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Compact operators on Musielak–Orlicz spaces

Abstract. Let $L^\varphi(\nu)$ be a Musielak–Orlicz space over a non-atomic σ-finite measure space (S, Σ, ν), determined by a Musielak–Orlicz function φ: $\boldsymbol{R}_+ \times S \to \boldsymbol{R}_+$, and let $L_a^\varphi(\nu)$ be its subspace consisting of ν-continuous elements. It is shown that every compact linear operator from $L_a^\varphi(\nu)$ into any complete topological vector space factors through the inclusion map $L_a^\varphi(\nu) \hookrightarrow L_a^{\hat\varphi}(\nu)$ where $\hat\varphi$ is the convex minorant of φ. It follows that a non-zero compact operator exists on $L_a^\varphi(\nu)$ if and only if

$$\liminf_{r\to\infty} r^{-1}\varphi(r, s) > 0$$

on a set of positive measure. Also, the Mackey topology of $L_a^\varphi(\nu)$ is the topology induced from $L_a^{\hat\varphi}(\nu)$. This extends some earlier results of N. J. Kalton concerning ordinary Orlicz function spaces.

1. Musielak–Orlicz spaces. Let (S, Σ, ν) be a positive measure space and $L^0(\nu)$ the linear space of all ν-equivalence classes of measurable scalar-valued functions on S. Also, let φ be a Musielak–Orlicz function, by which we mean here a function φ: $\boldsymbol{R}_+ \times S \to \boldsymbol{R}_+$ satisfying the following conditions: $\varphi(r, \cdot)$ is measurable for each $r \in \boldsymbol{R}_+$, and for all $s \in S$ the function $\varphi(\cdot, s)$ is nondecreasing and left-continuous on $(0, \infty)$, continuous at $r = 0$ and $\varphi(0, s) = 0$. Using φ, we define a functional

$$m_\varphi\colon L^0(\nu) \to \bar{\boldsymbol{R}}_+$$

(called the *modular generated by* φ) by

$$m_\varphi(x) = \int_S \varphi(|x(s)|, s)\, d\nu(s).$$

The Musielak–Orlicz space determined by φ is the linear space

$$L^\varphi(\nu) = \{x \in L^0(\nu)\colon\ m_\varphi(rx) < \infty \text{ for some } r > 0\}$$

equipped with the complete semimetrizable linear topology λ_φ defined by the F-seminorm

$$\|x\|_\varphi = \inf\{r > 0\colon\ m_\varphi(r^{-1}x) \leqslant r\}.$$

The sets $rB_\varphi(\varepsilon)$, where $r, \varepsilon > 0$ and

$$B_\varphi(\varepsilon) = \{x\colon\ m_\varphi(x) \leqslant \varepsilon\},$$

form a base of neighbourhoods of 0 in $(L^\varphi(v), \lambda_\varphi)$. Moreover, if $x, x_n \in L^\varphi(v)$ $(n \in N)$, then $x_n \to x$ (λ_φ) if and only if $m_\varphi(r(x-x_n)) \to 0$ as $n \to \infty$, $\forall r > 0$

The topology λ_φ is Hausdorff (and so $L^\varphi(v)$ is an F-space, i.e., a complete metrizable topological vector space) if and only if

$$\lim_{r\to\infty} \varphi(r, s) > 0 \quad \text{for a.a. } s \in S;$$

we shall say that φ is a *strict Musielak–Orlicz function* in this case.

Let

$$L_a^\varphi(v) = \{x \in L^\varphi(v)\colon\ m_\varphi(rx) < \infty \text{ for all } r > 0\}.$$

Then $L_a^\varphi(v)$ is a closed subspace of $L^\varphi(v)$, and if $x \in L^\varphi(v)$, then the following are equivalent:

(a) $x \in L_a^\varphi(v)$;

(b) x is order-continuous: $0 \leqslant x_n \leqslant |x|$, $x_n \downarrow 0 \Rightarrow \|x_n\|_\varphi \to 0$;

(c) x is v-continuous: $\Sigma \ni A_n \downarrow$, $v(\bigcap_n A_n) = 0 \Rightarrow \|x\chi_{A_n}\|_\varphi \to 0$;

(d) $\lim_{v(E)\to 0} \|x\chi_E\|_\varphi = 0$ and $\forall \varepsilon > 0$, $\exists A \in \Sigma$, $v(A) < \infty$: $\|x\chi_{S\setminus A}\|_\varphi \leqslant \varepsilon$.

