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# On the Existence and Local Asymptotic Stability of Solutions of Fractional Order Integral Equations

**Abstract.** In this paper, we present some results concerning the existence and the local asymptotic stability of solutions for a functional integral equation of fractional order, by using some fixed point theorems.

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*Key words and phrases:* Functional integral equation, Left-sided mixed Riemann-Liouville integral of fractional order, Solution, Local asymptotic stability, Fixed point.



**1. Introduction.** Integral equations are one of useful mathematical tools in both pure and applied analysis. This is particularly true for problems in mechanical vibrations and the related fields of engineering and mathematical physics. We can find numerous applications of differential and integral equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [2, 16, 19, 20, 21]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [1], Kilbas *et al.* [17], Lakshmikantham *et al.* [18], Miller and Ross [19], Podlubny [20].

During the last decade, many classes of integral equations have been considered including the local, global and the asymptotic behavior of solutions by Banaś *et al.* [3, 4, 5, 7, 8], Darwish *et al.* [10], Dhage [11, 12, 13, 14], and the references therein. In the most of them the main tool was the measure of noncompactness [6].

In [5], Banaś and Dhage studied the existence of solutions in the space of real functions defined, continuous and bounded on the half-line of the following nonlinear quadratic Volterra integral equation of fractional order

$$x(t) = f(t, x(\alpha(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma(s)))ds;\ t \in \mathbb{R}_+ := [0, \infty), \tag{1}$$

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where $\alpha,\ \beta,\ \gamma : \mathbb{R}_+ \to \mathbb{R}_+,\ f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions. The main tool used is the technique associated with certain measure of noncompactness related to monotonicity.

Motivated by the above paper, this paper deals with the existence of solutions to the following nonlinear quadratic Volterra integral equation of Riemann-Liouville fractional order

$$(2) \qquad u(t) = f(t, u(\alpha(t))) + \frac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t) - s)^{r-1} g(t, s, u(\gamma(s)))ds;\ if\ t \in \mathbb{R}_+,$$

where $\alpha, \beta, \gamma,\ f,\ g$ are as in (1), $r \in (0, \infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt,\ \xi > 0.$$

We prove the existence of solutions of equation (2) by using Schauder's fixed point theorem, and we obtain some results about the local asymptotic stability of solutions. Finally, an example illustrating the main result is presented in the last section.

**2. Preliminaries.** In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $L^1([0, b]);\ b > 0$ we denote the space of Lebesgue-integrable functions $u : [0, b] \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_0^b |u(t)|dt.$$

By $BC := BC(\mathbb{R}_+)$ we denote the Banach space of all bounded and continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC} = \sup_{t \in \mathbb{R}_+} |u(t)|.$$

For $u_0 \in BC$ and $\eta \in (0, \infty)$, we denote by $B(u_0, \eta)$, the closed ball in $BC$ centered at $u_0$ with radius $\eta$.

Definition 2.1 ([17]) Let $r > 0$. For $u \in L^1([0, b]);\ b > 0$ the expression

$$(I_0^r u)(t) = \frac{1}{\Gamma(r)} \int_0^t (t - s)^{r-1} u(s)ds,$$

is called the left-sided mixed Riemann-Liouville integral of order $r$.

In particular,

$$(I_0^0 u)(t) = u(t),\ (I_0^1 u)(t) = \int_0^t u(s)ds;\ \text{for almost all } t \in [0, b].$$

For instance, $I_0^r u$ exists for all $r > 0$, when $u \in L^1([0, b])$. Note also that when $u \in C([0, b])$, then $(I_0^r u) \in C([0, b])$, EXAMPLE 2.2 Let $\omega \in (-1, \infty)$ and $r \in (0, \infty)$, then

$$I_0^r t^\omega = \frac{\Gamma(1+\omega)}{\Gamma(1+\omega+r)} t^{\omega+r}, \text{ for almost all } t \in [0, b].$$

Let $G$ be an operator from $\Omega \subset BC$; $\Omega \neq \emptyset$ into itself and consider the solutions of equation

$$(Gu)(t) = u(t). \tag{3}$$

Now we review the concept of attractivity of solutions for equation (2).

DEFINITION 2.3 ([5]) Solutions of equation (3) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space $BC$ such that for arbitrary solutions $v = v(t)$ and $w = w(t)$ of equations (3) belonging to $B(u_0, \eta) \cap \Omega$ we have that

$$\lim_{t \to \infty} (v(t) - w(t)) = 0. \tag{4}$$

When the limit (4) is uniform with respect to $B(u_0, \eta) \cap \Omega$, solutions of equation (3) are said to be uniformly locally attractive (or equivalently that solutions of (3) are locally asymptotically stable).

