ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXII (1980)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXII (1980)

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# Countable codimensional subspaces of semiconvex spaces

**Abstract.** We obtain for semiconvex spaces analogues of three known results on the inheritance of certain barrelledness and bornological properties of locally convex spaces by subspaces of countable codimension.

**1. Introduction.** Following the papers of Saxon and Levin [13] and Valdivia [16] showing that a countable codimensional subspace of a barrelled space is again barrelled, there has been considerable interest in determining other properties of locally convex spaces which are similarly inherited. We refer the reader to [6], [11], [12], [16] and [17] for examples, noting in particular that Webb [17] has shown that countable barrelledness [5] is one such property (see also [6]).

Adasch und Ernst [2] and deWilde et Gerard-Houet [3] have also considered similar properties for certain topological vector spaces. The results of Iyahen [9] and Kadelburg [10] may also be relevant to the reader in this connection. Here we are concerned with this problem in semiconvex spaces [7]. In [7], Iyahen introduced corresponding notions of hyperbarrelledness and $\aleph_0$-hyperbarrelledness in such spaces. We show that hyperbarrelledness is inherited by subspaces of countable codimension; for the analogue of Webb's result we are led to a definition which is apparently rather stronger than Iyahen's $\aleph_0$-hyperbarrelledness. Valdivia showed in [16] that a countable codimensional subspace of an ultrabornological space (in the locally convex sense) is bornological. Our final result is a semiconvex version of this.

Our methods are extensions of the techniques developed by Valdivia in [15] and [16] and by Saxon and Levin in [13].

**2. Countable hyperbarrelledness.** In [7], Iyahen gives the following definitions.

* The first author's research was partly supported by an NRC grant. The second author acknowledges a travel grant under the Commonwealth University Interchange Scheme. AMS Subject Classification: 46A15.

(i) A semiconvex space $(E, \xi)$ is *hyperbarrelled* if each $\xi$-closed balanced semiconvex absorbent subset of $E$ is an $\xi$-neighbourhood of 0.

(ii) A semiconvex space $(E, \xi)$ is $\aleph_0$-*hyperbarrelled* if $V = \bigcap_{n=1}^{\infty} U_n$ is an $\xi$-neighbourhood of 0 whenever it is absorbent and there is $\lambda > 0$ such that each $U_n$ is a closed balanced $\lambda$-convex neighbourhood of 0.

Since a semiconvex space need not have a base of neighbourhoods of 0 consisting of sets which are $\lambda$-convex for some fixed $\lambda$ ([14], p. 179), it seems natural to modify (ii) by allowing $\lambda$ to vary with $n$. We do this by adapting Iyahen's idea of an *ultrabarrel of type* ($\alpha$) [8] to the semiconvex setting.

(iii) A *semiconvex ultrabarrel of type* ($\alpha$) in a (semiconvex) space $(E, \xi)$ is a system $\{U_n^{(k)}: n, k \in N\}$ of closed balanced semiconvex $\xi$-neighbourhoods of 0 such that $U_n^{(k+1)} + U_n^{(k+1)} \subseteq U_n^{(k)}$ $(n, k \in N)$ and $\bigcap_{n=1}^{\infty} U_n^{(k)}$ is a semiconvex absorbent set $(k \in N)$.

(iv) A semiconvex space $(E, \xi)$ is countably *hyperbarrelled* if, whenever $\{U_n^{(k)}: n, k \in N\}$ is a semiconvex ultrabarrel of type ($\alpha$), then $\bigcap_{n=1}^{\infty} U_n^{(k)}$ is an $\xi$-neighbourhood of 0 $(k \in N)$.

It is clear that a hyperbarrelled space is countably hyperbarrelled and that a countably hyperbarrelled space is $\aleph_0$-hyperbarrelled. Iyahen has shown in [8] that the strong dual of a metrizable locally convex space is countably ultrabarrelled and consequently it is countably hyperbarrelled. He also notes in [7] that such a space need not be (quasi-) hyperbarrelled. However,

THEOREM 1. *A separable countably hyperbarrelled space is hyperbarrelled.*

Proof. In a separable countably hyperbarrelled space $(E, \xi)$ let $\{x_m: m \in N\}$ be a dense subset, $\mathcal{U}$ be a base of balanced semiconvex neighbourhoods of 0, and $B$ be a closed balanced semiconvex absorbent set.

