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Additive and countably additive correspondences

Set-valued measures (or countably additive correspondences) have been intensively studied for their own sake, and in connection with their applications, for about fifteen years now. The main interest has been focused on those arising as integrals of correspondences (see e.g. [2], [7], [12], [13], [24], [39], [40], [43] and references therein). However, it seems that such important topics from the theory of vector measures as, e.g., the extension problem, convergence theorems, boundedness, have not been investigated in case of correspondences. The purpose of this paper is to fill in this gap, to some extent.

The first two sections are of preparatory character; in Section 2 much attention is paid to a theorem discovered by Radström [37] and generalized by Hörmander [26], but in the finite dimensional case essentially due to Minkowski (cf. [5]), which appears to be a very useful tool in studying bounded-valued correspondences. This was apparently observed first by Debreu [12], then used also by Valadier [43]; see also [25].

In Section 3 we discuss additivity, countable additivity and some other properties of correspondences. Though most of the results stated here remain valid for group-valued correspondences, we restrict ourselves, in Section 3 and throughout the paper, to those taking on values in locally convex vector spaces.

In Section 4 our main task is to prove the extension Theorem 4.5, an analogue (also as concerns the method used) of the corresponding result for vector or group valued measures [16].

In Section 5 we seek conditions which ensure boundedness of the range of a set correspondence. Also, we establish here the Saks type decomposition of correspondences, and generalize a result of Diestel [14].

The results in Section 6 are of Lapunov-type: they were inspired by results of Tweddle [42], Schmeidler [39], [40] and Artstein [2] (cf. also [8], [9]).

Section 7 contains the Vitali–Hahn–Saks and Nikodym type theorems on convergent sequences of correspondences, and an analogue of Nikodym's uniform boundedness theorem. All of these are almost immediate consequences of the Minkowski–Radström–Hörmander theorem.

In Section 8 we consider the problem of the existence of a control measure for an additive correspondence, and in connection with this the existence of additive selections.

1. Basic notation used throughout this paper is as follows.

X denotes a Hausdorff locally convex topological vector space, τ its topology, X' the topological dual of X. U is a generic notation for a member of a base $\mathscr{U}$ of *absolutely convex closed* neighborhoods of the origin in X. $\mathscr{A}(X)$ is the family of all non-empty subsets of X; $\mathscr{B}(X)$, $\mathscr{C}(X)$ and $\mathscr{K}(X)$ are the families of bounded, closed, and closed bounded convex sets in $\mathscr{A}(X)$, respectively. Instead of $\{0\}$ we will usually write 0. The closure [resp. convex hull, closed convex hull] of a set A in X is denoted $\bar{A}$ or A^- [resp. $\operatorname{co} A$, $\overline{\operatorname{co}}\, A$].

For the moment, let $\approx$ denote the equivalence relation on $\mathscr{A}(X)$ defined by

$$A \approx B \quad \text{iff} \quad \bar{A} = \bar{B}.$$

Then the quotient $\mathscr{A}(X)/\approx$ may be identified with $\mathscr{C}(X)$. Further, the usual addition $(A, B) \to A+B$ in $\mathscr{A}(X)$ agrees with $\approx$, that is $A_1 \approx B_1$, $A_2 \approx B_2$ imply $A_1+A_2 \approx B_1+B_2$. Therefore, $(\mathscr{A}(X), +)$ admits a factorization by $\approx$, and the resulting quotient semigroup may be identified with $(\mathscr{C}(X), \overset{*}{+})$, where the *new* addition is defined by

$$A \overset{*}{+} B = (A+B)^- \quad (= (\bar{A}+\bar{B})^-)$$

(cf. [26], p. 182, and [25]). Recall that $A \overset{*}{+} B = A+B$ if one of the sets A, B is closed and the other compact ([31], § 15.6 (10)). The symbol ${}^*\sum$ used for finite sums or series will indicate that the addition involved is $\overset{*}{+}$.

N denotes the set of positive integers, $\mathscr{P}(N)$ the family of all its subsets; $R = (-\infty, \infty)$.

$\mathscr{R}$ denotes a ring of subsets of a set S. Whenever it is clear from a context that a set, say E, is (or should be) taken from $\mathscr{R}$, we avoid writing "$E \in \mathscr{R}$" or so.

M always denotes an X-valued correspondence defined on $\mathscr{R}$, i.e. a function $\mathscr{R} \to \mathscr{A}(X)$, such that

$$M(\emptyset) = 0;$$

then $M^{\check{}}$, $\bar{M}$, $\operatorname{co} M$, $\overline{\operatorname{co}}\, M$ are the correspondences defined by the formulas

$$M^{\check{}}(E) = \bigcup \{M(F) \mid F \subset E\},$$

$$\bar{M}(E) = \big(M(E)\big)^-, \qquad (\operatorname{co} M)(E) = \operatorname{co} M(E), \qquad (\overline{\operatorname{co}}\, M)(E) = \overline{\operatorname{co}}\, M(E).$$

Any additional conditions on X, $\mathscr{R}$, M will be imposed explicitly only when necessary.

The reader is referred to [16]–[20] for information about submeasures, Fréchet–Nikodym topologies, and some terminology used there for usual set functions. This terminology will be extended here to case of correspondences.

2. A topology on $\mathscr{A}(X)$, the Minkowski–Radström–Hörmander theorem and series of sets. Let $\mathfrak{U}$ be the invariant uniformity on X compatible with τ. Then the *exponential extension* of $\mathfrak{U}$ to $\mathscr{A}(X)$ is defined to be the uniformity $\tilde{\mathfrak{U}}$ on $\mathscr{A}(X)$ determined by the families

$$\mathscr{E}(U) = \{(A, B) \in \mathscr{A}(X) \times \mathscr{A}(X) | A \subset B \overset{*}{+} U, B \subset A \overset{*}{+} U\}, \qquad U \in \mathscr{U}$$

as a base of vicinities. ($\overset{*}{+}$ may be replaced by $+$.)

The topology on $\mathscr{A}(X)$ associated with $\tilde{\mathfrak{U}}$, the *uniform exponential extension* of τ, will be denoted $\tilde{\tau}$. The symbols $\tilde{\mathfrak{U}}$ and $\tilde{\tau}$ will be also used to denote the induced uniformity and topology on any subfamily of $\mathscr{A}(X)$ (cf. [6], Exercises to Chapter 2; [32], [33]). Thus if $A_0 \in \mathscr{A}(X)$, then the families

$$\mathscr{E}(U, A_0) = \{A \in \mathscr{A}(X) | (A, A_0) \in \mathscr{E}(U)\}, \qquad U \in \mathscr{U},$$

form a base of $\tilde{\tau}$-neighborhoods of A_0 in $\mathscr{A}(X)$.

We note that the mappings $(A, B) \to A + B$, $A \to \overline{A}$, $A \to \mathrm{co}A$, $A \to \overline{\mathrm{co}}A$ are uniformly continuous.

If $\approx$ is the relation considered in Section 1, then $(\mathscr{C}(X), \tilde{\tau})$ may be identified with $(\mathscr{A}(X), \tilde{\tau})/\approx$, and $(\mathscr{C}(X), \tilde{\mathfrak{U}})$ with the separated uniform space associated with $(\mathscr{A}(X), \tilde{\mathfrak{U}})$.

If X is metrizable, then $(\mathscr{A}(X), \tilde{\mathfrak{U}})$ is semimetrizable, e.g. via the known Hausdorff distance between sets; if, besides, X is complete, then so is $\mathscr{C}(X)$. In the particular case where τ is determined by a norm $\| \ \|$, the Hausdorff distance is given by the formula

$$\|A, B\| = \inf\{t > 0 | A \subset B + tU_1, B \subset A + tU_1\},$$

where $U_1 = \{x \in X | \ \|x\| \leqslant 1\}$.

The families of convex, or bounded, or compact members of $\mathscr{C}(X)$, as well as the intersections of these families (and many others) are closed subsemigroups of $(\mathscr{C}(X), \overset{*}{+}; \tilde{\tau})$. On $\mathscr{K}(X)$, the addition $\overset{*}{+}$ together with the usual multiplication $(t, A) \to tA$ by non-negative reals (which is associative and distributive with respect to $\overset{*}{+}$) define the algebraic structure of an "abstract" convex cone.

The semigroup $\mathscr{K}(X)$ has the significant property that the law of cancellation

(1c) $$A \overset{*}{+} B = C \overset{*}{+} B \Rightarrow A = C$$

holds in it. This fact, first observed by Radström for the usual addition $+$ in a normed space X, allowed him to embed algebraically and isometrically some "cones" of closed convex subsets of X, endowed with the Hausdorff distance, into appropriately defined normed vector spaces [37].

Soon afterwards, this was generalized by Hörmander [26] who proved that if (X, τ) is a locally convex vector space over R, then $(\mathscr{K}(X), \tilde{\tau})$ can be identified with the cone

$$\mathscr{K} = \{K_A | A \in \mathscr{K}(X)\}$$

of the real vector space

$$\mathscr{X} = \{K_A - K_B | A, B \in \mathscr{K}(X)\}$$

equipped with the locally convex topology of uniform convergence on τ-equicontinuous subsets of X'. And he observed that when $\dim X < \infty$ this result is essentially due to Minkowski (cf. [5]). Here K_A: $X' \to R$ is the support function of A which is defined by

$$K_A(x') = \sup\{x'(x) | x \in A\}.$$

The original construction of Radström, in which X' does not appear has been recently modified in [25] to treat that general case. It seems worthwhile to present briefly a version of that construction in which neither X' nor the seminorms defining τ (as in [25]) will be used.

We start by proving that:

(lcg) *If $A \in \mathscr{A}(X)$, $B \in \mathscr{B}(X)$ and C is a closed convex subset of X, then*

$$A + B \subset C \overset{*}{+} B \Rightarrow A \subset C$$

(cf. [37], Lemma 1). Evidently, (lcg) $\Rightarrow$ (lc).

Take any $U \in \mathscr{U}$. Then

$$A + B \subset C + B + U.$$

Let $a \in A$, and choose any $b_1 \in B$. Then

$$a + b_1 = c_1 + b_2 + u_1 \quad \text{for some } c_1 \in C, b_2 \in B, u_1 \in U,$$

$$a + b_2 = c_2 + b_3 + u_2 \quad \text{for some } c_2 \in C, b_3 \in B, u_2 \in U,$$

and so forth. Hence

$$a = \frac{1}{n}(c_1 + \ldots + c_n) + \frac{1}{n}(b_{n+1} - b_1) + \frac{1}{n}(u_1 + \ldots + u_n), \qquad n \in N.$$

Hence, by boundedness of B and convexity of C and U,

$$a \in C + 2U.$$

Since C is closed, we conclude that $a \in C$. This proves (lcg).