If φ is convex, i.e., $\varphi(\cdot, s)$ is convex for a.a. $s \in S$, then the topology λ_φ is also defined by the seminorm

$$|||x|||_\varphi = \inf\{r > 0\colon\ m_\varphi(r^{-1}x) \leqslant 1\}.$$

Thus if φ is in addition strict, then $L_a^\varphi(v)$ is a Banach space.

We shall denote by $\hat{\varphi}$ the convex minorant of φ, i.e., the Musielak–Orlicz function $\hat{\varphi}\colon \boldsymbol{R}_+ \times S \to \boldsymbol{R}_+$ such that, for every $s \in S$, $\hat{\varphi}(\cdot, s)$ is the largest convex function smaller than $\varphi(\cdot, s)$ on $\boldsymbol{R}_+$. [Measurability of $\hat{\varphi}(r, \cdot)$ follows from the fact that $\hat{\varphi}(r, s) = \varphi^{**}(r, s)$, where $\varphi^*(r, s) = \sup\{qr - \varphi(q, s)\colon q \in \boldsymbol{Q}_+\}$ ($\varphi^*(\cdot, s)$ is the conjugate of $\varphi(\cdot, s)$ in the sense of Young) and $\varphi^{**} = (\varphi^*)^*$.]

For more information on Musielak–Orlicz spaces, see [8], [10] and [12]; an abstract characterization of such spaces can be found in [11], [7] and especially [12].

The following theorem from [2] will be of key importance in the proof of our main result in the next Section.

THEOREM 1. *Let (S, Σ, v) be a σ-finite measure space and L a linear subspace of $L^0(v)$, which is assumed to be solid, i.e., if $x \in L^0(v)$, $y \in L$ and $|x| \leqslant |y|$, then $x \in L$.*

Suppose that a functional $m: L \to \bar{\boldsymbol{R}}_+$ satisfies the following conditions:
$(m1)$ $m(0) = 0$;
$(m2)$ $x, y \in L$, $|x| \leqslant |y| \Rightarrow m(x) \leqslant m(y)$: *in particular*, $m(x) = m(|x|)$:
$(m3)$ $x, x_n \in L$ $(n \in \boldsymbol{N})$ *and* $0 \leqslant x_n \uparrow x \Rightarrow m(x_n) \to m(x)$;
$(m4)$ $x, y \in L$, $|x| \wedge |y| = 0 \Rightarrow m(x+y) = m(x) + m(y)$.

Then there exists a function ψ: $\boldsymbol{R}_+ \times S \to \bar{\boldsymbol{R}}_+$ such that $\psi(r, \cdot)$ is measurable for every $r \in \boldsymbol{R}_+$, $\psi(\cdot, s)$ is nondecreasing and left-continuous on $(0, \infty)$ and $\psi(0, s) = 0$ for a.a. $s \in S$, and

$$m(x) = \int_S \psi(|x(s)|, s)\, dv(s), \qquad \forall x \in L.$$

Remark 1. Additional properties of m imply corresponding properties of the representing kernel ψ. Thus, for instance,
(a) if m is $\boldsymbol{R}_+$-valued, then $\psi(r, s) < \infty$ for a.a. $s \in S$ and all $r \in \boldsymbol{R}_+$;
(b) if m is convex, then so is ψ.

2. Compact linear operators on $L_a^\varphi(v)$. Throughout the remainder of this paper we shall assume that the measure space (S, Σ, v) is nonatomic and σ-finite and that φ: $\boldsymbol{R}_+ \times S \to \boldsymbol{R}_+$ is a strict Musielak–Orlicz function. (See, however, Remark 3.) We shall write λ_φ for the topology on $L_a^\varphi = L_a^\varphi(v)$ induced from $L^\varphi = L^\varphi(v)$, and write $B_\varphi(\varepsilon)$ instead of $B_\varphi(\varepsilon) \cap L_a^\varphi$. If $x \in L_a^\varphi$, we write

$$N_x = \{y \in L_a^\varphi\colon |y| \leqslant |x|\}.$$

Finally, if $h \in L^\infty = L^\infty(v)$, we let M_h denote the multiplication operator $x \to hx$; clearly, it maps continuously and linearly L_a^φ into itself, and $M_h(N_x) \subset N_x$ for all $x \in L_a^\varphi$ provided that $\|h\|_\infty \leqslant 1$.