We can find a sequence $(U_n)$ in $\mathcal{U}$ such that $U_{n+1} + U_{n+1} \subseteq U_n$ $(n \in N)$ and

$$(*) \qquad \{x_m: m \in N\} \backslash B = \{x_m: m \in N\} \backslash \bigcap_{n=1}^{\infty} (B + U_n).$$

Choose $\beta \geq 1$ such that $B$ is $\beta$-convex and let

$$V_n^{(k)} = \mathrm{cl}\, \{\beta^{1-k} B + U_{n+k-1}\} \quad (n, k \in N).$$

Since $V_n^{(k)} \subseteq \beta^{1-k} B + U_{n+k-1} + U_{n+k-1} \subseteq \beta^{1-k} B + U_{n+k-2}$ $(n = 2, 3, \ldots, k \in N)$, it follows that $\bigcap_{n=1}^{\infty} V_n^{(k)} = \bigcap_{n=1}^{\infty} (\beta^{1-k} B + U_{n+k-1})$ for each $k \in N$. Since

$\bigcap_{n=1}^{\infty} V_n^{(k)} \supseteq \beta^{1-k}B$, it is absorbent $(k \in N)$; it is also $\beta$-convex $(k \in N)$ for

$$\begin{aligned}\bigcap_{n=1}^{\infty} V_n^{(k)} + \bigcap_{n=1}^{\infty} V_n^{(k)} &\subseteq \bigcap_{n=2}^{\infty} (\beta^{1-k}B + U_{n+k-1} + \beta^{1-k}B + U_{n+k-1}) \\ &\subseteq \bigcap_{n=2}^{\infty} (\beta\beta^{1-k}B + U_{n+k-2}) \subseteq \bigcap_{n=2}^{\infty} (\beta\beta^{1-k}B + \beta U_{n+k-2}) \\ &= \beta \bigcap_{n=1}^{\infty} V_n^{(k)}.\end{aligned}$$

It is now clear that $\{V_n^{(k)}\colon n, k \in N\}$ is a semiconvex ultrabarrel of type $(\alpha)$ in $(E, \xi)$ and consequently $\bigcap_{n=1}^{\infty} V_n^{(1)} = \bigcap_{n=1}^{\infty} (B + U_n)$ is an $\xi$-neighbourhood of 0. Finally

$$\begin{aligned}\operatorname{int} \bigcap_{n=1}^{\infty} (B + U_n) &\subseteq \operatorname{cl}\Big\{\Big(\operatorname{int} \bigcap_{n=1}^{\infty} (B + U_n)\Big) \cap \{x_m\colon m \in N\}\Big\} \\ &\subseteq \operatorname{cl}\big\{B \cap \{x_m\colon m \in N\}\big\} \qquad (\text{by } (*)) \\ &\subseteq B,\end{aligned}$$

which shows that $B$ is an $\xi$-neighbourhood of 0.

We have attempted above to justify considering countable hyperbarrelledness rather than $\aleph_0$-hyperbarrelledness in general. However, the two definitions coincide in the important special case below.

THEOREM 2. *Let $(E, \xi)$ be a semiconvex space in which there is a base of balanced neighbourhoods of 0 consisting of sets which are $\lambda$-convex for some fixed $\lambda > 0$. Then $(E, \xi)$ is countably hyperbarrelled if and only if it is $\aleph_0$-hyperbarrelled.*