Now consider the product-semigroup $\mathscr{K}^2(X) = \mathscr{K}(X) \times \mathscr{K}(X)$ and the equivalence relation $\simeq$ on it, defined by

$$(A_1, B_1) \simeq (A_2, B_2) \quad \text{iff} \quad A_1 \overset{*}{+} B_2 = B_1 \overset{*}{+} A_2.$$

The addition in $\mathscr{K}^2(X)$ agrees with $\simeq$, hence $R(X) = \mathscr{K}^2(X)/\simeq$ is a semigroup under the addition $\overset{*}{+}$ defined by

$$(A, B)_{\simeq} \overset{*}{+} (C, D)_{\simeq} = (A \overset{*}{+} C, B \overset{*}{+} D)_{\simeq};$$

$(E, F)_{\simeq}$ denotes the element of $R(X)$ determined by (E, F). Since $(A, B)_{\simeq} \overset{*}{+} (B, A)_{\simeq} = (0, 0)_{\simeq} = 0_{\simeq}$, the zero of $R(X)$, $(B, A)_{\simeq}$ is the negative of $(A, B)_{\simeq}$ relative to $\overset{*}{+}$; notation: $(B, A)_{\simeq} = \overset{*}{-}(A, B)_{\simeq}$. Thus $(R(X), \overset{*}{+})$ is an abelian group. Since the mapping $A \to (A, 0)_{\simeq}$ is an isomorphism of $(\mathscr{K}(X), \overset{*}{+})$ into $(R(X), \overset{*}{+})$, we can identify each $A \in \mathscr{K}(X)$ with $(A, 0)_{\simeq} \in R(X)$. Since $(A, B)_{\simeq} = (A, 0)_{\simeq} \overset{*}{+} (0, B)_{\simeq} = A \overset{*}{-} B$, we see that

$$R(X) = \mathscr{K}(X) \overset{*}{-} \mathscr{K}(X).$$

It is now obvious that $R(X)$ is a smallest, up to isomorphism, abelian group which contains $\mathscr{K}(X)$.

We should note that the negative $\overset{*}{-}A$ in $R(X)$ of a member A of $\mathscr{K}(X)$ does not coincide with $-A = \{-a \mid a \in A\}$ unless A has only one point.

For each $U \in \mathscr{U}$ let

$$E(U) = \{(A, B)_{\simeq} \in R(X) \mid (A, B) \in \mathscr{E}(U)\};$$

$E(U)$ is well-defined because using (lcg) one easily verifies that if $(A, B) \in \mathscr{E}(U)$, $(C, D) \in \mathscr{K}^2(X)$ and $(C, D) \simeq (A, B)$, then $(C, D) \in \mathscr{E}(U)$. Then $\{E(U) \mid U \in \mathscr{U}\}$ is a base of symmetric neighborhoods of $0_{\simeq}$ for a group topology, $r(\tau)$, on $R(X)$. It is Hausdorff, for if $\mathscr{X} = (A, B)_{\simeq} \in E(U)$, that is $A \subset B \overset{*}{+} U$ and $B \subset A \overset{*}{+} U$, for every U, then $A = B$, so that $\mathscr{X} = 0_{\simeq}$.

It is also easy to see that $r(\tau)$ induces $\tilde{\tau}$ on $\mathscr{K}(X)$. In fact, for each $A \in \mathscr{K}(X)$, the neighborhood $(A \overset{*}{+} E(U)) \cap \mathscr{K}(X)$ of A in $(\mathscr{K}(X), r(\tau))$ coincides with the neighborhood $\mathscr{E}(A, U) \cap \mathscr{K}(X)$ of A in $(\mathscr{K}(X), \tilde{\tau})$. From this, and the definition of $r(\tau)$, it follows that the invariant uniformity on $R(X)$ compatible with $r(\tau)$ is the *finest* one which induces $\tilde{\mathfrak{U}}$ on $\mathscr{K}(X)$.

The group $R(X)$ can be now converted into a vector space over R by defining multiplication by the formulas

$$t\cdot(A, B)_{\simeq} = (tA, tB)_{\simeq} \quad \text{if } t \geqslant 0, \quad t\cdot(A, B)_{\simeq} = (-tB, -tA)_{\simeq} \text{ if } t < 0.$$

This is a unique multiplication which induces the usual multiplication by non-negative reals on $\mathscr{K}(X)$ and satisfies the condition $(-1)\cdot(A, B)_{\simeq} = \overset{*}{-}(A, B)_{\simeq}$.

Since the neighborhoods $E(U)$ of $0_{\simeq}$ in $R(X)$ are convex and absorbing, $R(X)$ is a Hausdorff locally convex topological vector space under $r(\tau)$.

Thus we have established the following

2.1. THEOREM (Minkowski–Radström–Hörmander). *There exists a Hausdorff locally convex topological vector space $(R(X), r(\tau))$ over R such that*

(a) $\mathscr{K}(X) \subset R(X)$ *and* $R(X) = \mathscr{K}(X) \overset{*}{-} \mathscr{K}(X)$;

(b) $R(X)$ *induces on* $\mathscr{K}(X)$ *its algebraic structure of an "abstract" convex cone*;

(c) $r(\tau)$ *is the finest group topology on* $R(X)$ *whose invariant uniformity induces on* $\mathscr{K}(X)$ *its uniformity* $\tilde{\mathfrak{U}}$ *(and topology* $\tilde{\tau}$*).*

If (X, τ) *is metrizable [and complete], then so is* $(R(X), r(\tau))$ *[and* $\mathscr{K}(X)$ *is a complete, hence closed, convex cone in* $R(X)$*]. If* (X, τ) *is normed by* $\| \ \|$, *then so is* $(R(X), r(\tau))$, *and a norm defining* $r(\tau)$ *can be chosen so that the distance in* $R(X)$ *between any two members* A, B *of* $\mathscr{K}(X)$ *is equal to their Hausdorff distance* $\|A, B\|$. (1)

This theorem seems to be of some importance for the theory of correspondences (see [12], [25], [43]), because in some situations it enables one to replace correspondences by suitable vector-valued functions which are usually more handy.

Concerning the definitions and propositions stated below, see [4], [25], [36].

2.2. DEFINITIONS. Let (A_n) be a sequence of members of $\mathscr{A}(X)$. Then $\sum_{n=1}^{\infty} A_n$ denotes the set of all x such that for some $a_n \in A_n$ $(n \in N)$ the series $\sum_{n=1}^{\infty} a_n$ converges in (X, τ) to x.

We say that the *series* $\sum_{n=1}^{\infty} A_n$ *τ-converges* [resp. *$\tilde{\tau}$-converges*], if every series $\sum_{n=1}^{\infty} a_n$ $(a_n \in A_n)$ τ-converges in X [resp. if the sequence $S_n = \sum_{i=1}^{n} A_i$

(1) Added in proof. R. Urbański has recently extended this result to *non-locally* convex spaces X (to appear in Bull. Acad. Polon. Sci.).

$\tilde{\tau}$-converges in $\mathscr{A}(X)$]. We write then $(\tau)\sum_{n=1}^{\infty} A_n$ [resp. $(\tilde{\tau})\sum_{n=1}^{\infty} A_n$]; this symbol will also denote the τ-sum [resp. $\tilde{\tau}$-*sum*] of the series, by which we understand the set $\sum_{n=1}^{\infty} A_n$ defined above [resp. the unique closed set $S_0 \in (\tilde{\tau})\lim_{n\to\infty} S_n$].

We say that the series $\sum_{n=1}^{\infty} A_n$ satisfies the *Cauchy condition* (with respect to $\tilde{\tau}$) if

$$\text{(Cc)} \qquad \forall U, \exists k, \forall n, m\colon n \geqslant m \geqslant k \Rightarrow \sum_{i=m}^{n} A_i \subset U.$$

We will write $A_n \to 0$ if $A_n \to 0$ in $\tilde{\tau}$, that is $\forall U, \exists k, \forall n\colon n \geqslant k \Rightarrow A_n \subset U$.

It is obvious that the $\tilde{\tau}$-convergence of a series $\sum_{n=1}^{\infty} A_n$ is equivalent to the convergence of the series $^*\sum_{n=1}^{\infty} \bar{A}_n$ in the Hausdorff topological semigroup $\big(\mathscr{C}(X), \overset{*}{+}; \tilde{\tau}\big)$, and $(\tilde{\tau})\sum_{n=1}^{\infty} A_n = (\tilde{\tau})\lim_{n\to\infty} {}^*\sum_{i=1}^{n} \bar{A}_i$ in $\mathscr{C}(X)$.

2.3. Proposition. (a) *If* $\sum_{n=1}^{\infty} A_n \neq \emptyset$ *and the series* $\sum_{n=1}^{\infty} A_n$ $\tilde{\tau}$-*converges, then*

$$(\tilde{\tau})\sum_{n=1}^{\infty} A_n = \Big(\sum_{n=1}^{\infty} A_n\Big)^{-}.$$

(b) *If a series* $\sum_{n=1}^{\infty} A_n$ τ-*converges, then it satisfies* (Cc), *hence all the series* $\sum_{n=1}^{\infty} a_n$ $(a_n \in A_n)$ *converge uniformly, i.e.*

$$\forall U, \exists k, \forall n\colon n \geqslant k \Rightarrow \sum_{i=n}^{\infty} A_i \subset U,$$

so that the series $\sum_{n=1}^{\infty} A_n$ $\tilde{\tau}$-*converges* (*to* $\big(\sum_{n=1}^{\infty} A_n\big)^{-}$).

(b′) *If* $A_n \in \mathscr{B}(X)$, $\forall n$, *and the series* $\sum_{n=1}^{\infty} A_n$ $\tilde{\tau}$-*converges, then it satisfies* (Cc), *its* $\tilde{\tau}$-*sum is in* $\mathscr{B}(X) \cap \mathscr{C}(X)$, *the series* $^*\sum_{n=1}^{\infty} \overline{\mathrm{co}} A_n$ $\tilde{\tau}$-*converges in* $\mathscr{K}(X)$ *and* $(\tilde{\tau})\,{}^*\sum_{n=1}^{\infty} \overline{\mathrm{co}}\, A_n = \overline{\mathrm{co}}\,(\tau)\sum_{n=1}^{\infty} A_n$.

(c) *Suppose* X *is sequentially complete. Then if a series* $\sum_{n=1}^{\infty} A_n$ *satisfies* (Cc), *it is* τ-*convergent.*

Proof. Parts (a), (b), (c) are known (and easy). (b′) It is obvious that the $\tilde{\tau}$-sum S_0 of our series is closed and bounded. Now, given U, there is k such that $S_n = \sum_{i=1}^{n} A_i \in \mathscr{E}(S_0, U)$ for $n \geqslant k$. Then $\overline{\mathrm{co}}S_n = \big(\sum_{i=1}^{n} \overline{\mathrm{co}}A_i\big)^{-} \in \mathscr{E}(\overline{\mathrm{co}}\,S_0, U)$ for $n \geqslant k$. It follows that the series ${}^*\sum_{n=1}^{\infty} \overline{\mathrm{co}}\,A_n$ $\tilde{\tau}$-converges in $\mathscr{K}(X)$ to $\overline{\mathrm{co}}\,S_0$, hence also in $\big(R(X), r(\tau)\big)$. Hence, given U, there is k such that $\sum_{i=m}^{n} A_i \subset {}^*\sum_{i=m}^{n} \overline{\mathrm{co}}\,A_i \subset U$ for $n \geqslant m \geqslant k$, i.e. (Cc) is satisfied.