Proposition 1. *Let Y be a topological vector space, T: $L_a^\varphi \to Y$ a continuous linear operator, and u a strictly positive function in L_a^φ (i.e., a weak order unit of L_a^φ).*

Then the following statements are equivalent:
(a) *$T(N_x)$ is precompact for every $x \in L_a^\varphi$.*
(b) *$T(N_u)$ is precompact.*
(c) *$T(F)$ is precompact for every bounded uniformly v-continuous subset F of L_a^φ.*

Note. The latter assumption on F in (c) means that if $\Sigma \ni A_n \downarrow$ and $v(\bigcap_n A_n) = 0$, then $\|x\chi_{A_n}\|_\varphi \to 0$ uniformly for $x \in F$. Moreover, if F has this property, then it is bounded in L_a^φ if and only if it is bounded in L^0 (with the topology of convergence in measure on sets of finite measure).

Proof. (b) $\Rightarrow$ (c): Let V be a neighbourhood of 0 in Y, and choose $\delta > 0$ so that $Tx \in V$ whenever $x \in L_a^\varphi$ and $\|x\|_\varphi \leqslant \delta$. Since v is σ-finite and F

is uniformly ν-continuous, we may find $A \in \Sigma$ such that $\nu(A) < \infty$ and $\|x\chi_{S\setminus A}\|_\varphi \leqslant \delta$, $\forall x \in F$. Next, L^0-boundedness of F implies that $\nu(\{s \in A: |x(s)| > ku(s)\}) \to 0$ as $k \to \infty$, uniformly for $x \in F$. Therefore, by uniform ν-continuity of F again, we may find k so large that, denoting $A_x = \{s \in A: |x(s)| > ku(s)\}$, we have $\|x\chi_{A_x}\|_\varphi \leqslant \delta$, $\forall x \in F$. Now, for every $x \in F$, we have $\bar{x} = x\chi_{A\setminus A_x} \in N_{ku}$, where $T(N_{ku})$ is precompact by (b), and

$$Tx - T\bar{x} = T(x\chi_{A_x}) + T(x\chi_{S\setminus A}) \in V + V.$$

Thus for every neighbourhood U of 0 in Y there is a precompact subset P of Y such that $T(F) \subset P + U$, and so $T(F)$ itself must be precompact.

A continuous linear operator T: $L_a^\varphi \to Y$ such that $T(N_x)$ is precompact for every x in L_a^φ will be called *K-compact*. Evidently, if T is compact (i.e., maps a neighbourhood of 0 to a precompact set), then it is K-compact.

Our main result extends Theorem 2.1 in Kalton's paper [4], and is obtained by a slight modification of his argument.

THEOREM 2. *If T: $L_a^\varphi \to Y$ is a K-compact operator, where Y is a topological vector space, then T is continuous when the topology λ_φ of L_a^φ is replaced by the topology induced from $L_a^{\hat\varphi} \supset L_a^\varphi$.*

In consequence, if Y is a complete Hausdorff TVS, then T factors as follows

$$\begin{tikzcd} L_a^\varphi \arrow[rr, "T"] \arrow[dr, "J"'] & & Y \\ & L_a^{\hat\varphi} \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

where J is the natural inclusion map and $\hat{T}$ is a K-compact operator.

Proof. Let γ be the initial topology on L_a^φ for the family of all K-compact operators T: $L_a^\varphi \to Y$, i.e., the weakest linear topology under which all these operators are continuous. It follows easily that if A: $L_a^\varphi \to L_a^\varphi$ is a linear map such that TA is K-compact whenever T: $L_a^\varphi \to Y$ is K-compact, then A: $(L_a^\varphi, \gamma) \to (L_a^\varphi, \gamma)$ is continuous. In particular, using the observations preceding Proposition 1, we see that M_h: $(L_a^\varphi, \gamma) \to (L_a^\varphi, \gamma)$ is continuous for every $h \in L^\infty$. Moreover, it is clear that N_x is γ-precompact for every $x \in L_a^\varphi$.