Proof. We have already noted that countable hyperbarrelledness implies $\aleph_0$-hyperbarrelledness. Suppose that $(E, \xi)$ is $\aleph_0$-hyperbarrelled and let $\mathscr{U}$ be a base of balanced $\lambda$-convex $\xi$-neighbourhoods of 0. If $\{U_n^{(k)}\colon n, k \in N\}$ is a semiconvex ultrabarrel of type $(\alpha)$ in $(E, \xi)$, we can find $\beta_k > 0$ such that $\bigcap_{n=1}^{\infty} U_n^{(k)}$ is $\beta_k$-convex $(k \in N)$ and $V_n^{(k)} \in \mathscr{U}$ such that $V_n^{(k)} \subseteq U_n^{(k)}$ $(n, k \in N)$. Then $W_n^{(k)} = \operatorname{cl}\{V_n^{(k+2)} + \bigcap_{m=1}^{\infty} U_m^{(k+1)}\}$ is a $\max(\lambda, \beta_{k+1})$-convex $\xi$-neighbourhood of 0 $(n, k \in N)$ and $\bigcap_{n=1}^{\infty} W_n^{(k)}$ is absorbent $(k \in N)$. By hypothesis $\bigcap_{n=1}^{\infty} W_n^{(k)}$ is an $\xi$-neighbourhood of 0 $(k \in N)$. But

$$\bigcap_{n=1}^{\infty} W_n^{(k)} \subseteq \bigcap_{n=1}^{\infty} (U_n^{(k+2)} + U_n^{(k+2)} + U_n^{(k+1)}) \subseteq \bigcap_{n=1}^{\infty} U_n^{(k)} \qquad (k \in N).$$

This completes the proof.

A simple extension of the proof of [7], Theorem 4.2, establishes

THEOREM 3. *Any sc-inductive limit of countably hyperbarrelled spaces is countably hyperbarrelled.*

Combining the methods of [1] and [7], we have a second permanence property.

THEOREM 4. *Any product of countably hyperbarrelled spaces is countably hyperbarrelled.*

Proof. Let $(E_\gamma)_{\gamma\in\Gamma}$ be a non-empty family of countably hyperbarrelled spaces over the same scalar field. Let $\{U_n^{(k)}\colon k\in N\}$ be a semiconvex ultrabarrel of type $(\alpha)$ in $\Pi\{E_\gamma\colon \gamma\in\Gamma\}$ and fix $h\in N$. By [7], Lemma 4.1, there is a finite subset $\Gamma_0$ of $\Gamma$ such that $\Pi\{E_\gamma\colon \gamma\in\Gamma\backslash\Gamma_0\}\subseteq\bigcap_{n=1}^{\infty}U_n^{(h+1)}$. We suppose that $\Gamma_0$ and $\Gamma\backslash\Gamma_0$ are non-empty, the proof being easily modified in the remaining cases. Let $V_n^{(k)}=U_n^{(k+h)}\cap\Pi\{E_\gamma\colon \gamma\in\Gamma_0\}$ $(n,k\in N)$. Clearly $\{V_n^{(k)}\colon n,k\in N\}$ is a semiconvex ultrabarrel of type $(\alpha)$ in $\Pi\{E_\gamma\colon \gamma\in\Gamma_0\}$. By Theorem 3, $\Pi\{E_\gamma\colon \gamma\in\Gamma_0\}$ is countably hyperbarrelled so that $\bigcap_{n=1}^{\infty}V_n^{(k)}$ is a neighbourhood of 0 in $\Pi\{E_\gamma\colon \gamma\in\Gamma_0\}$ $(k\in N)$. Then $\bigcap_{n=1}^{\infty}V_n^{(k)}+\Pi\{E_\gamma\colon \gamma\in\Gamma\backslash\Gamma_0\}$ is a neighbourhood of 0 in $\Pi\{E_\gamma\colon \gamma\in\Gamma\}$ $(k\in N)$. Since

$$\bigcap_{n=1}^{\infty}V_n^{(1)}+\Pi\{E_\gamma\colon \gamma\in\Gamma\backslash\Gamma_0\}\subseteq\bigcap_{n=1}^{\infty}U_n^{(h+1)}+\bigcap_{n=1}^{\infty}U_n^{(h+1)}\subseteq\bigcap_{n=1}^{\infty}U_n^{(h)},$$

it follows that $\bigcap_{n=1}^{\infty}U_n^{(h)}$ is a neighbourhood of 0 in $\Pi\{E_\gamma\colon \gamma\in\Gamma\}$. The result now follows since $h\in N$ is arbitrary.