Actually, (Cc) is an easy consequence of (lcg). Indeed, if $S_n \in \mathscr{E}(S_0, U)$ for $n \geqslant k$, then $S_n \subset S_0 \overset{*}{+} U$ and $S_0 \subset S_m \overset{*}{+} U$ for $n > m \geqslant k$. Hence $S_m + \sum_{i=m+1}^{n} A_i \subset S_m \overset{*}{+} 2U$, so that $\sum_{i=m+1}^{n} A_i \subset 2U$ for $n > m \geqslant k$, by (lcg).

2.4. Corollary. *Suppose X is sequentially complete. Then a series of members of $\mathscr{B}(X)$ is τ-convergent iff it is $\tilde{\tau}$-convergent.*

The concepts of the *unconditional* τ- or $\tilde{\tau}$-*convergence* and the *subseries* τ- or $\tilde{\tau}$-*convergence* of a series $\sum_{n=1}^{\infty} A_n$, and of the $\tilde{\tau}$-*summability* of a family $(A_i)_{i\in I}$, are quite analogous to those in the "point" case (see e.g. [3], [5], [10], [25]). We say that a family $(A_i)_{i\in I}$ satisfies the *Cauchy condition* (with respect to $\tilde{\tau}$) *for summability* if

$$\text{(Ccs)} \qquad \forall U, \exists e_U, \forall e\colon e \cap e_U = \emptyset \Rightarrow \sum_{i\in e} A_i \subset U;$$

e_U, e denote finite subsets of I. If (Ccs) holds and $I = N$, we also say that the series $\sum_{n=1}^{\infty} A_n$ is *unconditionally Cauchy*.

The statements in the proposition below are either straightforward or can be proved by applying the Minkowski–Radström–Hörmander theorem (or (lcg)).

2.5. Proposition. (a) *If a series $\sum_{n=1}^{\infty} A_n$ is unconditionally [resp. subseries] τ-convergent, then it is unconditionally [resp. subseries] $\tilde{\tau}$-convergent and unconditionally Cauchy.*

(b) *If $A_n \in \mathscr{B}(X)$, $\forall n$, and the series $\sum_{n=1}^{\infty} A_n$ is unconditionally $\tilde{\tau}$-convergent, then it is unconditionally Cauchy and*

$$(\tilde{\tau}) \sum_{n=1}^{\infty} A_{p(n)} = (\tilde{\tau}) \sum_{n=1}^{\infty} A_n$$

for every permutation p of N. Moreover, ${}^*\sum_{n=1}^{\infty} \overline{\mathrm{co}}\,A_n$ is unconditionally $\tilde{\tau}$-convergent in $\mathscr{K}(X)$.

(c) *Suppose X is sequentially complete. Then for a series* $\sum_{n=1}^{\infty} A_n$, *where* $A_n \in \mathscr{B}(X)$, $\forall n$, *all the kinds of convergence introduced after* 2.4., *and* (Ccs), *are equivalent.*

In view of the Minkowski–Radström–Hörmander theorem it is also clear that $\tilde{\tau}$-summability in $\mathscr{B}(X)$ has "usual" properties ([6], Chapter 3).

Remark. It is well known that $\mathscr{B}(X)$ is the same for all locally convex topologies on X which are compatible with the (given via τ) duality (X, X') ([31], 20.11(7)), and also that the closure of a convex subset of X is the same for all of them ([31], 20.8(6)). Therefore the family $\mathscr{K}(X)$ and the addition $\overset{*}{+}$ in it do not depend on a choice of such a topology, and neither does $\big(R(X), \overset{*}{+}\big)$. It would be interesting to know somewhat more about the relationship between the corresponding *topologies* on $R(X)$ (it is trivial that $\tau_1 \subset \tau_2 \Rightarrow r(\tau_1) \subset r(\tau_2)$ and to represent the resulting duals of $R(X)$. Hörmander's functional representation of $R(X)$ seems to be more suitable for those purposes.

Another question is whether $\mathscr{K}(X)$ is the largest subsemigroup of $\big(\mathscr{C}(X), \overset{*}{+}\big)$ in which the law of cancellation holds. And what about $\mathscr{B}(X)$ in (lcg)?

Further, are there any conditions other than metrizability which jointly with completeness or sequential completeness of X imply completeness or sequential completeness of $\mathscr{C}(X)$ or $\mathscr{K}(X)$? This is pertinent to the extension problem for additive correspondences (4.5).

3. Additivity of correspondences. A correspondence M may be considered as a mapping of $\mathscr{R}$ into either the semigroup $\big(\mathscr{A}(X), +\big)$ or, if sets with the same closure are identified, the semigroup $\big(\mathscr{C}(X), \overset{*}{+}\big)$. Accordingly, we have two types of additivity, and then two types of countable additivity of a correspondence. More precisely:

3.1 Definition. We say that M is

(a) *additive* [resp. *subadditive*] if

$$\overline{M}(E_1 \cup E_2) = [\text{resp.} \subset]\; M(E_1) \overset{*}{+} M(E_2) \quad (= \overline{M}(E_1) \overset{*}{+} \overline{M}(E_2))$$

whenever E_1, E_2 are disjoint;

(b) *Countably additive*, or *σ-additive* [resp. *σ-subadditive*], if

$$\overline{M}\Big(\bigcup_{n=1}^{\infty} E_n\Big) = [\text{resp.} \subset]\; (\tilde{\tau}) \sum_{n=1}^{\infty} M(E_n) \quad \Big(= (\tilde{\tau})^{*} \sum_{n=1}^{\infty} \overline{M}(E_n)\Big)$$

for every disjoint sequence $(E_n) \subset \mathscr{R}$ whose union is in $\mathscr{R}$;

(c) *exhaustive* [resp. *semiexhaustive*] if $M(E_n) \to 0$ for every infinite disjoint sequence $(E_n) \subset \mathscr{R}$ [whose union is in $\mathscr{R}$];

(d) *order-continuous*, if $M(E_n) \to 0$ for every sequence $(E_n) \subset \mathscr{R}$ such that $E_n \downarrow \emptyset$;

(e) *strictly additive* [*subadditive*, *σ-additive*, *σ-subadditive*], if the equalities or inclusions in (a), (b) hold when $^-$ and * are omitted and $(\tilde{\tau})$ is replaced by (τ);

(f) *semicountably additive*, or *sσ-additive* [resp. *semicountably subadditive*, or *sσ-subadditive*], if

$$M\Big(\bigcup_{n=1}^{\infty} E_n\Big) = [\text{resp.} \subset] \text{ the set } \sum_{n=1}^{\infty} M(E_n)$$

for every disjoint sequence $(E_n) \subset \mathscr{R}$ whose union is in $\mathscr{R}$.

Though it is quite obvious, note that M has any of the properties introduced in (a) and (b) iff $\bar{M}$ has the corresponding property.

3.2. PROPOSITION. *Suppose M is additive. Then M is exhaustive [semiexhaustive] iff for every disjoint sequence $(E_n) \subset \mathscr{R}$ [whose union is in $\mathscr{R}$] the series $\sum_{n=1}^{\infty} M(E_n)$ is unconditionally Cauchy.*

Proof. *Only if.* Otherwise, for some U there is a sequence (e_k) of pairwise disjoint finite subsets of N such that $\sum_{i \in e_k} M(E_i) \not\subset U$. Then $\bar{M}(F_k) \not\subset U$, where $F_k = \bigcup_{i \in e_k} E_i$ $(k \in N)$. This contradicts the exhaustivity of M.

In the case when M is semiexhaustive the sequence $(G_n) \equiv (E_{i_1}, F_1, E_{i_2}, F_2, \ldots)$, where $\{i_1, i_2, \ldots\} = N \setminus \bigcup_k e_k$, yields a contradiction because its union belongs to $\mathscr{R}$ and $M(G_n) \not\to 0$.

Part *if* is trivial.

3.3. PROPOSITION. *M is additive and order-continuous iff M is σ-additive and semiexhaustive.*

Proof. *If.* We will show more, namely that $M^{\vee}$ is order-continuous. If this is false, we can find a sequence $(E_n) \subset \mathscr{R}$ such that $E_n \downarrow \emptyset$ and for some U, $M^{\vee}(E_n) \not\subset U$ $(n \in N)$. Hence for each n there is $F_n \subset E_n$ such that $M(F_n) \not\subset U$. Since

$$\bar{M}(F_n) = (\tilde{\tau}) \sum_{k=n}^{\infty} M\big(F_n \cap (E_k \setminus E_{k+1})\big),$$

for m large enough we have

$$\bar{M}(F_n \setminus E_{m+1}) = {}^{*}\sum_{k=n}^{m} M\big(F_n \cap (E_k \setminus E_{k+1})\big) \not\subset U.$$

Now one defines easily a sequence $1 = k_1 < k_2 < \ldots$ such that

$$\bar{M}(F_{k_n} \setminus E_{k_{n+1}}) \not\subset U, \qquad n \in N.$$

Denote $A_n = F_{k_n} \setminus E_{k_{n+1}}$, $B_n = E_{k_n} \setminus F_{k_n}$ $(n \in N)$. Then the sequence $(C_n) = (B_1, A_1, B_2, A_2, \ldots)$ is disjoint, its union is $E_1 \in \mathscr{R}$, and $M(C_n) \not\to 0$. A contradiction with the semiexhaustivity of M.

Only if. First we show that M is semiexhaustive. Let (E_n) be a disjoint sequence in $\mathscr{R}$ whose union belongs to $\mathscr{R}$, and let F_n denote $\bigcup_{k=n}^{\infty} E_k$ $(n \in N)$. Then

$$\bar{M}(F_n) = M(E_n) \overset{*}{+} M(F_{n+1}),$$

and $M(F_n) \to 0$, $M(F_{n+1}) \to 0$ by order-continuity of M. It follows that $M(E_n) \to 0$.

On the other hand, by additivity,

$$\bar{M}(F_1) = \sum_{k=1}^{n} M(E_k) \overset{*}{+} M(F_{n+1}), \qquad n \in N,$$

hence

$$\bar{M}\Big(\bigcup_{n=1}^{\infty} E_n\Big) = (\tilde{\tau}) \sum_{n=1}^{\infty} M(E_n).$$

Thus M is σ-additive.

The trivial example: $M(E) = X$ if $E \neq \emptyset$, $= 0$ if $E = \emptyset$, shows that σ-additivity does not imply semiexhaustivity, in general.