Now, let β be the linear topology on L_a^φ for which the sets $\overline{rB_\varphi(\varepsilon)}^\gamma$, where $r, \varepsilon > 0$, form a base of neighbourhoods of 0. Clearly $\gamma \subset \beta \subset \lambda_\varphi$. We are going to prove that:

(*) There exists a Musielak–Orlicz function ψ: $\boldsymbol{R}_+ \times S \to \boldsymbol{R}_+$ such that $\psi \leqslant \varphi$ and $\beta = \lambda_\psi | L_a^\varphi$, and

(**) for a.a. $s \in S$ the function $\psi(\cdot, s)$ is convex.

Define

$$m\colon L_a^\varphi \to \boldsymbol{R}_+$$

by

$$m(x) = \inf\{\varepsilon > 0:\ x \in \overline{B_\varphi(\varepsilon)}^{\gamma}\}.$$

Evidently $m \leqslant m_\varphi$. Since, for every $\varepsilon > 0$,

$$\{x \in L_a^\varphi:\ m(x) \leqslant \varepsilon\} = \bigcap_{\eta > \varepsilon} \overline{B_\varphi(\eta)}^{\gamma},$$

m is γ-lower-semicontinuous on L_a^φ. Now, observing that

$$\overline{B_\varphi(\varepsilon)}^{\gamma} \subset \{x \in L_a^\varphi:\ m(x) \leqslant \varepsilon\} \subset \overline{B_\varphi(\eta)}^{\gamma}$$

if $0 < \varepsilon < \eta$, we infer that the sets $r\{x \in L_a^\varphi:\ m(x) \leqslant \varepsilon\}$ $(r, \varepsilon > 0)$ form a base at 0 for β.

Thus in order to prove $(*)$ it suffices to find a Musielak–Orlicz function ψ such that $m = m_\varphi | L_a^\varphi$. This will follow from Theorem 1 if we show that our functional m satisfies conditions $(m1)$–$(m4)$, with $L = L_a^\varphi$. Since this can be done by a verbatim repetition of the arguments used in [4], Lemma 2.1, we omit them here. Thus, applying Theorem 1 and Remark 1(a), we get a representing function ψ: $\boldsymbol{R}_+ \times S \to \boldsymbol{R}_+$ possessing properties stated in Theorem 1 and such that $m = m_\psi$ on L_a^φ. Moreover, $m \leqslant m_\varphi$ is easily seen to imply $\psi(r, s) \leqslant \varphi(r, s)$ for a.a. $s \in S$ and all $r \in \boldsymbol{R}_+$; in particular, $\psi(\cdot, s)$ is continuous at $r = 0$ for a.a. $s \in S$. This proves $(*)$.

Now we pass to $(**)$. In view of Remark 1 (b) it suffices to prove that m is convex or, as easily seen, that the function $r \to m(rx)$: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ is convex for every $x \in L_a^\varphi$.

Fix an $x \in L_a^\varphi$. Then there exists a countably generated σ-subalgebra $\Sigma_x \subset \Sigma$ such that the measure space $(S, \Sigma_x, \nu | \Sigma_x)$ is nonatomic and x as well as all the functions $\varphi(r, \cdot)$ are Σ_x-measurable. We may also assume that $\nu(S) < \infty$ (because it is enough to consider functions x whose support is of finite measure). In view of the well-known Carathéodory theorem on isomorphic measure spaces, we may therefore assume that our measure space (S, Σ, ν) is simply the interval $[0, 1]$ with Lebesgue measurable subsets and Lebesgue measure.