**3. Subspaces of countable codimension.** The following two lemmas are fundamental for the proofs of the results referred to in § 1 (cf. [13], § 2 Proposition, § 3 Lemma; [15], Lemma 1).

LEMMA 1. *Let $(E,\xi)$ be a countably hyperbarrelled space and let $C$ be a closed balanced semiconvex subset of $(E,\xi)$ whose span $F$ has at most countable codimension in $E$. Then $F$ is closed in $(E,\xi)$.*

Proof. We give the proof for the infinite codimensional case; the finite case involves only a notational change. Let $e_1, e_2, \ldots$ be a basis for a supplement of $F$ in $E$. For each $n\in N$ choose $\delta_n>0$ and let $J_n$ be the set of scalars of modulus at most $\delta_n$. Let $x_0\in\operatorname{cl}F$ and let $\mathscr{F}$ be any filter in $F$ converging to $x_0$. Choose a base $\mathscr{U}$ of balanced semiconvex neighbourhoods of 0 in $E$ and let $\mathscr{G}$ be the filter in $E$ with base $\{X+U\colon X\in\mathscr{F},\ U\in\mathscr{U}\}$. $\mathscr{G}$ also converges to $x_0$. Suppose $\mathscr{G}$ does not induce a filter on $D_n=nC+$

$+\sum_{r=1}^{n} J_r e_r$ for any $n \in N$. Then there are sequences $(X_n)$ in $\mathscr{F}$, $(U_n)$ in $\mathscr{U}$ such that for each $n \in N$,

(i) $(x_n + U_n) \cap D_n = \emptyset$, (ii) $U_{n+1} + U_{n+1} + U_{n+1} \subseteq U_n$.

Choose $\beta > 0$ such that $C$ is $\beta$-convex and for $n, k \in N$ let $W_n^{(k)} = \mathrm{cl}\,\{n\beta^{-k} C + (\sum_{r=1}^{n} U_{r+k} \cap 2^{-k} J_r e_r) + U_{n+k}\}$. Then:

(a) each $W_n^{(k)}$ is a closed balanced semiconvex neighbourhood of 0 in $(E, \xi)$;

(b) $W_n^{(k+1)} + W_n^{(k+1)} \subseteq W_n^{(k)}$ $(n, k \in N)$;

(c) $\bigcap_{n=1}^{\infty} W_n^{(k)} \supseteq \beta^{-k} C + \bigcup_{m=1}^{\infty} (\sum_{r=1}^{m} U_{r+k} \cap 2^{-k} J_r e_r)$ which is absorbent $(k \in N)$.

We show:

(d) $\bigcap_{n=1}^{\infty} W_n^{(k)}$ is semiconvex $(k \in N)$.

Let $x, y \in \bigcap_{n=1}^{\infty} W_n^{(k)}$. Then, for each $n \in N$,

$$x, y \in \mathrm{cl}\,\{n\beta^{-k} C + (\sum_{r=1}^{n} U_{r+k} \cap 2^{-k} J_r e_r) + U_{n+k}\}$$

and so for $n \geqslant 2$,

$$x + y \in \mathrm{cl}\,\{n\beta^{-k} C + (\sum_{r=1}^{n} U_{r+k} \cap 2^{-k} J_r e_r) + U_{n+k} + n\beta^{-k} C + (\sum_{r=1}^{n} U_{r+k} \cap 2^{-k} J_r e_r) + U_{n+k}\}$$

$$\subseteq \mathrm{cl}\,\{\beta n\beta^{-k} C + 2(\sum_{r=1}^{n-1} U_{r+k} \cap 2^{-k} J_r e_r) + 2(U_{n+k} \cap 2^{-k} J_n e_n) + U_{n+k} + U_{n+k}\}$$