3.4. PROPOSITION. *Suppose M is strictly additive [and its values are, respectively, sequentially closed, or sequentially complete (in particular compact)]. Then M is order-continuous iff M is $s\sigma$-subadditive and semiexhaustive [resp. $s\sigma$-additive and semiexhaustive; strictly σ-additive].*

Proof. *If.* Let E_n, F_n be as in the *only if* part of the proof of 3.3. Then, for any U, there is by Proposition 3.2 some k such that $\sum_{i=m}^{n} M(E_i) \subset U$ if $n \geqslant m \geqslant k$. Hence, by $s\sigma$-subadditivity of M, $M(F_m)$ is contained in the set $\sum_{i=m}^{\infty} M(E_i)$, which in turn is contained in U; $m \geqslant k$. Since $M(F_1) = \sum_{i=1}^{m-1} M(E_i) + M(F_m)$, it follows that $\sum_{i=1}^{m-1} M(E_i) \in \mathscr{E}\big(M(F_1), U\big)$ for $m \geqslant k$. Thus $\bar{M}(F_1) = (\tilde{\tau}) \sum_{n=1}^{\infty} M(E_n)$, and this means that M is σ-additive. Therefore Proposition 3.3 can be applied to deduce that M is order-continuous.

Only if. Again, let E_n, F_n be as above. Then

$$M(F_n) = M(E_n) + M(F_{n+1}), \qquad n \in N,$$

and therefore, starting with an arbitrary element $y_1 \in M(F_1)$ we can successively define elements x_1, y_2, then $x_2, y_3, \ldots$ so that

$$x_n \in M(E_n), \quad y_n \in M(F_n)$$

and

$$y_n = x_n + y_{n+1}; \qquad n \in N.$$

Hence

$$y_1 = \sum_{k=1}^{n} x_k + y_{n+1}, \qquad n \in N,$$

and since $y_{n+1} \to 0$ by order-continuity of M, $y_1 = \sum_{k=1}^{\infty} x_k$. Thus $M(F_1) \subset \sum_{n=1}^{\infty} M(E_n)$. Semiexhaustivity of M follows from 3.3.

Now let $u_n \in M(E_n)$ $(n \in N)$. Then, picking out any $v_n \in M(F_n)$ $(n \in N)$, we have

$$\sum_{k=1}^{n} u_k = v_{n+1} \in M(F_1)$$

and $v_{n+1} \to 0$ by order-continuity. Therefore, if the series $\sum_{k=1}^{\infty} u_k$ converges [and it is certainly so when $M(F_1)$ is sequentially complete] and $M(F_1)$ is sequentially closed, the sum of the series belongs to $M(F_1)$. Hence the alternative assertions of the proposition follow.

The following is an immediate consequence of the Orlicz–Pettis theorem (see e.g. [21], [28]).

3.5. Proposition. *Let $\mathscr{R}$ be a σ-ring. If M is strictly σ-additive when X is endowed with the topology $\sigma(X, X')$, then M is strictly σ-additive under the original topology of X.*

The next two results show how strictly additive or σ-additive correspondences may be obtained from subadditive ones.

3.6. Proposition. *If M is strictly subadditive, then the formula*

$$\text{(a)} \qquad M^a(E) = \bigcup \left\{ \sum_{i=1}^{n} M(E_i) \mid (E_i)_{i=1}^{n} \text{ is a finite decomposition of } E \right\}$$

defines the least (with respect to $\subset$) strictly additive correspondence M^a: $\mathscr{R} \to \mathscr{A}(X)$ such that $M(E) \subset M^a(E)$, $E \in \mathscr{R}$. If M is $s\sigma$-subadditive, then so is M^a.

3.7. Proposition. *Suppose X is sequentially complete and suppose M is strictly additive, $s\sigma$-subadditive and semiexhaustive. Then the correspondence M^σ: $\mathscr{R} \to \mathscr{A}(X)$ defined by the formula*

$$(\sigma) \qquad M^\sigma(E) = \bigcup \left\{ \sum_{n=1}^{\infty} M(E_n) \mid (E_n) \text{ is a decomposition of } E \right\}$$

is strictly σ-additive.

Proof. In view of Proposition 3.2, if (A_n) is a disjoint sequence in $\mathscr{R}$ whose union is in $\mathscr{R}$, then the series $\sum_{n=1}^{\infty} M(A_n)$ converges unconditionally.

Let $E \in \mathscr{R}$ be the union of a disjoint sequence $(F_k) \subset \mathscr{R}$, and let $x \in M^\sigma(E)$. By definition, there exists a decomposition (E_n) of E and elements $x_n \in M(E_n)$ such that $x = \sum_{n=1}^{\infty} x_n$. Since $E_n = \bigcup_{k=1}^{\infty} (E_n \cap F_k)$, using $s\sigma$-subadditivity of M we find $x_{nk} \in M(E_n \cap F_k)$ such that $x_n = \sum_{k=1}^{\infty} x_{nk}$. The family $\{x_{nk} | n, k \in N\}$, as well as each family $\{x_{nk} | n \in N\}$ $(k \in N)$, is summable in X. Hence for each k the series $\sum_{n=1}^{\infty} x_{nk}$ converges unconditionally, and its sum, y_k, belongs to $M^\sigma(F_k)$. Now

$$x = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x_{nk} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} x_{nk} = \sum_{k=1}^{\infty} y_k,$$

so that $x \in \sum_{k=1}^{\infty} M^\sigma(F_k)$. Thus we have proved that

$$M^\sigma(E) \subset \sum_{k=1}^{\infty} M^\sigma(F_k),$$

i.e. $s\sigma$-subadditivity of M^σ.

Conversely, suppose $z_k \in M^\sigma(F_k)$. Then for each k there is a decomposition $(F_{kj})_{j=1}^{\infty}$ of F_k, and $z_{kj} \in M(F_{kj})$ such that $z_k = \sum_{j=1}^{\infty} z_{kj}$. The sets F_{kj} form a decomposition of E, hence the family $\{z_{kj} | k, j \in N\}$ is summable in X and

$$\sum_{k=1}^{\infty} z_k = \sum_{k,j \in N} z_{kj} \in M^\sigma(E).$$

Thus the series $\sum_{k=1}^{\infty} M^\sigma(F_k)$ τ-converges (unconditionally) and its sum is contained in $M^\sigma(E)$. This, together with $s\sigma$-subadditivity, is nothing else but strict σ-additivity of M^σ.

It is easily seen that M^σ is the least (with respect to $\subset$) σ-additive correspondence for which $M(E) \subset M^\sigma(E)$, $E \in \mathscr{R}$.

3.8. Proposition. (a) *If M has any of the following properties: additivity; subadditivity; exhaustivity; semiexhaustivity; additivity and order-continuity; strict additivity; strict subadditivity; $s\sigma$-subadditivity, then so does each of the correspondences $M^\vee$ and* co$M$.

(b) *If M is σ-additive, then so is* co$M$.

(c) *If $\mathscr{R}$ is a σ-ring and M is strictly σ-additive, then so is $M^\vee$.*

Proof. The only non-obvious implication in (a), (M is additive and order-continuous) $\Rightarrow$ ($M^{\check{}}$ is order-continuous), is a consequence of Proposition 3.3 and the *if* part of its proof. Implications (b) and (c) are easy to prove.

3.9. PROPOSITION. *Suppose f is a continuous linear mapping from X into another Hausdorff locally convex space Y, and let the correspondence $fM\colon \mathscr{R} \to \mathscr{A}(Y)$ be defined by the formula*

$$(fM)(E) = f[M(E)].$$

Then if M has any of the properties introduced by Definition 3.1, *except $s\sigma$-additivity, then so does fM, respectively.*

In order to include in our discussion correspondences of the type considered in Example 3.12 below, we formulate the following

3.10. DEFINITION. M is said to be *quasi-additive* [resp. *strictly quasi-additive*], if it is subadditive [strictly subadditive] and

(q) $\quad M(E) \subset \big(M(E\cup F) - M(F)\big)^{-}$ [resp. $M(E) \subset M(E\cup F) - M(F)$]

whenever E, F are disjoint.

3.11. PROPOSITION. *If M is quasi-additive, then so is $M^{\check{}}$; if in addition M is order-continuous, then so is $M^{\check{}}$, and both M and $M^{\check{}}$ are semi-exhaustive.*

Proof. The first assertion is straightforward. Now assume M is order-continuous and let $E_n \downarrow \emptyset$. Then $M(E_n \setminus E_{n+1}) \subset \big(M(E_n) - M(E_{n+1})\big)^{-}$ by (q), hence $M(E_n\setminus E_{n+1}) \to 0$ so that M is semiexhaustive. Suppose $M^{\check{}}$ is not order-continuous. Then we can assume that, for some U, and some $F_n \subset E_n$, $M(F_n) \not\subset 2U$ $(n \in N)$. Since $F_n \cap E_k \downarrow \emptyset$ $(k \to \infty)$, for k large enough we have $M(F_n \cap E_k) \subset U$, hence $M(F_n \setminus E_k) \not\subset U$ by subadditivity of M. Our proof can be now completed similarly as in the *if* proof of 3.3.

3.12. EXAMPLE. Let K be a family of (point-valued) additive set functions $\mu\colon \mathscr{R} \to X$. Then the correspondence $M = M_K\colon \mathscr{R} \to \mathscr{A}(X)$ given by the formula

$$M(E) = \{\mu(E) \mid \mu \in K\}$$

is strictly quasi-additive.

If each $\mu \in K$ is σ-additive [resp., if K is uniformly exhaustive ([16], §4)], then M is $s\sigma$-subadditive [resp. exhaustive]. If $\mathscr{R}$ is a σ-ring and each $\mu \in K$ is σ-additive [resp. exhaustive], then M has the following property:

($*$) If (E_n) is a disjoint sequence in $\mathscr{R}$ and, for some r, $x \in M(E_r)$, then there exists a σ-additive set function $\nu\colon \mathscr{P}(N) \to X$ such that

$$\nu(e) \in M\Big(\bigcup_{n\in e} E_n\Big) \quad \text{for every } e \in \mathscr{P}(N), \text{ and } \nu(\{r\}) = x$$

[resp.

$(**) = (*)$ with "σ-additive" replaced by "additive exhaustive"].

Formula 3.6(a) for $M = M_K$ takes the form

$$M^a(E) = \Big\{\sum_{i=1}^{n} \mu_i(E_i) \,\big|\, \big((\mu_i, E_i)\big)_{i=1}^n \text{ is a finite sequence in } K\times\mathscr{R}, \quad (E_i)_{i=1}^n \text{ is a decomposition of } E\Big\},$$

which reminds one of the known formulas for the supremum of a family of real-valued additive set functions (cf. [21], III. 7).

