COMMENTATIONES MATHEMATICAE Vol. 52, No. 1 (2012), 61-71

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A Common Fixed Point Theorem for Set-valued Contraction Mappings in Menger Space

Abstract. The aim of this paper is to prove a common fixed point theorem for even number of single-valued and two set-valued mappings in complete Menger space using implicit relation. Our result improves and extends the result of Chen and Chang [Common fixed point theorems in Menger spaces, Int. J. Math. Math. Sci. 2006, Art. ID 75931, 15 pp].

2010 Mathematics Subject Classification: Primary: 47H10, Secondary: 54H25.

Key words and phrases: t-norm, probabilistic metric space, set-valued mappings, weakly compatible mappings, implicit relation.

1. Introduction. There have been a number of generalizations of metric space. One such generalization is Menger space introduced in 1942 by Karl Menger [15] who used distribution functions instead of nonnegative real numbers as values of the metric. Schweizer and Sklar [23, 15] studied this concept and then the important development of Menger space theory was due to Sehgal and Bharucha-Reid [25]. It is also of fundamental importance in probabilistic functional analysis, nonlinear analysis and applications [2, 3]. In 1991, Mishra [17] formulated the definition of compatible maps in probabilistic metric space (shortly PM-space). This condition has further been weakened by introducing the notion of weakly compatible mappings by Singh and Jain [26] in PM-spaces. It is worth to mention that every pair of compatible maps is weakly compatible, but the converse is not always true.

In 1976, Caristi [1] proved a fixed point theorem. Since the Caristi's fixed point theorem does not require the continuity of the mapping, it has applications in many fields. In 1993, Zhang et al. [27] proved a set-valued Caristi's theorem in probabilistic metric spaces. Chuan [7] brought forward the concept of Caristi type hybrid fixed point in Menger PM-space. Various authors proved some fixed point theorems for multi-valued mappings in probabilistic metric spaces (see [4, 22, 9, 10, 21]). Recently,

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Chen and Chang [6] proved a common fixed point theorem for four single valued and two set-valued mappings in complete Menger space using the notion of compatibility. In [22], Razani and Shirdaryazdi proved a common fixed point theorem for even number of single valued mappings in complete Menger space. In [6, 18, 22], the results have been proved for continuous mappings.

In fixed point theory many authors (see [5, 13, 14, 16, 19, 20, 26]) used implicit relations as a tool to find common fixed point of mappings. These observations motivated us to prove a common fixed point theorem for even number of single-valued and two set-valued mappings in complete Menger space using implicit relation. Our results never require continuity of one or more mappings.

2. Preliminaries.

Definition 2.1 ([24]) A mapping $* : [0,1] \times [0,1] \to [0,1]$ is t-norm if $*$ is satisfying the following conditions:

1. $*$ is commutative and associative;
2. $a * 1 = a$ for all $a \in [0,1]$;
3. $c * d \geqslant a * b$ whenever $c \geqslant a$ and $d \geqslant b$ and $a, b, c, d \in [0,1]$.

Definition 2.2 ([24]) A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$.

We shall denote by $\Im$ the set of all distribution functions defined on $[-\infty, \infty]$ while $H(t)$ will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leqslant 0; \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \Im$ is called a probabilistic distance on X and the value of $\mathcal{F}$ at $(x, y) \in X \times X$ is represented by $F_{x,y}$.

Definition 2.3 ([24]) A PM-space is an ordered pair $(X, \mathcal{F})$, where X is a non-empty set of elements and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

1. $F_{x,y}(t) = H(t)$ for all $t > 0$ if and only $x = y$;
2. $F_{x,y}(0) = 0$;
3. $F_{x,y}(t) = F_{y,x}(t)$;
4. if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t + s) = 1$.

The ordered triple $(X, \mathcal{F}, *)$ is called a Menger space if $(X, \mathcal{F})$ is a PM-space, $*$ is a t-norm and the following inequality holds:

$$F_{x,y}(t + s) \geqslant F_{x,z}(t) * F_{z,y}(s),$$

for all $x, y, z \in X$ and $t, s > 0$.

Every metric space (X, d) can always be realized as a PM-space by considering $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$. So PM-spaces offer a wider framework than that of metric spaces and are better suited to cover even wider statistical situations.