LEMMA 2.4 ([9]) *Let $D \in BC$. Then $D$ is relatively compact in $BC$ if the following conditions hold:*
*(a) $D$ is uniformly bounded in $BC$,*
*(b) The functions belonging to $D$ are almost equicontinuous on $\mathbb{R}_+$,*
*i.e. equicontinuous on every compact interval of $\mathbb{R}_+$,*
*(c) The functions from $D$ are equiconvergent, that is, given $\epsilon > 0$, there corresponds $T(\epsilon) > 0$ such that $|u(t) - u(+\infty)| < \epsilon$ for any $t \geqslant T(\epsilon)$ and $u \in D$.*

**3. Main Results.** In this section, we are concerned with the existence and global asymptotic stability of solutions for the equation (2). The following hypotheses will be used in the sequel.

($H_1$) The functions $\alpha, \beta, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\lim_{t\to\infty} \alpha(t) = \infty$.

($H_2$) The function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist positive constants $M, L$ such that $M < L$ and

$$|f(t, u) - f(t, v)| \leqslant \frac{M|u - v|}{(1 + \alpha(t))(L + |u - v|)}, \text{ for } t \in \mathbb{R}_+ \text{ and for } u, v \in \mathbb{R}.$$

($H_3$) The function $t \to f(t, 0)$ is bounded on $\mathbb{R}_+$ with $f^* = \sup_{t\in\mathbb{R}_+} f(t, 0)$ and $\lim_{t\to\infty} |f(t, 0)| = 0$.

($H_4$) The function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist functions $p, q : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t, s, u)| \leqslant \frac{p(t)q(s)}{1 + \alpha(t) + |u|}, \text{ for } t, s \in \mathbb{R}_+ \text{ and for } u \in \mathbb{R}.$$

Moreover, assume that

$$\lim_{t\to\infty} p(t)\int_0^{\beta(t)} (\beta(t)-s)^{r-1}q(s)ds = 0.$$

THEOREM 3.1 *Assume that hypotheses $(H_1)-(H_4)$ hold. Then the equation (2) has at least one solution in the space $BC$. Moreover, solutions of equation (2) are locally asymptotically stable.*

PROOF Set $d^* := \sup_{t\in\mathbb{R}_+} d(t)$ where

$$d(t) = \frac{p(t)}{\Gamma(r)}\int_0^{\beta(t)} (\beta(t)-s)^{r-1}q(s)ds.$$

From hypothesis $(H_4)$, we infer that $d^*$ is finite.
Let us define the operator $N$, such that for any $u \in BC$

$$(Nu)(t) = f(t,u(\alpha(t))) + \frac{1}{\Gamma(r)}\int_0^{\beta(t)} (\beta(t)-s)^{r-1}g(t,s,u(\gamma(s)))ds;\ t\in\mathbb{R}_+. \tag{5}$$

By considering conditions of theorem we infer that $N(u)$ is continuous on $\mathbb{R}_+$. Now we prove that $N(u) \in BC$ for any $u \in BC$. For arbitrarily fixed $t \in \mathbb{R}_+$ we have

$$\begin{aligned}
|(Nu)(t)| &= \Big|f(t,u(\alpha(t))) + \frac{1}{\Gamma(r)}\int_0^{\beta(t)} (\beta(t)-s)^{r-1}g(t,s,u(\gamma(s)))ds\Big| \\
&\leqslant \Big|f(t,u(\alpha(t))) - f(t,0) + f(t,0)\Big| \\
&+ \Big|\frac{1}{\Gamma(r)}\int_0^{\beta(t)} (\beta(t)-s)^{r-1}g(t,s,u(\gamma(s)))ds\Big| \\
&\leqslant \frac{M|u(\alpha(t))|}{(1+\alpha(t))(L+|u(\alpha(t))|)} + \Big|f(t,0)\Big| \\
&+ \frac{p(t)}{\Gamma(r)}\int_0^{\beta(t)} \frac{(\beta(t)-s)^{r-1}q(s)}{1+\alpha(t)+|u(\gamma(s))|}ds \\
&\leqslant \frac{M\|u\|}{L+\|u\|} + f^* + d^*.
\end{aligned}$$

Thus

$$\|N(u)\| \leqslant M + f^* + d^*. \tag{6}$$

Hence $N(u) \in BC$. Equation (6) yields that $N$ transforms the ball $B_\eta := B(0,\eta)$ into itself where $\eta = M + f^* + d^*$. We shall show that $N : B_\eta \to B_\eta$ satisfies the assumptions of Schauder's fixed point theorem [15]. The proof will be given in several steps.