Countable additivity of each $\mu \in K$ and uniform exhaustivity of K may not ensure (semi)exhaustivity of M^a. However, if $X = (X, \|\ \|)$ is a Banach space and there exists an exhaustive additive $\lambda\colon \mathscr{R}\to[0, \infty]$ such that

$$\|\mu(E)\| \leqslant \lambda(E), \qquad E \in \mathscr{R},\ \mu\in K,$$

then M^a is exhaustive. In this case $M^\sigma = (M^a)^\sigma$ (Proposition 3.7) can be defined, and M^σ is the least strictly σ-additive correspondence which "contains" all $\mu \in K$.

If X is finite dimensional, the given condition is also necessary for M^a to be exhaustive.

If p is a continuous seminorm on X, we define a function $\check{p}M$: $\mathscr{R}\to[0, \infty]$ by the formula

$$(\check{p}M)(E) = \sup\{p(x) \mid x \in M^{\vee}(E)\}.$$

Evidently, $\check{p}M = \check{p}M^{\vee}$, $\check{p}M(\emptyset) = 0$ and $\check{p}M$ is non-decreasing: $E \subset F \Rightarrow \check{p}M(E) \leqslant \check{p}M(F)$.

3.13. Proposition. *If M is subadditive, then $\check{p}M$ is a submeasure. If, moreover, M is σ-subadditive or $s\sigma$-subadditive [resp. exhaustive; semiexhaustive; quasi-additive and order-continuous], then $\check{p}M$ is σ-subadditive [resp. exhaustive; semiexhaustive; order-continuous].*

4. Continuity and extensions. Let Γ be a Frechet–Nikodym (FN-) topology on $\mathscr{R}$. In particular, Γ can be the FN-topology $\Gamma(\eta)$ determined by a submeasure η on $\mathscr{R}$; recall that $\Gamma(\eta)$ is semimetrizable, for instance by the semimetric $(E, F)\to\inf\{1, \eta(E\Delta F)\}$ (Δ denotes the symmetric difference). We write $(\mathscr{R}, \eta)$ to indicate that $\mathscr{R}$ is endowed with the topology $\Gamma(\eta)$.

4.1. Definition. M is said to be *Γ-continuous*, $M \ll \Gamma$, if M is continuous as a mapping from $(\mathscr{R}, \Gamma)$ into $\big(\mathscr{A}(X), \tilde{\tau}\big)$. If $\Gamma = \Gamma(\eta)$, we write $M \ll \eta$ instead of $M \ll \Gamma(\eta)$.

Suppose M is subadditive. Then the families

$$\mathscr{V}(U) = \{E \in \mathscr{R} \mid M^{\vee}(E) \subset U\}, \qquad U \in \mathscr{U}$$

form a base of neighborhoods of $\emptyset$ for an FN-topology, $\Gamma(M)$, on $\mathscr{R}$ ([16], 1.5). This topology obviously depends only upon M and the topology of X. If $\{p_i \mid i \in I\}$ is a family of seminorms on X determining the topology of X, then the corresponding family of submeasures $\check{p}_i M$, $i \in I$, determines $\Gamma(M)$. It is evident that M is $\Gamma(M)$-continuous at $\emptyset$, this, however, does not imply continuity on $\mathscr{R}$. (Counterexample: Let $\mathscr{R}$ be the family of all subsets of $N \cup \{\infty\}$ and $M(E) = \{\sum_{n \in E} 2^{-n}\}$ if $\infty \notin E$, $= R$ if $E = \{\infty\}$, $= [0, 1]$ otherwise.)

4.2. Proposition. *Suppose M is quasi-additive and Γ is an FN-topology on $\mathscr{R}$. If M is Γ-continuous at $\emptyset$, then $M\colon (\mathscr{R}, \Gamma) \to (\mathscr{A}(X), \tilde{\tau})$ is uniformly continuous. Moreover, $\Gamma(M)$ is the weakest FN-topology on $\mathscr{R}$ under which M is continuous. Thus $M \ll \Gamma$ iff $\Gamma(M) \subset \Gamma$.*

Proof. Given U, there exists a normal ([16], §1) neighborhood $\mathscr{W}$ of $\emptyset$ in $(\mathscr{R}, \Gamma)$ such that $E \in \mathscr{W} \Rightarrow M(E) \subset U$. (Hence $\mathscr{W} \subset \mathscr{V}(U)$. It follows that $\Gamma(M) \subset \Gamma$.)

Let E, $F \in \mathscr{R}$ and $E \triangle F \in \mathscr{W}$. Then $M(E) \subset \big(M(E \setminus F) + M(E \cap F)\big)^- \subset \big(M(E \setminus F) + \big(M(F) - M(F \setminus E)\big)^-\big)^- = \big(M(F) + M(E \setminus F) - M(F \setminus E)\big)^- \subset M(F) + 3U$, and similarly $M(F) \subset M(E) + 3U$. Hence M is uniformly Γ-continuous.

We will write $M_1 \ll M_2$ when $M_1 \ll \Gamma(M_2)$, and $M_1 \sim M_2$ when $M_1 \ll M_2$ and $M_2 \ll M_1$ (cf. [16], [19]).

Some properties of M yield corresponding properties of $\Gamma(M)$. For example

4.3. Proposition. *If M is quasi-additive and order-continuous [exhaustive], then the FN-topology $\Gamma(M)$ is order-continuous [exhaustive].*

This is in part a consequence of Proposition 3.11, in part obvious. The above two propositions enable us to apply some results concerning the relation $\Gamma_1 \subset \Gamma_2$ among FN-topologies to obtain analogous results about continuity of correspondences. For example from [19], 1.1, we get immediately the following

4.4. Proposition. *Suppose $\mathscr{R}$ is a σ-ring, M is quasi-additive and order-continuous, and η is an arbitrary submeasure on $\mathscr{R}$. Then $M \ll \eta$ iff $M(E) = 0$ whenever $\eta(E) = 0$.*

In the particular case when $M = M_K$ (Example 3.12), $M \ll \Gamma$ means simply that the functions $\mu \in K$ are Γ-equicontinuous.

Now we proceed to the extension problem for correspondences.

4.5. Theorem. *Let X be metrizable and complete, and let $\mathscr{R}_0$ be the σ-ring generated by $\mathscr{R}$.*

If $M\colon \mathscr{R} \to \mathscr{C}(X)$ is quasi-additive, order-continuous and exhaustive, then there exists a unique quasi-additive order-continuous (hence exhaustive) *correspondence* $M_0\colon \mathscr{R}_0 \to \mathscr{C}(X)$ *such that*

$$M_0(E) = M(E), \qquad E \in \mathscr{R}.$$

In particular, if M is σ-additive (or strictly σ-additive) and exhaustive, then the extension M_0 is σ-additive and exhaustive.

Proof. Since the FN-topology $\Gamma = \Gamma(M)$ is order-continuous and exhaustive (4.3), there exists a unique order-continuous FN-topology Γ_0 on $\mathscr{R}_0$ which induces Γ on $\mathscr{R}$ ([16], 8.3 or 7.2). Since $\mathscr{R}$ is dense in $(\mathscr{R}_0, \Gamma_0)$ ([16], 8.2), $\mathscr{C}(X)$ is complete under the Hausdorff distance and M is uniformly Γ-continuous (4.2), there exists a unique Γ_0-continuous extension $M_0\colon \mathscr{R}_0 \to \mathscr{C}(X)$ of M.

M_0 is quasi-additive. In fact, let $E, F \in \mathscr{R}_0$, $E \cap F = \emptyset$. Since Γ and Γ_0 are semimetrizable and $\mathscr{R}$ is dense in $(\mathscr{R}_0, \Gamma_0)$, there are sequences (E_n), $(F_n) \subset \mathscr{R}$ such that

$$E_n \to E, \qquad F_n \to F \quad \text{in } \Gamma_0.$$

Then also

$$E_n \cup F_n \to E \cup F \quad \text{in } \Gamma_0.$$

We can assume that $E_n \cap F_n = \emptyset$ $(n \in N)$. (Otherwise replace E_n, F_n by $E_n \setminus F_n$, $F_n \setminus E_n$, respectively.) From

$$M(E_n \cup F_n) \subset \big(M(E_n) + M(F_n)\big)^-, \qquad M(E_n) \subset \big(M(E_n \cup F_n) - M(F_n)\big)^-,$$

by passing to the limits as $n \to \infty$ and employing continuity of the mappings $(A, B) \to (A+B)^-$, $(A, B) \to (A-B)^-$, we get

$$M_0(E \cup F) \subset \big(M_0(E) + M_0(F)\big)^-, \qquad M_0(E) \subset \big(M_0(E \cup F) - M_0(F)\big)^-.$$

So indeed M_0 is quasi-additive.

Since Γ_0 is order-continuous (and exhaustive), so is, obviously, M_0. It is also evident that $\Gamma_0 = \Gamma(M_0)$.

Now suppose $M_1\colon \mathscr{R}_0 \to \mathscr{C}(X)$ is another quasi-additive order-continuous extension of M. Then $\Gamma_1 = \Gamma(M_1)$ is an order-continuous FN-topology on $\mathscr{R}_0$, $\mathscr{R}$ is dense in $(\mathscr{R}_0, \Gamma_1)$ and $M \ll \Gamma_1$. It follows readily that Γ_1 induces Γ on $\mathscr{R}$, hence by uniqueness of Γ_0 we have $\Gamma_1 = \Gamma_0$. Thus $M_1 \ll \Gamma_0$, hence $M_1 = M_0$.

To prove the remaining part of the theorem it suffices to observe that in view of Proposition 3.3 only additivity of M_0 must be verified. But this can be done quite similarly as in the case of quasi-additivity above.

Remark. It is not clear whether M_0 need be strictly additive (and hence strictly σ-additive, 3.4) if M is. This is certainly the case when M is compact-valued, because then additivity coincides with strict additivity, but in general the answer seems to be "no".

Let us note also that for additive $M\colon \mathscr{R} \to \mathscr{K}(X)$ the extension M_0: $\mathscr{R}_0 \to \mathscr{K}(X)$ can be easily obtained by applying the Minkowski–Radström–Hörmander theorem.

4.6. Corollary. *If $P\colon \mathscr{R} \to \mathscr{C}(X)$ is another quasi-additive [additive] correspondence, $P(E) \subset M(E)$ for every $E \in \mathscr{R}$, and P_0 denotes the unique quasi-additive [additive] order-continuous extension of P on $\mathscr{R}_0$, then $P_0(E) \subset M_0(E)$ for every $E \in \mathscr{R}_0$.*

Therefore, if $M = \overline{M_K}$ (Example 3.12), where each $\mu \in K$ is σ-additive and K is uniformly exhaustive, and K_0 denotes the family of σ-additive extensions of functions from K on $\mathscr{R}_0$, then $M_{K_0} \subset M_0$. Since $M_{K_0} | \mathscr{R} = M_K$, it follows that $M_0 = \overline{M_{K_0}}$.