Definition 2.4 ([3]) Let $(X, \mathcal{F}, *)$ be a Menger space and A be a non-empty subset of X. Then A is said to be probabilistically bounded if

$$\sup_{t>0} \inf_{x,y \in A} F_{x,y}(t) = 1.$$

If X itself is probabilistically bounded, then X is said to be a probabilistically bounded space.

Throughout this paper, $\mathcal{B}(X)$ will denote the family of non-empty bounded subsets of a Menger space $(X, \mathcal{F}, *)$. For all $A, B \in \mathcal{B}(X)$ and for every $t > 0$, we define

$$_D F_{A,B}(t) \quad = \quad \sup\{F_{a,b}(t); a \in A, b \in B\}$$

and

$$_\delta F_{A,B}(t) \quad = \quad \inf\{F_{a,b}(t); a \in A, b \in B\}.$$

If the set A consists of a single point a, we write

$$_\delta F_{A,B}(t) \quad = \quad {}_\delta F_{a,B}(t).$$

If the set B also consists of a single point b, we write

$$_\delta F_{A,B}(t) \quad = \quad F_{a,b}(t).$$

It follows immediately from the definition that

$$\begin{aligned} {}_\delta F_{A,B}(t) &= {}_\delta F_{B,A}(t) \geqslant 0, \\ {}_\delta F_{A,B}(t) &= 1 \Leftrightarrow A = B = \{a\}, \end{aligned}$$

for all $A, B \in \mathcal{B}(X)$.

Definition 2.5 ([24]) Let $(X, \mathcal{F}, *)$ be a Menger space with continuous t-norm.

1. A sequence $\{x_n\}$ in X is said to converge to a point x in X if and only if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $\mathbb{N}$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geqslant \mathbb{N}$.
2. A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $\mathbb{N}$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \geqslant \mathbb{N}$.
3. A Menger space in which every Cauchy sequence is convergent is said to be complete.

The following definition is on the lines of Jungck and Rhoades [12].

Definition 2.6 The mappings $f : X \to X$ and $g : X \to \mathcal{B}(X)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is $gu = \{fu\}$ for some $u \in X$ then $fgu = gfu$ (Note that the term $gu = \{fu\}$ implies that gu is a singleton).

Remark 2.7 If mappings $f : X \to X$ and $g : X \to \mathcal{B}(X)$ of a Menger space $(X, \mathcal{F}, *)$ are compatible then they are weakly compatible but the converse need not be true.

3. Implicit Relation.

In 2008, Imdad and Ali [11] used the following implicit relation for the existence of common fixed points of the involved mappings.

Let Ψ be the class of all real continuous functions $\psi : [0,1]^4 \to \mathbb{R}$ satisfying the following conditions:

(R-1) For every $u > 0$, $v \geqslant 0$ with $\psi(u, v, u, v) \geqslant 0$ or $\psi(u, v, v, u) \geqslant 0$, we have $u > v$.

(R-2) $\psi(u, u, 1, 1) < 0$, for all $u > 0$.

Example 3.1 ([11]) Define $\psi : [0,1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - \phi\left(\min\{t_2, t_3, t_4\}\right)$, where $\phi : [0,1] \to [0,1]$ is a continuous function such that $\phi(s) > s$ for $0 < s < 1$.

Example 3.2 ([11]) Define $\psi : [0,1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - a \min\{t_2, t_3, t_4\}$, where $a > 1$.

Example 3.3 ([11]) Define $\psi : [0,1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - at_2 - \min\{t_3, t_4\}$, where $a > 0$.

Example 3.4 ([11]) Define $\psi : [0,1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - at_2 - bt_3 - ct_4$, where $a > 1$, $b, c \geqslant 0 (\neq 1)$.

Example 3.5 ([11]) Define $\psi : [0,1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - at_2 - b(t_3 + t_4)$, where $a > 1$, $b \geqslant 0 (\neq 1)$.

Example 3.6 ([11]) Define $\psi : [0,1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1^3 - at_2t_3t_4$, where $a > 1$.