(note that $U_{r+k} \cap 2^{-k} J_r e_r$ is actually convex since $U_{r+k}$ is balanced)

$$\subseteq \mathrm{cl}\left\{\left(\frac{\beta n}{n-1}\right)(n-1)\beta^{-k} C + 2(\sum_{r=1}^{n-1} U_{r+k} \cap 2^{-k} J_r e_r) + 2(U_{n+k} + U_{n+k} + U_{n+k})\right\}$$

$$\subseteq \mathrm{cl}\,\{2\beta(n-1)\beta^{-k} C + 2(\sum_{r=1}^{n-1} U_{r+k} \cap 2^{-k} J_r e_r) + 2U_{n+k-1}\} \quad \text{(by (ii))}$$

$$\subseteq \max(2\beta, 2)\,\mathrm{cl}\,\{(n-1)\beta^{-k} C + (\sum_{r=1}^{n-1} U_{r+k} \cap 2^{-k} J_r e_r) + U_{n+k-1}\}$$

$$= \max(2\beta, 2)\, W_{n-1}^{(k)}.$$

Consequently

$$\bigcap_{n=1}^{\infty} W_n^{(k)} + \bigcap_{n=1}^{\infty} W_n^{(k)} \subseteq \max(2\beta, 2) \bigcap_{n=2}^{\infty} W_{n-1}^{(k)} = \max(2\beta, 2) \bigcap_{n=1}^{\infty} W_n^{(k)}.$$

Since $(E, \xi)$ is countably hyperbarrelled we now deduce that $\bigcap_{n=1}^{\infty} W_n^{(k)}$ is an $\xi$-neighbourhood of 0 $(k \in N)$. Choose $X \in \mathscr{F}$ such that $X - X \subseteq \bigcap_{n=1}^{\infty} W_n^{(2)}$ and for each $n \in N$, choose $Y_n \in \mathscr{F}$ such that $Y_n - Y_n \subseteq W_n^{(2)}$. Note that

$$W_n^{(2)} + W_n^{(2)} \subseteq \mathrm{cl}\,\{n\beta^{-1} C + (\sum_{r=1}^{n} 2^{-1} J_r e_r) + U_{n+1}\}$$
$$\subseteq n\beta^{-1} C + (\sum_{r=1}^{n} 2^{-1} J_r e_r) + U_n \quad (n \in N).$$

Let $z \in Y_n + W_n^{(2)}$, $x \in X_n \cap Y_n$. Then for some $y \in Y_n$, $u \in W_n^{(2)}$ we have

$$z = y + u = x + (y - x) + u \in X_n + W_n^{(2)} + W_n^{(2)}$$
$$\subseteq X_n + n\beta^{-1} C + (\sum_{r=1}^{n} 2^{-1} J_r e_r) + U_n.$$

Consequently by (i), $z \notin n\beta^{-1} C + \sum_{r=1}^{n} 2^{-1} J_r e_r$ so that

$$(*) \qquad (Y_n + W_n^{(2)}) \cap (n\beta^{-1} C + \sum_{r=1}^{n} 2^{-1} J_r e_r) = \emptyset \quad (n \in N).$$

Suppose $w \in X$. Then we can find $n_0$ such that $w \in n_0 \beta^{-1} C$. For any $y \in Y_{n_0}$ we must have $w - y \notin W_{n_0}^{(2)}$ by $(*)$, which implies that $w - y \notin \bigcap_{n=1}^{\infty} W_n^{(2)}$. Since $X - X \subseteq \bigcap_{n=1}^{\infty} W_n^{(2)}$ we now deduce that $y \notin X$. This shows that $X \cap Y_{n_0} = \emptyset$ which is impossible since $X, Y_{n_0} \in \mathscr{F}$. Consequently $\mathscr{G}$ must induce a filter on $D_{n_1}$ say. Now $D_{n_1}$ is closed, being the sum of a closed set and a compact set, and so $x_0 \in D_{n_1}$. This shows that the closure of $F$ is contained in $F + \bigcup_{n=1}^{\infty} \sum_{r=1}^{n} J_r e_r$ for *any* choice of the $J_n$. Since the intersection of all these sets is $F$, we deduce that $F$ is closed.