Let (r_n) be the sequence of Rademacher functions on $[0, 1]$, i.e., a sequence of independent random variables such that $\nu\{r_n = 1\} = \nu\{r_n = -1\} = \frac{1}{2}$. Note that $r_n x \in N_x$, $\forall n \in \boldsymbol{N}$. We claim that there exist nets $(i(\alpha))_{\alpha \in A}$ and $(j(\alpha))_{\alpha \in A}$ in $\boldsymbol{N}$ such that

$$j(\alpha) > i(\alpha) \to \infty \quad \text{and} \quad (r_{i(\alpha)} - r_{j(\alpha)})x \to 0 \ (\gamma).$$

Consider the sequence $((r_n x, n))_{n \in \boldsymbol{N}}$ in the precompact space $(N_x, \gamma) \times \bar{\boldsymbol{R}}$, and let A be the directed set $\mathscr{U} \times \boldsymbol{N}$, where $\mathscr{U}$ is a base at 0 for γ in L_a^φ. Then for each $\alpha = (U, k)$ in A there are $i(\alpha), j(\alpha) \in \boldsymbol{N}$ such that $j(\alpha) > i(\alpha) \geqslant k$ and $r_{i(\alpha)} x - r_{j(\alpha)} x \in U$. This proves our claim.

For every $\alpha \in A$ denote

$$E_\alpha^0 = \{s\colon r_{i(\alpha)} - r_{j(\alpha)}(s) = 0\},$$
$$E_\alpha^+ = \{s\colon r_{i(\alpha)} - r_{j(\alpha)}(s) = 2\},$$
$$E_\alpha^- = \{s\colon r_{i(\alpha)} - r_{j(\alpha)}(s) = -2\}.$$

Then S is the disjoint union of these sets and $\nu(E_\alpha^0) = \frac{1}{2}$, $\nu(E_\alpha^+) = \nu(E_\alpha^-) = \frac{1}{4}$. Fix $0 < a \leqslant b$ and set

$$x_\alpha = bx + \tfrac{1}{2} a (r_{i(\alpha)} - r_{j(\alpha)}) x$$
$$= [b\chi_{E_\alpha^0} + (b+a)\chi_{E_\alpha^+} + (b-a)\chi_{E_\alpha^-}] x.$$

Then

$$m(x_\alpha) = m(bx\chi_{E_\alpha^0}) + m((b+a)x\chi_{E_\alpha^+}) + m((b-a)x\chi_{E_\alpha^-})$$

and since $x_\alpha \to bx$ (γ) and m is γ-lower-semicontinuous, we obtain

$$(+) \qquad m(bx) \leqslant \liminf_\alpha m(x_\alpha).$$

On the other hand, since

$$\chi_{E_\alpha^0} \to \tfrac{1}{2}\chi_S, \quad \chi_{E_\alpha^+} \to \tfrac{1}{4}\chi_S, \quad \chi_{E_\alpha^-} \to \tfrac{1}{4}\chi_S \quad \text{in } \sigma(L^\infty, L^1),$$

we have

$$m(bx\chi_{E_\alpha^0}) = \int_S \psi(b|x(s)|, s)\chi_{E_\alpha^0}(s)\,d\nu(s) \to \tfrac{1}{2}\int_S \psi(b|x(s)|, s)\,d\nu(s)$$
$$= \tfrac{1}{2} m(bx),$$

and similarly

$$m((b+a)x\chi_{E_\alpha^+}) \to \tfrac{1}{4} m((b+a)x), \qquad m((b-a)x\chi_{E_\alpha^-}) \to \tfrac{1}{4} m((b-a)x).$$

Using $(+)$ we therefore obtain

$$m(bx) \leqslant \tfrac{1}{2} m(bx) + \tfrac{1}{4} m((b+a)x) + \tfrac{1}{4} m((b-a)x),$$

and so

$$m(bx) \leqslant \tfrac{1}{2} m((b+a)x) + \tfrac{1}{2} m((b-a)x).$$

Since we already know that the function $r \to m(rx)$ is left-continuous (by $(m3)$), the latter inequality proves that this function is convex on $\boldsymbol{R}_+$. We have thus verified $(**)$.