4. Results.

THEOREM 4.1 *Let $(X, \mathcal{F}, *)$ be a complete Menger space where $*$ is a continuous t-norm. Let $P_1, P_2, \ldots, P_{2n} : X \to X$ be single-valued mappings and let $A, B : X \to \mathcal{B}(X)$ two set-valued mappings. If the following conditions are satisfied:*

(1) $A(X) \subseteq P_2P_4 \ldots P_{2n}(X), B(X) \subseteq P_1P_3 \ldots P_{2n-1}(X)$;

(2) One of $P_1P_3 \ldots P_{2n-1}(X)$ or $P_2P_4 \ldots P_{2n}(X)$ is a closed subset of X;

(3) The pairs $(A, P_1P_3 \ldots P_{2n-1})$ and $(B, P_2P_4 \ldots P_{2n})$ are weakly compatible; Suppose that

$$\left\{ \begin{array}{c} P_1(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})P_1, \\ P_1P_3(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})P_1P_3, \\ \vdots \\ P_1 \ldots P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1 \ldots P_{2n-3}, \\ A(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})A, \\ A(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})A, \\ \vdots \\ AP_{2n-1} = P_{2n-1}A, \\ P_2(P_4 \ldots P_{2n}) = (P_4 \ldots P_{2n})P_2, \\ P_2P_4(P_6 \ldots P_{2n}) = (P_6 \ldots P_{2n})P_2P_4, \\ \vdots \\ P_2 \ldots P_{2n-2}(P_{2n}) = (P_{2n})P_2 \ldots P_{2n-2}, \\ B(P_4 \ldots P_{2n}) = (P_4 \ldots P_{2n})B, \\ B(P_6 \ldots P_{2n}) = (P_6 \ldots P_{2n})B, \\ \vdots \\ BP_{2n} = P_{2n}B; \end{array} \right\}$$

(4) There exists $\psi \in \Psi$ such that

$$\psi \left(\begin{array}{c} {}_{\delta}F_{Ax,By}(t), F_{P_1P_3 \ldots P_{2n-1}x, P_2P_4 \ldots P_{2n}y}(t), \\ {}_{\delta}F_{Ax, P_1P_3 \ldots P_{2n-1}x}(t), {}_{\delta}F_{By, P_2P_4 \ldots P_{2n}y}(t) \end{array} \right) \geqslant 0, \tag{1}$$

for all $x, y \in X$ and $t > 0$. Then there exists a point $z \in X$ such that $\{z\} = \{P_1z\} = \{P_2z\} = \ldots = \{P_{2n}z\} = Az = Bz$.

PROOF Let x_0 be an arbitrary point in X. By (1), we choose a point $x_1 \in X$ such that $y_0 = P_2P_4 \ldots P_{2n}x_1 \in Ax_0$. For this point x_1 there exists a point $x_2 \in X$ such that $y_1 = P_1P_3 \ldots P_{2n-1}x_2 \in Bx_1$, and so on. Continuing in this manner we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$y_{2n} = P_2P_4 \ldots P_{2n}x_{2n+1} \in Ax_{2n}, y_{2n+1} = P_1P_3 \ldots P_{2n-1}x_{2n+2} \in Bx_{2n+1},$$

for $n = 0, 1, 2, \ldots$. Now, using inequality (1) with $x = x_{2n}$ and $y = x_{2n+1}$, we get

$$\psi \left(\begin{array}{c} {}_{\delta}F_{Ax_{2n}, Bx_{2n+1}}(t), F_{P_1P_3 \ldots P_{2n-1}x_{2n}, P_2P_4 \ldots P_{2n}x_{2n+1}}(t), \\ {}_{\delta}F_{Ax_{2n}, P_1P_3 \ldots P_{2n-1}x_{2n}}(t), {}_{\delta}F_{Bx_{2n+1}, P_2P_4 \ldots P_{2n}x_{2n+1}}(t) \end{array} \right) \geqslant 0,$$

$$\psi \left(F_{y_{2n}, y_{2n+1}}(t), F_{y_{2n-1}, y_{2n}}(t), F_{y_{2n}, y_{2n-1}}(t), F_{y_{2n+1}, y_{2n}}(t) \right) \geqslant 0.$$

Using (R-1), we get

$$F_{y_{2n},y_{2n+1}}(t) > F_{y_{2n-1},y_{2n}}(t). \tag{2}$$

Thus $\{F_{y_{2n},y_{2n+1}}(t), n \geqslant 0\}$ is a bounded strictly increasing sequence of positive real numbers in $[0,1]$ and therefore tends to a limit $L(t) \leqslant 1$. We claim that $L(t) = 1$. For if $L(t_0) < 1$ for some t_0, then letting $n \to \infty$ in inequality (2), we get $L(t_0) > L(t_0)$ a contradiction. Hence $L(t) = 1$ for all $t > 0$.