5. Boundedness.

5.1. Definition. M is said to be *bounded-valued* if $M\colon \mathscr{R} \to \mathscr{B}(X)$, and *bounded* if its range $\bigcup \{M(E) | E \in \mathscr{R}\}$ is bounded.

5.2. Theorem. *Suppose M is quasi-additive, bounded-valued and satisfies the condition:*

(bs) *For every disjoint sequence $(E_n) \subset \mathscr{R}$ and any choice of elements $x_n \in M(E_n)$ $(n \in N)$, the sequence (x_n) is bounded (equivalently, the set $\bigcup_{n=1}^{\infty} M(E_n)$ is bounded).*

Then M is bounded.

Proof. We can assume that $\mathscr{R}$ is an algebra on S. [Otherwise $\mathscr{R}_0 = \mathscr{R} \cup \{S \setminus E | E \in \mathscr{R}\}$ is an algebra $\supset \mathscr{R}$ and by setting $M(S\setminus E) = -M(E)$ for $E \in \mathscr{R}$, we extend M to a quasi-additive bounded-valued correspondence on $\mathscr{R}_0$, which also satisfies (bs).]

Suppose M is not bounded. Then there exists U such that

$$M^{\vee}(S) \not\subset kU \quad \text{for every } k \in N. \tag{1}$$

Since $M(S)$ is bounded, we can find $k_1 \in N$ such that

$$M(S) \subset k_1 U.$$

In view of (1), there exists E_1 such that

$$M(E_1) \not\subset 2k_1 U.$$

Since by quasi-additivity of M, $M(E_1) \subset \big(M(S) - M(S\setminus E_1)\big)^{-}$, we deduce that

$$M(S\setminus E_1) \not\subset k_1 U.$$

Evidently, either

$$M^{\vee}(E_1) \not\subset kU, \quad \forall k \in N$$

or

$$M^{\vee}(S\setminus E_1) \not\subset kU, \qquad \forall k \in N;$$

let A_1 denote any of the sets E_1, $S\setminus E_1$ for which this is true, and let $B_1 = S\setminus A_1$.

Thus

$$M^{\vee}(A_1) \not\subset kU, \qquad \forall k \in N; \qquad M(A_1) \not\subset k_1 U; \qquad M(B_1) \not\subset k_1 U.$$

Repeating the above reasoning with A_1 in place of S, we find a decomposition

$$A_1 = A_2 \cup B_2, \qquad A_2 \cap B_2 = \emptyset$$

with

$$M^{\vee}(A_2) \not\subset kU, \qquad \forall k \in N; \qquad M(A_2) \not\subset k_2 U; \qquad M(B_2) \not\subset k_2 U,$$

where $k_2 \in N$, $k_2 > k_1$.

Continuing in this manner, we get an infinite disjoint sequence $(B_n) \subset \mathscr{R}$ and a sequence $(k_n) \subset N$ such that

$$M(B_n) \not\subset k_n U$$

which contradicts (bs).

5.3. Corollary. *If M is quasi-additive (in particular additive or strictly additive), bounded-valued and exhaustive, then M is bounded.*

As we shall see in what follows, a large part of the theory of additive bounded-valued correspondences can be reduced to the theory of usual additive set functions. This is based on a construction described below, in which the Minkowski–Radström–Hörmander theorem (2.1) is applied in an essential way (cf. [12], [25], [43]).

Suppose M is additive and bounded-valued. Then the correspondence $M^* = \overline{\mathrm{co}}\, M$ is also additive (3.8) and its values are in $\mathscr{K}(X)$, the family of non-empty closed bounded convex subsets of X. Therefore M^* can be considered as an additive set function taking values in the space $R(X)$ from the Minkowski–Radström–Hörmander theorem. If M is σ-additive, or order-continuous, or semiexhaustive, or exhaustive, then so is, respectively, M^* (3.8). Since for a point-valued additive set function σ-additivity is equivalent to order-continuity and implies semiexhaustivity, it is immediately seen that the following improvement of Proposition 3.3 holds. (Propositions 2.3 and 2.5 can be also applied here.)

5.4. Theorem. *Suppose M is bounded-valued. Then: (M is σ-additive) $\Leftrightarrow$ (M is additive and σ-subadditive) $\Leftrightarrow$ (M is additive and order-continuous) $\Rightarrow$ M is semiexhaustive.*

As an immediate consequence of Corollary 5.3 and Theorem 5.4 we get

5.5. COROLLARY. *If $\mathscr{R}$ is a σ-ring and M is σ-additive* (*in particular strictly σ-additive*) *and bounded-valued, then M is bounded.*

Actually, this result is readily derived from the well-known fact that a vector measure on a σ-ring is bounded (see e.g. [16], 4.12). (Boundedness of M^* in $R(X)$ is equivalent to boundedness of M.)

By a direct application of Theorem 5.2 we establish our next result.

5.6. COROLLARY. *If $\mathscr{R}$ is a σ-ring and M is $s\sigma$-additive and bounded-valued, then M is bounded.*

Proof. Let (E_n) be a disjoint sequence in $\mathscr{R}$, E its union, and let $x_n \in M(E_n)$ $(n \in N)$. Take any $z \in M\big(\bigcup_{n=1}^{\infty} E_n\big)$. Then, by $s\sigma$-additivity of M, $z = \sum_{n=1}^{\infty} y_n$ $(y_n \in M(E_n))$ and $z_m = \sum_{n=m}^{\infty} y_n \in M\big(\bigcup_{n=m}^{\infty} E_n\big)$ for every m. Now, given U, there exist $r, s \in N$ such that $M(E) \subset rU$ and $z_n \in U$ for $n \geqslant s$. Then $\sum_{k=1}^{n} x_k \in M(E) - z_{n+1} \subset (r+1)U$, hence $x_n \in 2(r+1)U$ for all $n > s$. It follows that the sequence (x_n) is bounded and thus condition (bs) in 5.2 is satisfied.

Countable additivity of a correspondence M is usually understood as $s\sigma$-additivity in our sense ([2], [13], [39]), but to obtain some of the deepest results the requirement that M be bounded-valued is frequently imposed (see e.g. [2], [39], [43]). In this case, as Corollary 5.6 shows, M is bounded. It follows that if a sequence $(E_n) \subset \mathscr{R}$ is disjoint and $x_n \in M(E_n)$ $(n \in N)$, then the set

$$\Big\{\sum_{n\in e} x_n \mid e \text{ is a finite subset of } N\Big\}$$

is bounded. Now, there are spaces, called O-spaces, where this property of a sequence (x_n) implies convergence (unconditional) of the series $\sum_{n=1}^{\infty} x_n$ (see [20], [21], [30], and references therein), for example finite dimensional spaces.

Hence, if X is an O-space, our series $\sum_{n=1}^{\infty} x_n$ $(x_n \in M(E_n))$ will *converge* and its sum will belong to $M(\sum_{n=1}^{\infty} E_n)$, by definition of $s\sigma$-additivity. Thus the following theorem holds.

5.7. THEOREM. *If X is an O-space, $\mathscr{R}$ is a σ-ring and M is $s\sigma$-additive and bounded-valued, then M is strictly σ-additive* (*and bounded*).

So far our results in this Section do not cover the case when $M = M_K$ (Example 3.12), that is, none of them implies the Nikodym theorem on uniform boundedness ([10], [18], [22], [41]).

This gap is filled in by

5.8. PROPOSITION. *Suppose $\mathscr{R}$ is a σ-ring, M is quasi-additive, bounded-valued and has property* $(*)$ *or* $(**)$ *from Example* 3.12. *Then M is bounded.*

Proof. Let $x_n \in M(E_n)$ $(n \in N)$, where (E_n) is a disjoint sequence in $\mathscr{R}$. For each n there is, by 3.12 $(**)$ ($\Leftarrow$ 3.12 $(*)$), an additive exhaustive function $\mu_n \colon \mathscr{P}(N) \to X$ such that

$$(+) \qquad \mu_n(e) \in M(\bigcup_{n\in e} E_n), \qquad e \in \mathscr{P}(N)$$

and $\mu_n(\{n\}) = x_n$.

Condition $(+)$ implies boundedness of the set $\{\mu_n(e) \mid n \in N\}$ for every $e \in \mathscr{P}(N)$.

By the Nikodym type theorem ([18], Theorem 2) the set $\{\mu_n(e) \mid n \in N, e \in \mathscr{P}(N)\}$ is bounded, hence so is its subset $\{x_n \mid n \in N\}$.

Thus condition (bs) is 5.2 is satisfied, so that M is bounded.

5.9. DEFINITIONS. Suppose M is subadditive. Members of $\mathscr{N}(M) = \{E \in \mathscr{R} \mid M^{\vee}(E) = 0\}$ are called *M-null.*

A set $A \in \mathscr{R}$ is called an *atom* of M if $M^{\vee}(A) \neq 0$ and for every $B \subset A$ either $M^{\vee}(B) = 0$ or $M^{\vee}(A \setminus B) = 0$. This means that $M^{\vee}(A) \neq 0$ and there exists $X_0 \in \mathscr{A}(X)$ such that for every $B \subset A$

$$M^{\vee}(B) = 0 \qquad \text{or} \qquad \overline{M^{\vee}(B)} = \overline{X_0}$$

or, provided M is quasi-additive,

$$M(B) = 0 \qquad \text{or} \qquad \overline{M}(B) = \overline{X}_0.$$

(If M is strictly subadditive or strictly quasi-additive, respectively, closure can be omitted.)

M is said to be *non-atomic* if it has no atoms. We say that M satisfies *countable chain condition,* (ccc) if every family of mutually disjoint sets $E \in \mathscr{R}$ with $M(E) \neq 0$ is at most countable (cf. [19]).

For example if X is metrizable and $M \colon \mathscr{R} \to \mathscr{A}(X)$ is additive and exhaustive, or if there is a finite measure $\nu \colon \mathscr{R} \to [0, \infty)$ such that $\nu(E) = 0 \Rightarrow M(E) = 0$, then M satisfies (ccc).

An easy application of the Kuratowski–Zorn Principle shows that if $\mathscr{R}$ is a σ-ring and M satisfies (ccc), then there exists a set $S_0 \in \mathscr{R}$ such that $M(E) = 0$ if $E \cap S_0 = \emptyset$. It follows that then we can assume that $\mathscr{R}$ is a σ-algebra on S_0.

We say that M is *bounded* [*unbounded*] *on a set* $B \in \mathscr{R}$ if $M^{\vee}(B)$ is bounded [unbounded].

