order difference equations*, Glasnik Mat. 22 (42) (1987), 327–333.

[2] A. Drozdowicz, J. Popenda, *Asymptotic behavior of the solutions of the second order difference equation*, Proc. Amer. Math. Soc. 99, No 1 (1987), 135–140.

[3] T. G. Hallam, *Asymptotic behavior of the solutions of an n-th order nonhomogeneous ordinary differential equation*, Trans. Amer. Math. Soc. 122 (1966), 177–196.

[4] —, *Asymptotic behavior of the solutions of a nonhomogeneous singular equations*, J. Diff. Eq. 3 (1967), 135–152.

[5] J. W. Hooker, W. T. Patula, *A second order nonlinear difference equation, oscillation and asymptotic behavior*, J. Math. Anal. Appl. 91, No 1 (1983), 9–29.

[6] J. Popenda, *On the asymptotic behavior of the solutions of an n-th order difference equation*, Ann. Polonici Math. 44 (1984), 95–111.

[7] W. F. Trench, *Asymptotic behavior of solutions of an n-th order differential equation*, Rocky Mt. J. Math. 14 (1984), 441–450.

(INSTYTUT MATEMATYKI, POLITECHNIKA POZNAŃSKA, POZNAŃ)
INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF POZNAŃ
POZNAŃ, POLAND