Claim: $\{y_n\}$ is a Cauchy sequence in X. Now for $m \geqslant 1$,

$$F_{y_n,y_{n+m}}(t) \geqslant F_{y_n,y_{n+1}}\left(\frac{t}{2}\right) * F_{y_{n+1},y_{n+m}}\left(\frac{t}{2}\right).$$

This yields that

$$\begin{aligned}
\lim_{n\to\infty} F_{y_n,y_{n+m}}(t) &\geqslant \lim_{n\to\infty} F_{y_n,y_{n+1}}\left(\frac{t}{2}\right) * \lim_{n\to\infty} F_{y_{n+1},y_{n+m}}\left(\frac{t}{2}\right)\\
&= 1 * \lim_{n\to\infty} F_{y_{n+1},y_{n+m}}\left(\frac{t}{2}\right)\\
&= \lim_{n\to\infty} F_{y_{n+1},y_{n+m}}\left(\frac{t}{2}\right)\\
&\geqslant \lim_{n\to\infty}\left(F_{y_{n+1},y_{n+2}}\left(\frac{t}{4}\right) * F_{y_{n+2},y_{n+m}}\left(\frac{t}{4}\right)\right)\\
&= \lim_{n\to\infty} F_{y_{n+2},y_{n+m}}\left(\frac{t}{4}\right)\\
&\vdots\\
&\geqslant \lim_{n\to\infty}\left(F_{y_{n+m-2},y_{n+m-1}}\left(\frac{t}{2^{m-1}}\right) * F_{y_{n+m-1},y_{n+m}}\left(\frac{t}{2^{m-1}}\right)\right)\\
&= 1,
\end{aligned}$$

and thus $\lim_{n\to\infty} F_{y_n,y_{n+m}}(t) = 1$, since $*$ is continuous and $a*1 = a$ for all $a \in [0,1]$. Hence $\{y_n\}$ is a Cauchy sequence in X.

Now, suppose that $P_2P_4\ldots P_{2n}(X)$ is a closed subset of X, then for some $u \in X$ we have $z = P_2P_4\ldots P_{2n}(u) \in P_2P_4\ldots P_{2n}(X)$. Putting $x = x_{2n}$ and $y = u$ in inequality (1), we have

$$\psi\left(\begin{array}{c} {}_{\delta}F_{Ax_{2n},Bu}(t), F_{P_1P_3\ldots P_{2n-1}x_{2n},P_2P_4\ldots P_{2n}u}(t),\\ {}_{\delta}F_{Ax_{2n},P_1P_3\ldots P_{2n-1}x_{2n}}(t), {}_{\delta}F_{Bu,P_2P_4\ldots P_{2n}u}(t)\end{array}\right) \geqslant 0,$$

$$\psi\left(\begin{array}{c} {}_{\delta}F_{y_{2n},Bu}(t), F_{y_{2n-1},P_2P_4\ldots P_{2n}u}(t),\\ F_{y_{2n-1},y_{2n}}(t), {}_{\delta}F_{Bu,z}(t)\end{array}\right) \geqslant 0,$$

as $n \to \infty$, we have

$$\psi\left({}_{\delta}F_{z,Bu}(t), F_{z,z}(t), F_{z,z}(t), {}_{\delta}F_{Bu,z}(t)\right) \geqslant 0.$$

Using (R-1), we have ${}_{\delta}F_{z,Bu}(t) > 1$ for all $t > 0$, which contradicts. Hence $z = Bu$. Therefore, $Bu = \{z\} = \{P_2P_4 \dots P_{2n}u\}$. Since $(B, P_2P_4 \dots P_{2n})$ is weakly compatible pair we have $B(P_2P_4 \dots P_{2n}u) = (P_2P_4 \dots P_{2n})Bu$, hence $Bz = \{P_2P_4 \dots P_{2n}z\}$. Putting $x = x_{2n}$ and $y = z$ in inequality (1), we have

$$\psi\begin{pmatrix} {}_{\delta}F_{Ax_{2n},Bz}(t), F_{P_1P_3\dots P_{2n-1}x_{2n},P_2P_4\dots P_{2n}z}(t), \\ {}_{\delta}F_{Ax_{2n},P_1P_3\dots P_{2n-1}x_{2n}}(t), {}_{\delta}F_{Bz,P_2P_4\dots P_{2n}z}(t) \end{pmatrix} \geqslant 0,$$

$$\psi\begin{pmatrix} F_{y_{2n},P_2P_4\dots P_{2n}z}(t), F_{y_{2n-1},P_2P_4\dots P_{2n}z}(t), \\ F_{y_{2n-1},y_{2n}}(t), F_{P_2P_4\dots P_{2n}z,P_2P_4\dots P_{2n}z}(t) \end{pmatrix} \geqslant 0.$$