5.10. THEOREM. *Suppose $\mathscr{R}$ is a σ-ring and M is additive and order-continuous* ($=$ *σ-additive and exhaustive,* 3.3), *in particular strictly σ-additive, and satisfies* (ccc). *Then either M is bounded or there exists a finite number of pairwise disjoint sets $B, A_1, \ldots, A_k$ in $\mathscr{R}$ such that*

(a) *M is bounded on B,*

(b) *$A_1, \ldots, A_k$ are atoms of M on which M is unbounded.*

(c) *M vanishes outside the set $S_0 = B \cup A_1 \cup \ldots \cup A_k$.*

Proof. By the Kuratowski–Zorn Principle there exists a maximal family $\mathscr{B}$ of mutually disjoint sets E such that $M^{\vee}(E)$ is bounded and $\neq 0$. By (ccc), $\mathscr{B}$ is countable, say $\mathscr{B} = \{B_1, B_2, \ldots\}$. Let $B = \bigcup B_n$. Since M is σ-additive (3.8, 3.3), $\overline{M^{\vee}}(B) = (\tilde{\tau})\sum_{n=1}^{\infty} M^{\vee}(B_n)$ is bounded.

By the maximality of $\mathscr{B}$, if $E \cap B = \emptyset$, then either $M^{\vee}(E) = 0$ or $M^{\vee}(E)$ is unbounded.

Similarly as above, there exists a maximal family, $\mathscr{A}$, of mutually disjoint and disjoint from B, atoms of M, and $\mathscr{A} = \{A_1, A_2, \ldots\}$. By exhaustivity of M, the family $\mathscr{A}$ is finite; as we noted above, each $M^{\vee}(A_i)$ is unbounded.

Finally, condition (c) is satisfied, for otherwise non-atomicity of M outside of S_0 would imply the existence of an infinite disjoint sequence (E_n) such that $M^{\vee}(E_n)$ is unbounded $(n \in N)$, thus contradicting exhaustivity of M.

5.11. Corollary. *If $\mathscr{R}$ and M are as in 5.10 and M is non-atomic, then M is bounded.*

Now we give an analogue of the Saks decomposition ([22], IV. 9.7).

5.12. Theorem. *Under the same assumptions as in 5.10, for each U there exists a finite family $\mathscr{F}$ of mutually disjoint sets in $\mathscr{R}$ such that, for each $F \in \mathscr{F}$, either $M^{\vee}(F) \subset U$ or F is an atom of M and $M^{\vee}(F) \not\subset U$, and M vanishes outside of $\bigcup \mathscr{F}$.*

Proof. Let $\mathscr{E}$ be a maximal family of mutually disjoint sets E in $\mathscr{R}$ such that either $M^{\vee}(E) \subset U$ or E is an atom of M and $M^{\vee}(E) \not\subset U$; $\mathscr{E}$ is countable by (ccc). Since $M^{\vee}$ is additive and exhaustive (3.8, 3.3), by 3.2 there exists a finite subfamily $\mathscr{F}_0$ of $\mathscr{E}$ such that $M^{\vee}(\bigcup \mathscr{E}') \subset (\frac{1}{2})U$ for every finite subfamily $\mathscr{E}'$ of $\mathscr{E} \setminus \mathscr{F}_0$. Then, by σ-additivity of $M^{\vee}$, $M^{\vee}(F_0) \subset U$, where $F_0 = \cup(\mathscr{E} \setminus \mathscr{F}_0)$. It follows easily that $\mathscr{F} = \mathscr{F}_0 \cup \{F_0\}$ is as required.

We close this Section with a generalization of a result of Diestel [14] (cf. also [15], [21], [29], [30], [33]).

5.12. Theorem. *Suppose X is separable, $\mathscr{R}$ is a σ-ring and M is additive, compact-valued and bounded. Then M is exhaustive.*

Proof. Once more it is possible to apply the Minkowski–Radström–Hörmander theorem. Let D be a countable dense subset of X. Let C be a compact subset of X. Then for any closed U there exists a finite subset F of D such that

$$C \subset F + U;$$

we can assume that none of the points in F can be removed from F without affecting this inclusion. This implies

$$F \subset C + U.$$

It follows that

$$\mathrm{co}\, F \in \mathscr{E}(\overline{\mathrm{co}\, C}, U).$$

Hence the linear subspace $\mathscr{X}_0$ of the space $R(X)$ from the Minkowski–Radström–Hörmander theorem, spanned by all these $\overline{\mathrm{co}\, C}$'s, is separable.

The function $M^* = \overline{\mathrm{co}\, M}$: $\mathscr{R} \to \mathscr{X}_0$ is additive and bounded, and so it is exhaustive (apply, e.g., [29], Theorem 8). Hence M is exhaustive, too.

6. Convexity. The notions of an atom of M and of non-atomicity of M introduced by Definitions 5.9 are of purely algebraic character. For the purposes of the present section we assume the following

6.1. Definition. Given a topology τ on X, we say that M is *τ-non-atomic* if for each set E in $\mathscr{R}$ and each τ-neighborhood V of 0 in X there exists a finite decomposition $E_1, \ldots, E_r$ of E such that $M^{\check{}}(E_i) \subset V$ for $i = 1, \ldots, r$.

It is trivial that if M is τ-non-atomic and a topology τ_1 is weaker than τ, then M is also τ_1-non-atomic.

In what follows $\mathscr{R}$ will denote a *σ-ring* and M an *additive order-continuous correspondence $\mathscr{R} \to \mathscr{A}(X)$*.

In view of Theorem 5.12, if τ is the original topology of X and M satisfies (ccc), thus particularly if X is metrizable, then M is τ-non-atomic iff M is non-atomic in the sense of Definition 5.9. As we noted above then M will also be $\sigma(X, X')$-non-atomic. Now, it is easily seen that $\sigma(X, X')$-non-atomicity of M is equivalent to (usual) non-atomicity of all correspondences $x'M$ $(x' \in X')$; $x'M(E) = x'[M(E)]$, $E \in \mathscr{R}$. But, even when M is point-valued, usual non-atomicity of M may not imply $\sigma(X, X')$-non-atomicity [42]; the converse implication is trivial.

The following theorem is related to the results of Schmeidler ([39], Theorem 1.2; [40], Theorem 1) and Artstein ([2], Theorem 4.2); cf. also [8], [9].

6.1. Theorem. *If M is $\sigma(X, X')$-non-atomic, then the $\sigma(X, X')$-closure of $M(E)$ is convex for every $E \in \mathscr{R}$.*

Proof. (cf. [42], p. 67). We can assume that X is over R. Let $\xi = (x'_1, \ldots, x'_n)$ be a finite sequence in X'. We can consider ξ as a linear mapping $X \to R^n$, $\xi(x) = \big(x'_1(x), \ldots, x'_n(x)\big)$. Then the correspondence M_ξ: $\mathscr{R} \to \mathscr{A}(R^n)$, defined by equality

$$M_\xi(E) = \big(\xi M(E)\big)^-,$$

is additive and order-continuous (3.9), non-atomic, and compact-valued. Hence it is strictly additive, so also strictly σ-additive, by 3.4. Now we can apply any of the results of Schmeidler or Artstein mentioned above to see that values of M_ξ are convex.

Since

$$\text{the } \sigma(X, X')\text{-closure of } M(E) = \bigcap_{\xi} \xi^{-1}[M_\xi(E)],$$

where ξ varies over all finite sequences in X', the assertion of our theorem follows.

From the theorem we derive easily a generalization of a result of Tweddle [42] (cf. also [8], Théorème 2, and [39], Theorem 1.6, [40], Theorem 2).

6.2. Corollary. *If M is $\sigma(X, X')$-non-atomic, then the $\sigma(X, X')$-closure of the range of M is convex.*

Proof. Since $M^\vee$ is $\sigma(X, X')$-non-atomic, additive and order-continuous (3.8), for each $E \in \mathscr{R}$ the $\sigma(X, X')$-closure $M^\vee(E)^{-\sigma}$ of $M^\vee(E)$ is convex by 6.1.

Since $M^\vee(E_1)^{-\sigma} \cup M^\vee(E_2)^{-\sigma} \subset M^\vee(E_1 \cup E_2)^{-\sigma}$ for any $E_1, E_2 \in \mathscr{R}$, the set

$$A = \bigcup_{E \in \mathscr{R}} M^\vee(E)^{-\sigma}$$

is convex, and so is $A^{-\sigma}$. But $A^{-\sigma}$ is identical with the $\sigma(X, X')$-closure of the range of M.

6.3. Corollary. *If M is $\sigma(X, X')$-non-atomic and its values are weakly closed, in particular weakly compact, then they are convex.*

An open question is whether the result of Schmeidler [(40], Theorem 1) may be derived from Theorem 6.1 (see how Tweddle [42] obtains a result of Uhl).

7. Convergence of correspondences and uniform boundedness principle. In this Section $\mathscr{R}$ denotes a *σ-ring*. The proofs of the two results stated below are immediate if one applies the Minkowski–Radström–Hörmander theorem and indicated results from the theory of vector-valued set functions.

Our first result contains analogues of the Vitali–Hahn–Saks and Nikodym types theorems [17].

7.1. Theorem. *Let $(M_n)_{n \in N}$ be a sequence of σ-additive (here = additive and order-continuous, 5.4) [resp. additive and exhaustive] correspondences $\mathscr{R} \to \mathscr{B}(X)$ such that for each $E \in \mathscr{R}$ the sequence $\big(M_n(E)\big)_{n \in N}$ converges in $\big(\mathscr{A}(X), \tilde{\tau}\big)$; let $M_0(E)$ denote any of the sets in $(\tilde{\tau})\lim\limits_{n\to\infty} M_n(E)$.*

Then $M_0\colon \mathscr{R} \to \mathscr{B}(X)$, M_0 *is* σ*-additive* [*resp. additive and exhaustive*], *and the family* $\{M_n \mid n \geqslant 0\}$ *is uniformly* σ*-additive* (*here* = *uniformly order-continuous*) [*resp. uniformly exhaustive*].

Moreover, if Γ *is an* FN*-topology on* $\mathscr{R}$ *and* $M_n \ll \Gamma$ *for every* $n \in N$, *then also* $M_0 \ll \Gamma$ *and the family* $\{M_n \mid n \geqslant 0\}$ *is* Γ*-equicontinuous.*

The next result is a generalization of the Nikodym type theorem on uniform boundedness of a family of vector measures ([18], [29], Theorem 6; see also [10], [41]).

7.2. THEOREM. *Suppose* $\mathscr{M}$ *is a family of* σ*-additive* (*or additive and exhaustive*) *correspondences* $\mathscr{R} \to \mathscr{B}(X)$ *such that the set*

$$\bigcup \{M(E) \mid M \in \mathscr{M}\}$$

is bounded for every $E \in \mathscr{R}$. *Then the set*

$$\bigcup \{M(E) \mid M \in \mathscr{M},\ E \in \mathscr{R}\}$$

is also bounded.

Remark. On this occasion I would like to note that R. B. Darst was apparently the first who extended Nikodym's uniform boundedness principle to the case of scalar valued bounded finitely additive (hence exhaustive) set functions [10]. I was unaware of this fact while preparing my note [18]. The result of Darst was kindly pointed out to me by Professor J. K. Brooks.

