ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXIX (1990)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXIX (1990)

STANISŁAW MIKLOS (Wrocław)

Exactly $(n, 1)$-mappings on certain continua

Absrract. It is shown that every weakly nonunicoherent continuum is an exactly $(n, 1)$ image of some unicoherent continuum.

A surjective mapping (i.e. a continuous function) $f\colon X \to Y$ is *exactly* $(n, 1)$ if $f^{-1}(y)$ contains exactly n points for each $y \in Y$.

In [3], p. 351, Nadler and Ward asked: Which continua (i.e. compact connected metric spaces) are the images of some continuum under an exactly $(n, 1)$-mapping, where $2 \leqslant n < \infty$? In the same paper (Theorem 2 of [3], p. 353), it is shown that every nonhereditarily unicoherent continuum is an exactly $(n, 1)$ image of some nonunicoherent continuum.

In this paper, we sharpen the Nadler and Ward result and show that there is an exactly $(n, 1)$-mapping from unicoherent continua onto any weakly nonunicoherent continuum.

A continuum X is called *unicoherent* if for each two subcontinua A and B of X such that $X = A \cup B$ the intersection $A \cap B$ is connected. A nonunicoherent continuum is called *weakly nonunicoherent* if it contains a unicoherent continuum whose complement has a finite number of components which are arcs without their end points.

We need the following three auxiliary classes of mappings from an arc onto itself, each of which is exactly $(3, 1)$ outside the end points of the range.

Let A be an arc and let F_1, F_2 and F_3 be classes of mappings from A onto itself defined as follows (compare § 4 of [1], p. 483, and § 2 of [2], p. 80):

1° $f \in F_1$ if there is an increasing sequence of four distinct points a_0, a_1, a_2, a_3 of the arc $A = a_0 a_3$ ordered from a_0 to a_3 such that, for each $i = 0, 1, 2$, the partial mapping $f | a_i a_{i+1}$ is a homeomorphism from the arc $a_i a_{i+1}$ onto A with $f(a_2) = a_0$ and $f(a_1) = a_3$.

1980 *Mathematics Subject Classification.* Primary 54C10, 54F55.
Key words and phrases. Exactly $(n, 1)$-mapping, unicoherent continuum.

Then

$$\operatorname{card} f^{-1}(a_0) = \operatorname{card} f^{-1}(a_3) = 2, \quad \operatorname{card} f^{-1}(x) = 3 \quad \text{for each } x \in A \setminus \{a_0, a_3\};$$

2° $f \in F_2$ if there is an infinite increasing sequence of distinct points $a_0, a_1, a_2, \ldots$ of the arc $A = a_0 a$, ordered from a_0 to a, which is convergent to a and such that, for each $i = 1, 2, 3$ the partial mapping $f | a_i a_{i+1}$ is a member of F_1 with $f(a_i) = a_i$ and $f(a) = a$.

Then

$$\operatorname{card} f^{-1}(a_0) = 2, \quad \operatorname{card} f^{-1}(a) = 1, \quad \operatorname{card} f^{-1}(x) = 3 \quad \text{for each } x \in A \setminus \{a_0, a\};$$

3° $f \in F_3$ if there is a point $a_1 \in A \setminus \{a_0, a\}$, where $a_0 a = A$, such that $f | a_1 a_0$ and $f | a_1 a$ are members of F_2 with $f^{-1}(a) = a$ and $f^{-1}(a_0) = a_0$.

Then

$$\operatorname{card} f^{-1}(a_0) = \operatorname{card} f^{-1}(a) = 1, \quad \operatorname{card} f^{-1}(x) = 3 \quad \text{for each } x \in A \setminus \{a_0, a\};$$

Remark. Analogously as $F_1 = F_1^3$, $F_2 = F_2^3$, $F_3 = F_3^3$ we may define three classes F_1^5, F_2^5 and F_3^5 of mappings from an arc onto itself, which are exactly (5, 1) outside the end points of the range.