Taking limit as $n \to \infty$, we have

$$\psi\left(F_{z,P_2P_4\dots P_{2n}z}(t), F_{y_{2n-1},P_2P_4\dots P_{2n}z}(t), 1, 1\right) \geqslant 0,$$

which contradicts (R-2). Hence $z = P_2P_4 \dots P_{2n}z$. Therefore, $Bz = \{P_2P_4 \dots P_{2n}z\} = \{z\}$. Since $B(X) \subseteq P_1P_3 \dots P_{2n-1}(X)$, there exists $v \in X$ such that $\{P_1P_3 \dots P_{2n-1}v\} = Bz = \{P_2P_4 \dots P_{2n}z\} = \{z\}$. Putting $x = v$ and $y = z$ in inequality (1), we have

$$\psi\begin{pmatrix} {}_{\delta}F_{Av,Bz}(t), F_{P_1P_3\dots P_{2n-1}v,P_2P_4\dots P_{2n}z}(t), \\ {}_{\delta}F_{Av,P_1P_3\dots P_{2n-1}v}(t), {}_{\delta}F_{Bz,P_2P_4\dots P_{2n}z}(t) \end{pmatrix} \geqslant 0$$

$$\psi\left({}_{\delta}F_{Av,z}(t), F_{z,z}(t), {}_{\delta}F_{Av,z}(t), F_{z,z}(t)\right) \geqslant 0,$$

or

$$\psi\left({}_{\delta}F_{Av,z}(t), 1, {}_{\delta}F_{Av,z}(t), 1\right) \geqslant 0.$$

Using (R-1), we get ${}_{\delta}F_{Av,z}(t) > 1$ for all $t > 0$, which contradicts. Hence, $Av = \{z\}$. Since $Av = \{P_1P_3 \dots P_{2n-1}v\}$ and the pair $(A, P_1P_3 \dots P_{2n-1})$ is weakly compatible, we obtain $Az = A(P_1P_3 \dots P_{2n-1}v) = (P_1P_3 \dots P_{2n-1})Av = \{P_1P_3 \dots P_{2n-1}z\}$. Putting $x = z$ and $y = x_{2n+1}$ in inequality (1), we get

$$\psi\begin{pmatrix} {}_{\delta}F_{Az,Bx_{2n+1}}(t), F_{P_1P_3\dots P_{2n-1}z,P_2P_4\dots P_{2n}x_{2n+1}}(t), \\ {}_{\delta}F_{Az,P_1P_3\dots P_{2n-1}z}(t), {}_{\delta}F_{Bx_{2n+1},P_2P_4\dots P_{2n}x_{2n+1}}(t) \end{pmatrix} \geqslant 0,$$

$$\psi\left({}_{\delta}F_{Az,y_{2n+1}}(t), F_{z,y_{2n}}(t), {}_{\delta}F_{Az,z}(t), {}_{\delta}F_{y_{2n+1},y_{2n}}(t)\right) \geqslant 0.$$

Taking limit as $n \to \infty$, we have

$$\psi\left({}_{\delta}F_{Az,z}(t), F_{z,z}(t), {}_{\delta}F_{Az,z}(t), {}_{\delta}F_{z,z}(t)\right) \geqslant 0$$

$$\psi\left({}_{\delta}F_{Az,z}(t), 1, {}_{\delta}F_{Az,z}(t), 1\right) \geqslant 0.$$

Using (R-1), we get ${}_{\delta}F_{Az,z}(t) > 1$ for all $t > 0$, which contradicts. Hence, $Az = \{P_1P_3 \dots P_{2n-1}z\} = \{z\}$. Therefore, we get $Az = Bz = \{P_1P_3 \dots P_{2n-1}z\} = \{P_2P_4 \dots P_{2n}z\} = \{z\}$. Now we show that z is the fixed point of all the component mappings. Putting $x = P_3 \dots P_{2n-1}z, y = z, P_1' = P_1P_3 \dots P_{2n-1}$ and