Our other main tool is

LEMMA 2. *Let $F$ be a closed subspace of at most countable codimension in a countably hyperbarrelled space $(E, \xi)$ and let $G$ be any algebraic supplement of $F$ in $E$. Then $\xi$ induces the finest linear topology on $G$ and $(E, \xi)$ is the topological direct sum of $(F, \xi|_F)$ and $(G, \xi|_G)$.*

Proof. This is standard if the codimension of $F$ is finite ([4], Chapter 1,

§ 12, Corollary 3). Otherwise let $e_1, e_2, \dots$ be a basis of a supplement $G$ of $F$ in E. A base of neighbourhoods of 0 for the finest linear topology $\eta$ on $G$ is given by all sets of the form $\bigcup_{n=1}^{\infty} \sum_{r=1}^{n} J_r e_r$, where $J_n$ is the set of scalars of modulus at most $\delta_n$ for some arbitrary positive $\delta_n$ $(n \in N)$. Let $\mathscr{U}$ be a base of closed balanced semiconvex neighbourhoods of 0 for $\xi$. A base of neighbourhoods of 0 for the direct sum topology defined by $\xi|_F$ and $\eta$ is given by all sets of the form $U \cap F + \bigcup_{n=1}^{\infty} \sum_{r=1}^{n} J_r e_r$, where $U \in \mathscr{U}$ and the $J_n$ are as above. Choose any such set and choose $\beta > 0$ such that $U$ is $\beta$-convex.

Since $F$ is closed in $(E, \xi)$, it is closed in $E_n = F + L(e_1, e_2, \dots, e_n)$ $(n \in N)$ with the topology induced by $\xi$. ($L(e_1, e_2, \dots, e_n)$ denotes the linear span of $e_1, e_2, \dots, e_n$ $(n \in N)$.) Consequently $\xi|_{E_n}$ is the direct sum topology of $\xi|_F$ and the usual topology on $L(e_1, e_2, \dots, e_n)$. We can therefore find $U_n \in \mathscr{U}$ such that $E_n \cap U_n \subseteq \beta^{-1} U \cap F + \sum_{r=1}^{n} 2^{-1} J_r e_r$ $(n \in N)$. We may further assume that $U_{n+1} + U_{n+1} + U_{n+1} \subseteq U_n$ $(n \in N)$. Put $W_n^{(k)} = \operatorname{cl} \{\beta^{-k} U \cap \cap F + (\sum_{r=1}^{n} U_{r+k} \cap 2^{-k} J_r e_r) + U_{n+k}\}$ $(n, k \in N)$. It follows as in the proof of Lemma 1 that $\bigcap_{n=1}^{\infty} W_n^{(k)}$ is an $\xi$-neighbourhood of 0 $(k \in N)$. Now

$$\bigcap_{n=1}^{\infty} W_n^{(1)} \subseteq \bigcap_{n=1}^{\infty} \{\beta^{-1} U \cap F + (\sum_{r=1}^{n} U_{r+1} \cap 2^{-1} J_r e_r) + U_n\}.$$

Let $x \in \bigcap_{n=1}^{\infty} W_n^{(1)}$ and choose $m$ such that $x \in E_m$. Then since $x \in \beta^{-1} U \cap F + (\sum_{r=1}^{m} U_{r+1} \cap 2^{-1} J_r e_r) + U_m$, we have by linear independence

$$\begin{aligned} x &\in \beta^{-1} U \cap F + (\sum_{r=1}^{m} U_{r+1} \cap 2^{-1} J_r e_r) + U_m \cap E_m \\ &\subseteq \beta^{-1} U \cap F + (\sum_{r=1}^{m} U_{r+1} \cap 2^{-1} J_r e_r) + \beta^{-1} U \cap F + \sum_{r=1}^{m} 2^{-1} J_r e_r \\ &\subseteq U \cap F + \sum_{r=1}^{m} J_r e_r. \end{aligned}$$

Thus

$$\bigcap_{n=1}^{\infty} W_n^{(1)} \subseteq U \cap F + \bigcup_{n=1}^{\infty} \sum_{r=1}^{n} J_r e_r$$

which is therefore an $\xi$-neighbourhood of 0. Since $\xi$ is necessarily coarser than the direct sum topology, the proof is complete.

We now apply Lemmas 1 and 2 to obtain an analogue of Webb's result concerning countably barrelled spaces ([17], Theorem 6).