**Step 1:** *$N$ is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $B_\eta$. Then, for each $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
|(Nu_n)(t) - (Nu)(t)| &\leqslant |f(t, u_n(\alpha(t))) - f(t, u(\alpha(t)))| \\
&+ \frac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t) - s)^{r-1} |g(t, s, u_n(\gamma(s))) - g(t, s, u(\gamma(s)))| ds \\
&\leqslant \frac{M\|u_n - u\|}{(1 + \alpha(t))(L + \|u_n - u\|)}
\end{aligned}$$

$$+\frac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t) - s)^{r-1} \|g(t, s, u_n(\gamma(s))) - g(t, s, u(\gamma(s)))\| ds. \tag{7}$$

**Case 1.** If $t \in [0, T];\ T > 0$, then, since $u_n \to u$ as $n \to \infty$ and $g$ is continuous, (7) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as } n \to \infty.$$

**Case 2.** If $t > T;\ T > 0$, then from $(H_4)$ and (7) we get

$$\begin{aligned}
|(Nu_n)(t) - (Nu)(t)| &\leqslant \frac{M\|u_n - u\|}{L + \|u_n - u\|} \\
&+ \frac{p(t)}{\Gamma(r)} \int_0^{\beta(t)} \frac{(\beta(t) - s)^{r-1} q(s)(|u_n(\gamma(s))| + |u(\gamma(s))|)}{(1 + \alpha(t) + |u_n(\gamma(s))|)(1 + \alpha(t) + |u(\gamma(s))|)} ds \\
&\leqslant \frac{M\|u_n - u\|}{L + \|u_n - u\|} \\
&+ \frac{2\eta p(t)}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t) - s)^{r-1} q(s) ds
\end{aligned}$$

$$\leqslant \frac{M\|u_n - u\|}{L + \|u_n - u\|} + 2\eta d(t). \tag{8}$$

Since $u_n \to u$ as $n \to \infty$ and $t \to \infty$, then (8) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as } n \to \infty.$$

**Step 2**: *$N(B_\eta)$ is uniformly bounded.*
This is clear since $N(B_\eta) \subset B_\eta$ and $B_\eta$ is bounded.

**Step 3:** *$N(B_\eta)$ is equicontinuous on every compact interval $I$ of $\mathbb{R}_+$.*
Let $t_1, t_2 \in I$, $t_1 < t_2$ and let $u \in B_\eta$. Also without lose of generality suppose that $\beta(t_1) \leqslant \beta(t_2)$, thus we have

$$|(Nu)(t_2) - (Nu)(t_1)|$$

$$
\begin{aligned}
&\leqslant |f(t_2,u(\alpha(t_2)))-f(t_2,u(\alpha(t_1)))|+|f(t_2,u(\alpha(t_1)))-f(t_1,u(\alpha(t_1)))|\\
&+\frac{1}{\Gamma(r)}\Big|\int_0^{\beta(t_2)}(\beta(t_2)-s)^{r-1}[g(t_2,s,u(\gamma(s)))-g(t_1,s,u(\gamma(s)))]ds\Big|\\
&+\frac{1}{\Gamma(r)}\Big|\int_0^{\beta(t_2)}(\beta(t_2)-s)^{r-1}g(t_1,s,u(\gamma(s)))ds\\
&-\int_0^{\beta(t_1)}(\beta(t_2)-s)^{r-1}g(t_1,s,u(\gamma(s)))ds\Big|\\
&+\frac{1}{\Gamma(r)}\Big|\int_0^{\beta(t_1)}(\beta(t_2)-s)^{r-1}g(t_1,s,u(\gamma(s)))ds\\
&-\int_0^{\beta(t_1)}(\beta(t_1)-s)^{r-1}g(t_1,s,u(\gamma(s)))ds\Big|\\
&\leqslant \frac{M|u(\alpha(t_2))-u(\alpha(t_1))|}{(1+\alpha(t_2))(L+|u(\alpha(t_2))-u(\alpha(t_1))|)}\\
&+|f(t_2,u(\alpha(t_1)))-f(t_1,u(\alpha(t_1)))|\\
&+\frac{1}{\Gamma(r)}\int_0^{\beta(t_2)}(\beta(t_2)-s)^{r-1}\Big|g(t_2,s,u(\gamma(s)))-g(t_1,s,u(\gamma(s)))\Big|ds\\
&+\frac{1}{\Gamma(r)}\int_{\beta(t_1)}^{\beta(t_2)}(\beta(t_2)-s)^{r-1}\Big|g(t_1,s,u(\gamma(s)))\Big|ds\\
&+\frac{1}{\Gamma(r)}\int_0^{\beta(t_1)}\Big|(\beta(t_2)-s)^{r-1}-(\beta(t_1)-s)^{r-1}\Big|\times\Big|g(t_1,s,u(\gamma(s)))\Big|ds\\
&\leqslant \frac{M|u(\alpha(t_2))-u(\alpha(t_1))|}{L+|u(\alpha(t_2))-u(\alpha(t_1))|}+|f(t_2,u(\alpha(t_1)))-f(t_1,u(\alpha(t_1)))|\\
&+\frac{1}{\Gamma(r)}\int_0^{\beta(t_2)}(\beta(t_2)-s)^{r-1}\Big|g(t_2,s,u(\gamma(s)))-g(t_1,s,u(\gamma(s)))\Big|ds\\
&+\frac{p(t)}{\Gamma(r)}\int_{\beta(t_1)}^{\beta(t_2)}(\beta(t_2)-s)^{r-1}q(s)ds\\
&+\frac{p(t)}{\Gamma(r)}\int_0^{\beta(t_1)}\Big|(\beta(t_2)-s)^{r-1}-(\beta(t_1)-s)^{r-1}\Big|q(s)ds.
\end{aligned}
$$