$P_2^{'} = P_2P_4 \dots P_{2n}$ in inequality (1), we have

$$\psi\left(\begin{array}{c} \delta F_{AP_3\dots P_{2n-1}z,Bz}(t), F_{P_1^{'}P_3\dots P_{2n-1}z,P_2^{'}z}(t), \\ \delta F_{AP_3\dots P_{2n-1}z,P_1^{'}P_3\dots P_{2n-1}z}(t), \delta F_{Bz,P_2^{'}z}(t) \end{array}\right) \geqslant 0,$$

$$\psi\left(\begin{array}{c} F_{P_3\dots P_{2n-1}z,z}(t), F_{P_3\dots P_{2n-1}z,z}(t), \\ F_{P_3\dots P_{2n-1}z,P_3\dots P_{2n-1}z}(t), F_{z,z}(t) \end{array}\right) \geqslant 0,$$

$$\psi\left(F_{P_3\dots P_{2n-1}z,z}(t), F_{P_3\dots P_{2n-1}z,z}(t), 1, 1\right) \geqslant 0,$$

which contradicts (R-2). Hence, $P_3 \dots P_{2n-1}z = z$. Therefore, $P_1z = z$. Continuing this procedure, we have

$Az = \{P_1z\} = \{P_3z\} = \dots = \{P_{2n-1}z\} = \{z\}$.

Similarly, if we put $x = z, y = P_4 \dots P_{2n}z, P_1^{'} = P_1P_3 \dots P_{2n-1}$ and $P_2^{'} = P_2P_4 \dots P_{2n}$ in inequality (1), we get $P_4 \dots P_{2n}z = z$. Hence, $P_2z = z$. Continuing this procedure, we get

$Bz = \{P_2z\} = \{P_4z\} = \dots = \{P_{2n}z\} = \{z\}$.

Therefore z is a unique common fixed point of $P_1, P_2, \dots, P_{2n}, A$ and B.

The proof is similar when $P_1P_3 \dots P_{2n-1}(X)$ is assumed to be a closed subset of X.

Uniqueness: Let $w(\neq z)$ be another common fixed point of $P_1, P_2, \dots, P_{2n}, A$ and B. Putting $x = z$ and $y = w$ in inequality (1), we have

$$\psi\left(\begin{array}{c} \delta F_{Az,Bw}(t), F_{P_1P_3\dots P_{2n-1}z,P_2P_4\dots P_{2n}w}(t), \\ \delta F_{Az,P_1P_3\dots P_{2n-1}z}(t), \delta F_{Bw,P_2P_4\dots P_{2n}w}(t) \end{array}\right) \geqslant 0,$$

and so

$$\psi\left(F_{z,w}(t), F_{z,w}(t), F_{z,z}(t), F_{w,w}(t)\right) \geqslant 0,$$

or

$$\psi\left(F_{z,w}(t), F_{z,w}(t), 1, 1\right) \geqslant 0,$$

which contradicts (R-2), we get $F_{z,w}(t) = 1$, we have, $z = w$. Therefore, z is a unique common fixed point of $P_1, P_2, \dots, P_{2n}, A$ and B. ■

REMARK 4.2 Theorem 4.1 improves and extends the result of Chen and Chang [6] to even number of mappings. In [6], Chen and Chang proved a common fixed point theorem for four single-valued functions and two set-valued functions in complete Menger space using compatibility. Our main result is proved for even number of single-valued and two set-valued mappings using weak compatibility without any requirement of continuity of the involved mappings.

By setting $P_1P_3 \dots P_{2n-1} = S$ and $P_2P_4 \dots P_{2n} = T$ in Theorem 4.1, we get the following result for four mappings.

Corollary 4.3 *Let $(X, \mathcal{F}, *)$ be a complete Menger space where $*$ is a continuous t-norm. Let $S, T : X \to X$ be single-valued mappings and let $A, B : X \to \mathcal{B}(X)$ two set-valued mappings. If the following conditions are satisfied:*

(1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$;

(2) One of $S(X)$ and $T(X)$ is a closed subset of X;

(3) The pairs (A, S) and (B, T) are weakly compatible;

(4) There exists $\psi \in \Psi$ such that

$$\psi\left({}_{\delta}F_{Ax,By}(t), F_{Sx,Ty}(t), {}_{\delta}F_{Ax,Sx}(t), {}_{\delta}F_{By,Ty}(t)\right) \geqslant 0, \tag{3}$$

for all $x, y \in X$ and $t > 0$. Then there exists a point $z \in X$ such that $\{z\} = \{Sz\} = \{Tz\} = Az = Bz$.