Using the classes F_1, F_2 and F_3 we prove the following

PROPOSITION 1. *Each weakly nonunicoherent continuum Y is the image of some unicoherent continuum X_1 under an exactly* (3, 1)*-mapping.*

Proof. Since the continuum Y is weakly nonunicoherent, it contains an unicoherent continuum C whose complement $Y \setminus C$ has a finite number of components $A_1, A_2, \ldots, A_n$ which are arcs without their end points. Note that the continuum C can always be chosen in such a way that for each i the set $\operatorname{cl} A_i$ is not a simple closed curve. For each i we choose an arc B_i in A_i. Let $A = \{a_1, a_2, \ldots, a_s\}$ and $B = \{b_1, b_2, \ldots, b_s\}$, where $s = 2n$, be all end points of the arcs $\operatorname{cl} A_i$ and B_i, respectively, such that for each odd index $j = 1, 3, \ldots, s-1$ we have

$$b_j b_{j+1} = B_i \subset \operatorname{cl} A_i = a_j a_{j+1}, \quad a_j b_j \cap a_{j+1} b_{j+1} = \emptyset.$$

Put $T = C \cup \bigcup \{a_j b_j: j = 1, 2, \ldots, s\}$. Note that T is a unicoherent continuum. Denote by C_1 and C_2 copies of C and let $c_1 \in C_1$ and $c_2 \in C_2$ be copies of the points a_1 and a_2, respectively. Consider the free union $U = T \oplus C_1 \oplus C_2$ and let ϱ be the equivalence relation on U which identifies the points c_1, a_1 and c_2, a_2 only. Thus all equivalence classes of ϱ except $\{c_1, a_1\}$ and $\{c_2, a_2\}$ are one-point sets. Thus $X_1 = U/\varrho$ is a unicoherent continuum (in the adjunction topology). Let p be the natural projection of U into X_1.

Define a mapping g_1 from X_1 onto itself as follows: $g_1 | p(C \cup C_1 \cup C_2)$

is the identity mapping; for each j let $f = g_1 | p(a_j b_j)$ be a member of one of the classes F_1, F_2 and F_3, namely: a) if $j = 1$, then $f \in F_2$ with $f^{-1}(p(b_1)) = \{p(b_1)\}$; b) if $j = 2$, then $f \in F_1$; c) if $j > 2$ is odd, then $f \in F_3$; and d) if $j > 2$ is even, then $f \in F_2$ with $f^{-1}(p(a_j)) = \{p(a_j)\}$.

Further, define a mapping g_2: $X_1 \to Y$ as follows. The partial mappings $g_2 | p(C)$, $g_2 | p(C_1)$ and $g_2 | p(C_2)$ are the natural embeddings of these sets onto $C \subset Y$. The mapping $g_2 | p(a_j b_j)$ for each (even) $j = 2, 4, \ldots, s$ is again the natural embedding of $p(a_j b_j)$ onto $a_j b_j \subset T \subset Y$, while for each (odd) $j = 1, 3, \ldots, s-1$ it maps the arc $p(a_j b_j)$ homeomorphically onto the arc $a_j b_{j+1} = a_j b_j \cup b_j b_{j+1}$, where $g_2(p(a_j)) = a_j$ and $g_2(p(b_j)) = b_{j+1}$.

Finally, define a mapping g: $X_1 \to Y$ putting $g = g_2 g_1$.

Claim. g is exactly $(3, 1)$. In fact, it follows, directly from the definition of g, that $\operatorname{card} g^{-1}(y) = 3$ for each $y \in Y \setminus g(p(B))$. By the definition of g_1, the set $g_1^{-1}(p(b_j))$ consists of exactly two points for each even j, and $g_1^{-1}(p(b_j)) = \{p(b_j)\}$ for each odd j. Hence, by the definition of g_2, we have $\operatorname{card} g^{-1}(b_j) = 3$ for each j. This completes the proof. ■

The next proposition and the statement following it can be proved in the same way as Proposition 1.

PROPOSITION 2. *Each weakly nonunicoherent continuum is the image of some unicoherent continuum under an exactly* $(4, 1)$*-mapping.*

STATEMENT. *If Y is a weakly nonunicoherent continuum and if $i = 3, 4$, then there exists a mapping f_i from some unicoherent continuum X_i onto Y with the following property: there is a point y_i in Y such that*

$$\operatorname{card} f_i^{-1}(y_i) = i+1, \quad \operatorname{card} f_i^{-1}(y) = i \quad \textit{whenever } y \in Y \setminus \{y_i\}.$$

Using the classes F_1^3, F_2^3, F_3^3, F_1^5, F_2^5 and F_3^5 (see the remark) and arguing similarly as in the proof of Proposition 1 we obtain

PROPOSITION 3. *Each weakly nonunicoherent continuum is the image of some unicoherent continuum under an exactly* $(5, 1)$*-mapping.*

We need also the following fact whose easy proof is omitted.