From continuity of $\alpha,\beta,f,g$ and as $t_1\to t_2$, the right-hand side of the above inequality tends to zero.

**Step 4:** $N(B_\eta)$ *is equiconvergent.*

Let $t \in \mathbb{R}_+$ and $u \in B_\eta$, then we have

$$\begin{aligned}
|(Nu)(t)| &\leqslant |f(t,u(\alpha(t))) - f(t,0) + f(t,0)| \\
&+ \Big|\frac{1}{\Gamma(r)}\int_0^{\beta(t)} (\beta(t)-s)^{r-1} g(t,s,u(\gamma(s)))ds\Big| \\
&\leqslant \frac{M|u(\alpha(t))|}{(1+\alpha(t))(L+|u(\alpha(t))|)} + |f(t,0)| \\
&+ \frac{p(t)}{\Gamma(r)}\int_0^{\beta(t)} \frac{(\beta(t)-s)^{r-1}q(s)}{1+\alpha(t)+|u(\gamma(s))|}ds \\
&\leqslant \frac{M}{1+\alpha(t)} + |f(t,0)| \\
&+ \frac{1}{1+\alpha(t)}\left(\frac{p(t)}{\Gamma(r)}\int_0^{\beta(t)} (\beta(t)-s)^{r-1}q(s)ds\right) \\
&\leqslant \frac{M}{1+\alpha(t)} + |f(t,0)| + \frac{d^*}{1+\alpha(t)}.
\end{aligned}$$

Thus

$$|(Nu)(t)| \to 0, \ \text{as } t \to +\infty.$$

Hence, we get

$$|(Nu)(t) - (Nu)(+\infty)| \to 0, \ \text{as } t \to +\infty.$$

As a consequence of Steps 1 to 4 together with the Lemma 2.4, we can conclude that $N : B_\eta \to B_\eta$ is continuous and compact. From an application of Schauder's theorem [15], we deduce that $N$ has a fixed point $u$ which is a solution of the equation (2).

Now we should investigate uniform local attractivity for solutions of equation (26). Let us assume that $u_0$ is a solution of equation (2) with conditions of Theorem 3.1. Consider ball $B(u_0, \eta^*)$ with $\eta^* = \frac{LM^*}{L-M}$, where

$$M^* := \frac{1}{\Gamma(r)} \sup_{t\in\mathbb{R}_+} \left\{ \int_0^{\beta(t)} (\beta(t)-s)^{r-1} |g(t,s,u(\gamma(s))) - g(t,s,u_0(\gamma(s)))| ds; \ u \in BC \right\}.$$

Take $u \in B(u_0, \eta^*)$, we have

$$\begin{aligned}
|(Nu)(t) - u_0(t)| &= |(Nu)(t) - (Nu_0)(t)| \\
&\leqslant |f(t, u(\alpha(t))) - f(t, u_0(\alpha(t)))| \\
&+ \frac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t) - s)^{r-1} |g(t, s, u(\gamma(s))) - g(t, s, u_0(\gamma(s)))| ds \\
&\leqslant \frac{M\|u - u_0\|}{L + \|u - u_0\|} \\
&+ \frac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t) - s)^{r-1} |g(t, s, u(\gamma(s))) - g(t, s, u_0(\gamma(s)))| ds \\
&\leqslant \frac{M}{L}\|u - u_0\| + M^* \\
&\leqslant \frac{M}{L}\eta^* = \eta^*.
\end{aligned}$$