Now, we give an example which illustrates Corollary 4.3.

Example 4.4 Let $X = [0, 2]$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$ define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, *)$ be a complete Menger space, where $*$ is defined as $a * b = ab$ for all $a, b \in [0, 1]$. Let $\psi : [0, 1]^4 \to \mathbb{R}$ be defined as in Example 3.1 and define A, B, S and $T : X \to X$ by

$$A(x) = \begin{cases} \{1\}, & \text{if } x \in [0, 1]; \\ \{\frac{3}{2}\}, & \text{if } x \in (1, 2]. \end{cases} \quad B(x) = \begin{cases} \{1\}, & \text{if } x \in [0, 1]; \\ \{\frac{6}{5}\}, & \text{if } x \in (1, 2]. \end{cases}$$

$$S(x) = \begin{cases} \frac{3-x}{2}, & \text{if } x \in [0, 1]; \\ \frac{x+1}{2}, & \text{if } x \in (1, 2]. \end{cases} \quad T(x) = \begin{cases} 2 - x, & \text{if } x \in [0, 1]; \\ 1, & \text{if } x \in (1, 2]. \end{cases}$$

It is clear that $A(X) = \{1, \frac{3}{2}\} \subseteq T(X) = [1, 2]$, $B(X) = \{1, \frac{6}{5}\} \subseteq S(X) = [1, \frac{3}{2}]$. Here $S(X)$ and $T(X)$ are closed subsets of X. Then A, B, S and T satisfy all the conditions of Corollary 4.3 and have a unique common fixed point $1 \in X$ i.e. $\{1\} = \{S(1)\} = \{T(1)\} = A(1) = B(1)$. It may be noted in this example that the pairs (A, S) and (B, T) commute at coincidence point $1 \in X$. So the pairs (A, S) and (B, T) are weakly compatible. Now we show that the pairs (A, S) and (B, T) are not compatible, let us consider a sequence $\{x_n\}$ defined as $\{x_n\} = \{1 - \frac{1}{n}\}$ where $n \geqslant 1$, then $x_n \to 1$ as $n \to \infty$. Then $Ax_n, \{Sx_n\} \to \{1\}$ as $n \to \infty$ but $F_{ASx_n,SAx_n}(t) \to \frac{2t}{2t+1} \neq 1$ as $n \to \infty$. Thus the pair (A, S) is not compatible. Also $Bx_n, \{Tx_n\} \to \{1\}$ as $n \to \infty$ but $F_{BTx_n,TBx_n}(t) \to \frac{5t}{5t+1} \neq 1$ as $n \to \infty$. So the pair (B, T) is not compatible. All the mappings involved in this example are discontinuous even at the common fixed point 1.

By setting $A = B$ and $S = T$ in Corollary 4.3, we get the following result.

Corollary 4.5 *Let $(X, \mathcal{F}, *)$ be a complete Menger space where $*$ is a continuous t-norm. Let $S : X \to X$ be a single-valued mapping and let $A : X \to \mathcal{B}(X)$ be a set-valued mapping. If the following conditions are satisfied:*

(1) $A(X) \subseteq S(X)$;

(2) $S(X)$ is a closed subset of X;

(3) The pair (A, S) is weakly compatible;

(4) There exists $\psi \in \Psi$ such that

$$\psi\left({}_{\delta}F_{Ax,Ay}(t), F_{Sx,Sy}(t), {}_{\delta}F_{Ax,Sx}(t), {}_{\delta}F_{Ay,Sy}(t)\right) \geqslant 0, \tag{4}$$

for all $x, y \in X$ and $t > 0$. Then there exists a point $z \in X$ such that $\{z\} = \{Sz\} = Az$.

Remark 4.6 The conclusions of Theorem 4.1, Corollary 4.3 and Corollary 4.5 remain true if the implicit function is replaced by one of the implicit functions as defined in Examples 3.1–3.6 for all distinct $x, y \in X$. Also, it is noted that the results obtained by using various implicit functions are new for single-valued mappings and set-valued mappings in complete Menger spaces.

Acknowledgement. The authors are thankful to Professor Dr. Yeol Je Cho for his paper [21].

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(Received: 11.10.2011)