FACT. *For each integer $n > 5$ there are nonnegative integers s and t such that $n = 3s + 4t$.*

Now we are ready to prove the main result.

THEOREM. *If a continuum Y is weakly nonunicoherent and if $3 \leqslant n < \infty$, then there is an exactly $(n, 1)$-mapping from some unicoherent continuum X onto Y.*

Proof. For each $n = 3, 4, 5$ the assertion holds by Propositions 1, 2 and 3. So let $n > 5$ be given. Then, by the fact, $n = 3s + 4t$, where s and t are nonnegative integers. Note that one of the members s and t must be positive.

We may assume $s > 0$ (if $t > 0$ the proof is the same). By Proposition 1, there is an exactly (3, 1)-mapping g from some unicoherent continuum X_1 onto Y, and by the statement, for each $i = 3, 4$ there is a mapping f_i from some unicoherent continuum X_i onto Y with the property that there is a point y_i in Y such that

$$\operatorname{card} f_i^{-1}(y_i) = i+1, \quad \operatorname{card} f_i^{-1}(y) = i \quad \text{whenever } y \in Y\setminus\{y_i\}.$$

We choose points $x_1, x_1' \in X_1$, $x_3 \in X_3$ and $x_4 \in X_4$ with $g(x_1) = y_3$, $g(x_1') = y_4$, $f_3(x_3) = y_3$ and $f_4(x_4) = y_4$.

For each $j = 1, 2, \ldots, s-1$ let X_3^j be a copy of the continuum X_3 and $x_3^j \in X_3^j$ be a copy of the point x_3. Furthermore, if $t > 0$, for each $k = 1, 2, \ldots, t$ let X_4^k be a copy of the continuum X_4, and $x_4^k \in X_4^k$ be a copy of the point x_4. Consider the free union

$$V = X_1 \oplus \Big(\bigoplus_{j=1}^{s-1} X_3^j\Big) \oplus \Big(\bigoplus_{k=1}^{t} X_4^k\Big)$$

and let σ be the equivalence relation on V which identifies the points $x_1, x_3^1, \ldots, x_3^{s-1}$ and $x_1', x_4^1, \ldots, x_4^t$ only. Therefore $X = V/\sigma$ is a unicoherent continuum (in the adjunction topology).

Further, let p be the natural projection of V onto X, and q_3 and q_4 be the common extensions of the natural embeddings of X_3^j into X_3 (for each j) and of X_4^k into X_4 (for each k), respectively.

Now, define a mapping f from X onto Y as follows. If $x \in p(X_1)$, then $f(x) = g(X_1 \cap p^{-1}(x))$. For every $j = 1, 2, \ldots, s-1$ if $x \in p(X_3^j)$, then $f(x) = f_3\big(q_3(X_3^j \cap p^{-1}(x))\big)$. For every $k = 1, 2, \ldots, t$ if $x \in p(X_4^k)$, then $f(x) = f_4\big(q_4(X_4^k \cap p^{-1}(x))\big)$.

Since the statement guarantees $\operatorname{card} f^{-1}(y_i) = n$ for $i = 3, 4$, we conclude by construction of the unicoherent continuum X that f is an exactly $(n, 1)$-mapping. The theorem follows. ■

References

[1] P. W. Gilbert, *n-to-one mappings of linear graphs*, Duke Math. J. 9 (1942), 475–486.

[2] J. Mioduszewski, *On k-to-1 functions on the closed interval and the straight line*, Comment. Math. Prace Mat. 5 (1961), 79–93 (in Polish).

[3] S. B. Nadler, Jr., L. E. Ward, Jr., *Concerning exactly* $(n, 1)$ *images of continua*, Proc. Amer. Math. Soc. 87 (1983), 351–354.

(INSTYTUT MATEMATYKI, UNIWERSYTET WROCŁAWSKI, WROCŁAW)
INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW
WROCŁAW, POLAND