THEOREM 5. *Let $(E, \xi)$ be a countably hyperbarrelled space and let $F$ be a subspace of at most countable codimension. Then $(F, \xi|_F)$ is countably hyperbarrelled.*

Proof. Let $\{U_n^{(k)}: n, k \in N\}$ be a semiconvex ultrabarrel of type $(\alpha)$ in $(F, \xi|_F)$. Taking closures in $(E, \xi)$, we deduce from Lemma 1 that all the sets $\mathrm{cl} \bigcap_{n=1}^{\infty} U_n^{(k)}$ $(k \in N)$ have the same linear span $H$, which is a closed subspace of $(E, \xi)$. If $\mathscr{U}$ is a base of closed balanced semiconvex neighbourhoods of 0 in $(E, \xi)$ we can choose $V_n^{(k)} \in \mathscr{U}$ such that for all $n, k \in N$, $H \cap V_n^{(k)} \subseteq \mathrm{cl}\, U_n^{(k)}$, $V_{n+1}^{(k)} + V_{n+1}^{(k)} \subseteq V_n^{(k)}$, $V_n^{(k+1)} + V_n^{(k+1)} \subseteq V_n^{(k)}$. Let $W_n^{(k)} = \mathrm{cl} \{(\bigcap_{m=r}^{\infty} U_m^{(k)}) + H \cap V_n^{(k+1)}\}$ $(n, k \in N)$. We see as before that $\{W_n^{(k)}: n, k \in N\}$ is a semiconvex ultrabarrel of type $(\alpha)$ in $(H, \xi|_H)$ and further that $\bigcap_{n=1}^{\infty} W_n^{(k+1)} \subseteq \bigcap_{n=1}^{\infty} \mathrm{cl}\, U_n^{(k)}$ $(k \in N)$.

Let $G$ be a supplement of $H$ in $E$. It follows from Lemma 2 that if $B_n^{(k)} = W_n^{(k)} + G$ $(n, k \in N)$, then $\{B_n^{(k)}: n, k \in N\}$ is a semiconvex ultrabarrel of type $(\alpha)$ in $(E, \xi)$. For each $k \in N$ we have $F \cap \bigcap_{n=1}^{\infty} B_n^{(k+1)} = F \cap \bigcap_{n=1}^{\infty} W_n^{(k+1)}$ $\subseteq F \cap \bigcap_{n=1}^{\infty} \mathrm{cl}\, U_n^{(k)} = \bigcap_{n=1}^{\infty} U_n^{(k)}$ and consequently $\bigcap_{n=1}^{\infty} U_n^{(k)}$ is a neighbourhood of 0 in $(F, \xi|_F)$ for each $k \in N$. This completes the proof.

From Theorems 2 and 5 and our remarks in Section 2 we deduce immediately

COROLLARY. *Let $(E, \xi)$ be an $\aleph_0$-hyperbarrelled space for which there is $\lambda > 0$ such that $\xi$ has a base of balanced $\lambda$-convex neighbourhoods of 0. If $F$ is a subspace of $E$ of at most countable codimension, then $(F, \xi|_F)$ is $\aleph_0$-hyperbarrelled.*

*In particular, a subspace of at most countable codimension in the strong dual of a metrizable locally convex space is $\aleph_0$-hyperbarrelled in the induced topology.*

If in the setting of Theorem 5, $B$ is a closed balanced semiconvex absorbent subset of $(F, \xi|_F)$ and $H$ is now the linear span of the closure $A$ of $B$ in $(E, \xi)$, then $A + G$ is a closed balanced semiconvex absorbent subset of $(E, \xi)$. We can therefore establish similarly

THEOREM 6. *Let $(E, \xi)$ be a hyperbarrelled space and let $F$ be a subspace of at most countable codimension. Then $(F, \xi|_F)$ is hyperbarrelled.*

Finally we give an analogue of Corollary 1.3 of [16]. If $B$ is a balanced semiconvex bounded subset of a topological vector space $E$, then $E_B$ denotes the linear span of $B$ endowed with the locally bounded topology having $\{n^{-1}B\colon n \in N\}$ as a base of neighbourhoods of 0.