Thus we observe that $N$ is continuous function such that $N(B(u_0, \eta^*)) \subset B(u_0, \eta^*)$. Moreover, if $u$ is a solution of equation (2) then

$$\begin{aligned}
|u(t) - u_0(t)| &= |(Nu)(t) - (Nu_0)(t)| \\
&\leqslant |f(t, u(\alpha(t))) - f(t, u_0(\alpha(t)))| \\
&+ \frac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t) - s)^{r-1} |g(t, s, u(\gamma(s))) - g(t, s, u_0(\gamma(s)))| ds \\
&\leqslant \frac{M|u(\alpha(t)) - u_0(\alpha(t))|}{L + |u(\alpha(t)) - u_0(\alpha(t))|} \\
&+ \frac{p(t)}{\Gamma(r)} \int_0^{\beta(t)} \left( \frac{(\beta(t) - s)^{r-1} q(s)}{1 + \alpha(t) + |u(\gamma(s))|} + \frac{(\beta(t) - s)^{r-1} q(s)}{1 + \alpha(t) + |u_0(\gamma(s))|} \right) ds \\
&\leqslant \frac{M|u(\alpha(t)) - u_0(\alpha(t))|}{L + |u(\alpha(t)) - u_0(\alpha(t))|} + \frac{2p(t)}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t) - s)^{r-1} q(s) ds
\end{aligned}$$

$$\leqslant \frac{M}{L}|u(\alpha(t)) - u_0(\alpha(t))| + \frac{2p(t)}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t) - s)^{r-1} q(s) ds. \tag{9}$$

Since $\alpha(t) \to \infty$ as $t \to \infty$, then

$$\lim_{t\to\infty} |u(\alpha(t)) - u_0(\alpha(t))| = \lim_{t\to\infty} |u(t) - u_0(t)|.$$

Thus, by using (9), we deduce that

$$\lim_{t\to\infty} |u(t) - u_0(t)| \leqslant \lim_{t\to\infty} \frac{2Lp(t)}{(L - M)\Gamma(r)} \int_0^{\beta(t)} (\beta(t) - s)^{r-1} q(s) ds = 0.$$

Consequently, all solutions of equation (2) are locally asymptotically stable. ∎

**4. An Example.** As an application of our results we consider the following integral equation of fractional order

$$u(t) = \frac{1}{2(1+t)(1+|u(t)|)}$$

$$+\frac{1}{\Gamma(\frac{2}{3})}\int_0^t (t-s)^{\frac{-1}{3}} \frac{ln(1+s|u(t)|)}{(1+t+|u(s)|)^2(1+t^4)}ds;\ t \in \mathbb{R}_+, \tag{10}$$

where $r = \frac{2}{3},\ \alpha(t) = \beta(t) = \gamma(t) = t,$

$$f(t,u) = \frac{1}{2(1+t)(1+|u|)},\ t \in \mathbb{R}_+,\ u \in \mathbb{R},$$

and

$$g(t,s,u) = \frac{ln(1+s|u|)}{(1+t+|u|)^2(1+t^4)};\ t,s \in \mathbb{R}_+,\ u \in \mathbb{R}.$$

For each $t \in \mathbb{R}_+$ and $u, v \in \mathbb{R}$, we have

$$|f(t,u) - f(t,v)| \leqslant \frac{|u-v|}{2(1+t)(1+|u-v|)},$$

Then we can easily check that the assumptions of Theorem 3.1 are satisfied. In fact, we have that the function $f$ is continuous and satisfies assumption $(H_2)$, with $M = \frac{1}{2}$ and $L = 1$. Also $f$ satisfies assumption $(H_3)$ with $f^* = \frac{1}{2}$. Next, let us notice that the function $g$ satisfies assumption $(H_4)$, where $p(t) = \frac{1}{1+t^4}$ and $q(s) = s$. Also,

$$\int_0^{\beta(t)} (\beta(t)-s)^{r-1} q(s)ds = \frac{9}{10} t^{\frac{5}{3}},$$

and

$$\lim_{t\to\infty} p(t) \int_0^{\beta(t)} (\beta(t)-s)^{r-1} q(s)ds = \lim_{t\to\infty} \frac{9t^{\frac{5}{3}}}{10(1+t^4)} = 0.$$

Hence by Theorem 3.1, equation (10) has a solution defined on $\mathbb{R}_+$ and solutions of this equation are locally asymptotically stable.

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