8. Control measures and selections. The topological meaning of a control measure for a vector measure has been widely discussed in [19], and could be repeated now in case of correspondences, but we restrict ourselves to the definition of this notion and two existence theorems.

8.1. DEFINITION. A *control measure* for an additive correspondence M is any additive set function $\nu\colon \mathscr{R} \to [0, \infty]$ such that

$$M \sim \nu,$$

i.e. $\Gamma(M) = \Gamma(\nu)$.

If M is exhaustive [or order-continuous and $\mathscr{R}$ is a σ-ring], then its control measure, if exists, can be chosen to be bounded (hence exhaustive) [and σ-additive] (cf. [19], p. 209).

8.2. THEOREM. *Suppose* $\mathscr{R}$ *is a* σ*-ring and* M *is additive and order-continuous. If* M *satisfies* (ccc), *in particular if* X *is metrizable, then there exists a finite* σ*-additive control measure for* M.

(Cf. [43], assertion (2) of Théorème 23.)

Proof. In view of Theorem 5.10, without loss of generality we can assume that M is bounded. Then $M^* = \overline{\mathrm{co}}\, M\colon \mathscr{R} \to R(X)$ is a σ-additive vector measure which satisfies (ccc). By theorem 2.3 in [19] (see also [35]), there exists a finite σ-additive control measure ν for M^*. It is obvious that ν is a required control measure for M.

8.3. Corollary. *If X is metrizable and $M: \mathscr{R}$ (a ring) $\to \mathscr{A}(X)$ is additive and exhaustive, then there exists a bounded additive control measure for M.*

Proof. We can assume that X is complete and M is closed-valued. Then Theorem 4.5, used via the Stone representation theorem, similarly as in [19], Section 4, reduces our assertion to the order-continuous case, so that the preceding theorem can be applied.

8.4. Definition. A (point-valued) set function μ: $\mathscr{R} \to X$ is called a *selection* for M, if $\mu(E) \in M(E)$ for every E in $\mathscr{R}$.

Of course, if M is additive [σ-additive], we are especially interested in the existence of additive [σ-additive] selections for M.

If A is a subset of X, $\operatorname{Ext} A$ will denote the set of extreme points of A.

8.5. Lemma. *Suppose C_0, C_1, C_2 are convex subsets of X and $C_0 = C_1 + C_2$. Then if $x_0 \in \operatorname{Ext} C_0$, there exists exactly one pair of points $x_1 \in C_1$, $x_2 \in C_2$ such that $x_0 = x_1 + x_2$; these points are extreme points of C_1 and C_2 respectively.* (Cf. [36].)

8.6. Theorem. *Suppose $\mathscr{R}$ is an algebra of subsets of S and M is strictly additive and convex-valued. Then for each $x \in \operatorname{Ext} M(S)$ there exists a unique additive selection μ_x for M such that*

$$\mu_x(S) = x.$$

In addition, the selection μ_x has the property

$$\mu_x(E) \in \operatorname{Ext} M(E), \qquad E \in \mathscr{R}.$$

Moreover, if M is order-continuous [exhaustive], then μ_x is σ-additive [exhaustive].

Proof. For each $E \in \mathscr{R}$ let $\mu_x(E)$ denote the uniquely determined member x_1 of $M(E)$ such that for some $x_2 \in M(S \setminus E)$, $x = x_1 + x_2$; $\mu_x(E) \in \operatorname{Ext} M(E)$ (8.5). We claim that the selection μ_x for M thus defined is additive.

Let $E, F \in \mathscr{R}$, $E \cap F = \emptyset$. Then there exist elements $y_1 \in M(E)$, $y_2 \in M(F)$ and $y_3 \in M\big(S \setminus (E \cup F)\big)$ such that $x = y_1 + y_2 + y_3$. Since $y_1 \in M(E)$ and $y_2 + y_3 \in M(S \setminus E)$, we have $y_1 = \mu_x(E)$. Similarly $y_2 = \mu_x(F)$ and $y_1 + y_2 = \mu_x(E \cup F)$. Hence

$$\mu_x(E) + \mu_x(F) = \mu_x(E \cup F).$$

The last assertion of the theorem is obvious.

8.7. Corollary. *If $\mathscr{R}$ is an algebra of subsets of S and M is (strictly) additive weakly compact valued, then M admits at least one additive selection μ. The selection μ is σ-additive [exhaustive] if M is.*

Proof. With no loss of generality we can assume that X is complete. Then the values of the correspondence $\overline{\mathrm{co}}\, M$ are weakly compact convex subsets of X ([31], 20.6(3)), it is strictly additive (3.8) and $\mathrm{Ext}\,\overline{\mathrm{co}}\, M(E) \subset M(E)$ for every $E \in \mathscr{R}$ ([31], 25.1 (7)). We apply the preceding theorem.

We shall need the following result due to Husain and Tweddle ([27]).

8.8. Lemma. *If C_0, C_1, C_2 are compact convex subsets of X with $C_0 = C_1 + C_2$, then $\{x_1 \in \mathrm{Ext}\, C_1 \mid$ there exists $y \in \mathrm{Ext}\, C_2$ such that $x+y \in \mathrm{Ext}\, C_0\}$ is a dense subset of $\mathrm{Ext}\, C_1$.*

From Lemmas 8.5 and 8.8 we derive immediately the following

8.9. Corollary. *If $\mathscr{R}$ is an algebra of subsets of S and M is a (strictly) additive correspondence whose values are compact convex subsets of X, then the correspondence $E \to \mathrm{Ext}\, M(E)$, $E \in \mathscr{R}$, is quasi-additive.*

The next result is an easy consequence of Theorem 8.6, Lemma 8.8 and the Krein–Milman theorem.

8.10. Corollary. *Suppose $\mathscr{R}$, M are as above, with "compact" replaced by "weakly sompast". For each $x \in \mathrm{Ext}\, M(S)$ let μ_x be the selection for M established in Theorem 8.6. Then, for every $E \in \mathscr{R}$,*

$$M(E) = \overline{\mathrm{co}}\, \{\mu_x(E) \mid x \in \mathrm{Ext}\ M(S)\}.$$

8.11. Corollary. *If $\mathscr{R}$ is a σ-ring, M is weakly compact valued, σ-additive and satisfies* (ccc), *then*

$$(+) \qquad M(E) = \{\mu(E) \mid \mu \text{ is an additive selection for } M\}^-$$

for every $E \in \mathscr{R}$.

Proof. As noted in Section 5, (ccc) enables us to assume that $\mathscr{R}$ is a σ-algebra on a set S. Let $\{A_1, A_2, \ldots\}$ be a maximal, countable by (ccc), family of mutually disjoint atoms of M. Denote $S_a = \bigcup A_n$, $S_b = S \setminus S_a$. M is non-atomic on S_b, hence its values on subsets of S_b are convex (6.1). If μ_1 is a selection for M on S_b and $x_n \in M(A_n)$ ($n \in N$), then the set function $\mu \colon \mathscr{R} \to X$ defined by

$$\mu(E) = \mu_1(E \cap S_b) + \sum \{x_n \mid E \cap A_n \notin \mathscr{N}(M)\}$$

is a selection for μ on $\mathscr{R}$. In view of 8.10, the family of all μ which can be obtained in this way is large enough to guarantee (+). (Evidently, this family contains all selections for M.)

Remark. Our results on selections generalize those given in [24], where X was required to satisfy some extra condition (P). On the other hand, they are certainly not so deep as the results of Artstein ([2], Section 8).

8.12. Theorem. *If $\mathscr{R}$, M are as in Corollary 8.11, then there exists a σ-additive selection λ for M such that $\lambda \sim M$* (i.e. $\Gamma(\lambda) = \Gamma(M)$).

Proof. We start with the following observation: If $E \in \mathscr{R} \setminus \mathscr{N}(M)$, then there exists $F \subset E$, $F \in \mathscr{R} \setminus \mathscr{N}(M)$, and a σ-additive selection μ_F of $M_F = M|\mathscr{R}_F$, where $\mathscr{R}_F = \{A \in \mathscr{R} | A \subset F\}$, such that $\mathscr{N}(M_F) = \mathscr{N}(\mu_F)$.

Indeed, we can assume $M(E) \neq 0$, and then by 8.11 there exists a σ-additive selection μ for M_E such that $\mu(E) \neq 0$.

Let $\mathscr{A}$ be a maximal family of mutually disjoint sets A in $\mathscr{R}_E$ such that $\mu(A) = 0$ and $A \notin \mathscr{N}(M)$. By (ccc), $\mathscr{A}$ is at most countable, hence its union A_0 is in $\mathscr{R}_E$ and $\mu(A_0) = 0$. Then $F = E \setminus A_0$ and $\mu_F = \mu|\mathscr{R}_F$ are as required in our assertion. (In addition, $A \in \mathscr{N}(\mu_F)$ iff $\mu_F(A) = 0$.)

Now, there exists a maximal family $\mathscr{F}$ of mutually disjoint sets in $\mathscr{R} \setminus \mathscr{N}(M)$ with the property that for each $F \in \mathscr{F}$ there is a σ-additive selection μ_F for M_F such that $\mathscr{N}(M_F) = \mathscr{N}(\mu_F)$. $\mathscr{F}$ is at most countable, so that we can write $\mathscr{F} = \{F_1, F_2, \ldots\}$ $(F_i \cap F_j = \emptyset$ if $i \neq j)$. The maximality of $\mathscr{F}$ and the observation made above imply that M vanishes outside of $\bigcup \mathscr{F}$. It follows easily that the σ-additive set function $\lambda: \mathscr{R} \to X$ defined by $\lambda(E) = \sum_n \mu_{F_n}(E \cap F_n)$ is a selection for M which satisfies $\mathscr{N}(M) = \mathscr{N}(\lambda)$.

Since both M and λ satisfy (ccc), each of them has a finite σ-additive control measure (8.2; [19], 2.3). Hence $\mathscr{N}(M) = \mathscr{N}(\lambda)$ is equivalent to $\Gamma(M) = \Gamma(\lambda)$ ([19], 1.1).

By a theorem of Rybakov ([38]; see also [1], [16], [19], [44] for some improvements and generalizations), if λ is a σ-additive measure defined on a σ-ring $\mathscr{R}$ and taking values in a normed space X, then there is $x_0' \in X'$ such that $\lambda \sim x_0' \lambda$.

This, and Theorem 8.12, imply immediately the following analogue of the Rybakov's result for correspondences.

8.13. COROLLARY. *If X is a normed space, $\mathscr{R}$ a σ-ring, and M is σ-additive and weakly compact valued, then there exists $x_0' \in X'$ such that $M \sim x_0' M$.*

Postscript. After this paper was submitted for publication, the author learned that Pallu de la Barrière [3] obtained much stronger results on selections; he also proved a version of Corollary 6.3. The reader is also referred to some recent works by A. Costé for related results, and to the Thesis of Godet-Thobie [23] for an excellent exposition of the theory of multimeasures.

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