LEMMA 3. *Let $(E, \xi)$ be a semiconvex space and suppose there is a family $\mathscr{B}$ of balanced semiconvex bounded subsets of $(E, \xi)$ such that the spaces $E_B$ $(B \in \mathscr{B})$ are hyperbarrelled and $(E, \xi)$ is the sc-inductive limit of $\{E_B\colon B \in \mathscr{B}\}$ under the natural embedding mappings. Let $F$ be a subspace of $E$ of at most countable codimension. Then $(F, \xi|_F)$ is the sc-inductive limit of $\{F_{B\cap F}\colon B \in \mathscr{B}\}$ under the natural embedding mappings.*

Proof. Let $V$ be a balanced semiconvex absorbent subset of $F$ such that $V \cap F_{B\cap F}$ is a neighbourhood of 0 in $F_{B\cap F}$ for each $B \in \mathscr{B}$. Choose $\lambda > 0$ such that $V$ is $\lambda$-convex. Let $\mathscr{A}$ be the set of all finite sums of elements of $\mathscr{B}$. It is clear that for each $A \in \mathscr{A}$, $E_A$ is also hyperbarrelled and $V \cap F_{A\cap F}$ is a neighbourhood of 0 in $F_{A\cap F}$. Let $W[A]$ be the closure of $\lambda^{-1} V \cap F_{A\cap F}$ in $E_A$ $(A \in \mathscr{A})$. Since the closure of $\lambda^{-1} V \cap F_{A\cap F}$ in $F_{A\cap F}$ is contained in $\lambda^{-1} V \cap F_{A\cap F} + \lambda^{-1} V \cap F_{A\cap F} \subseteq V \cap F_{A\cap F}$, it follows that $W[A] \cap F = W[A] \cap F_{A\cap F} \subseteq V$ $(A \in \mathscr{A})$. Note also that $W[A]$ is $\lambda$-convex. We show that $W = \bigcup \{W[A]\colon A \in \mathscr{A}\}$ is $\lambda$-convex. If $x, y \in W$, we can choose $A_1, A_2 \in \mathscr{A}$ such that $x \in W[A_1]$, $y \in W[A_2]$. But then $x, y \in W[A_1 + A_2]$ and so $x + y \in \lambda W[A_1 + A_2]$. We have therefore shown that $W + W \subseteq \lambda W$.

Let $G$ be the span of $V + W$ and let $H$ be any supplement of $G$ in $E$. Then $V + W + H$ is a $\lambda$-convex balanced absorbent subset of $E$. For each $B \in \mathscr{B}$, the span of $W[B]$ is a closed, at most countable codimensional subspace $L(B)$ of $E_B$; we know further that $E_B$ is the topological direct sum of $L(B)$ and any of its supplements, each of which must have its finest vector topology (Lemmas 1 and 2). Now $(V + W + H) \cap E_B$ is a balanced semiconvex absorbent subset of $E_B$ and $(V + W + H) \cap L(B) \supseteq W[B]$ which is a neighbourhood of 0 in $L(B)$. It now follows that $(V + W + H) \cap E_B$ is a neighbourhood of 0 in $E_B$ $(B \in \mathscr{B})$. Consequently $V + W + H$ is a neighbourhood of 0 in $(E, \xi)$. Since

$$(V + W + H) \cap F = (V + W) \cap F = V + (W \cap F) \subseteq V + V \subseteq \lambda V,$$

it follows that $V$ is a neighbourhood of 0 in $F$. The result now follows.

THEOREM 7. *Let $(E, \xi)$ be a separated sequentially complete almost convex hyperbornological space [15] and let $F$ be a subspace of at most countable codimension. Then $(F, \xi|_F)$ is hyperbornological.*

Proof There is a family $\mathscr{B}$ of balanced semiconvex bounded subsets of $(E, \xi)$ such that each $E_B$ $(B \in \mathscr{B})$ is a complete metrizable locally bounded space, and therefore hyperbarrelled, and $(E, \xi)$ is the sc-inductive limit of $\{E_B\colon B \in \mathscr{B}\}$ under the natural embedding mappings [7]. The spaces $F_{B\cap F}$ $(B \in \mathscr{B})$ are hyperbornological since they are metrizable. The result now follows from Lemma 3 and [7], Theorem 3.2.

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