Since $\gamma \subset \lambda_\psi | L_a^\varphi$ and $\psi(r, s) \leqslant \hat{\varphi}(r, s)$ for a.a. $s \in S$ and all $r \in \boldsymbol{R}_+$, the first assertion of the theorem follows immediately. Now, if Y is complete, then $T\colon (L_a^\varphi, \lambda_{\hat{\varphi}} | L_a^\varphi) \to Y$ extends to a continuous linear operator $\hat{T}\colon L_a^{\hat{\varphi}} \to Y$ and we obviously have $T = J\hat{T}$. Finally, by an argument somewhat similar to that used in Proposition 1, it is not hard to see that $\hat{T}$ is K-compact.

COROLLARY 1. *The following statements are equivalent.*

(a) L_a^φ *admits a nonzero continuous linear functional.*

(b) L_a^φ *admits a nonzero compact operator with values in some* TVS.

(c) L_a^φ *admits a nonzero K-compact operator with values in some* TVS.

(d) $\liminf_{r\to\infty} r^{-1}\varphi(r, s) > 0$ *is satisfied on a set of positive measure.*

Proof. (a)$\Rightarrow$(b) and (b)$\Rightarrow$(c) are trivial. If (c) is assumed, then the topology γ in the proof of Theorem 2 is indiscrete, hence so are β and $\lambda_{\hat{\varphi}}$. Therefore the (measurable) set $\{s\in S:\ \lim_{r\to\infty}\hat{\varphi}(r, s) = \infty\}$ has positive measure. Since $\hat{\varphi} \leqslant \varphi$ and, by convexity of $\hat{\varphi}$, $r^{-1}\hat{\varphi}(r, s)$ is nondecreasing, this set is certainly contained in $\{s\in S:\ \liminf_{r\to\infty} r^{-1}\varphi(r, s) > 0\}$ (actually these two sets are equal). Thus (c)$\Rightarrow$(d).

(d)$\Rightarrow$(a): If φ satisfies (d), then there exist $E\in\Sigma$ with $\nu(E) > 0$ and constants $r_0 \geqslant 0$ and $c > 0$ such that

$$\varphi(r, s) \geqslant cr \quad \text{for all } s\in E \text{ and } r \geqslant r_0$$

which implies (using obvious notation) that $L_a^\varphi(E) \subset L^1(E)$, where the inclusion map is continuous. It follows that $L_a^\varphi(E)$ has a separating dual space, and this evidently implies (a). (Shortly: (d) implies that $\hat{\varphi}$ is nontrivial, hence, using density of L_a^φ in $(L_a^{\hat{\varphi}}, \lambda_{\hat{\varphi}})$, $\lambda_{\hat{\varphi}}|L_a^\varphi$ is a nontrivial seminormed topology weaker than λ_φ.)

Before proceeding to our next corollary, we recall that if X is a TVS, then the Mackey topology on X is the finest locally convex topology on X yielding the same continuous linear functionals as the original topology of X. If X is semimetrizable, then its Mackey topology is the finest locally convex topology weaker than the original topology. Moreover, in this case the Mackey topology is semimetrizable and it is a unique semimetrizable locally convex topology between the weak topology of X and the original topology. For the description of the Mackey topology on Orlicz sequence and function spaces see [5] (separable case) and [1] (general case); the case of Musielak–Orlicz sequence spaces is investigated in [9].

Applying Theorem 2 when Y is the space of scalars and using the above remarks, we obtain the following

COROLLARY 2. *The Mackey topology* μ_φ *of* L_a^φ *coincides with the seminormed topology* $\lambda_{\hat{\varphi}}|L_a^\varphi$ *induced from* $L_a^{\hat{\varphi}}$; *it is normed if and only if* $\liminf_{r\to\infty} r^{-1}\varphi(r, s) > 0$ *for a.a.* $s\in S$, *and in this case the completion of* $(L_a^\varphi, \mu_\varphi)$ *equals* $L_a^{\hat{\varphi}}$.

Remark 2. The equivalence of (a) and (b) in Corollary 1 does not hold for arbitrary F-spaces: There exist F-spaces whose dual space is trivial and which admit nontrivial compact operators, cf. [6]..

Remark 3. The assumption that the measure space (S, Σ, ν) is σ-finite is, in fact, superfluous in Theorem 2 (and its corollaries). Although we cannot obtain an integral representation of m when ν is arbitrary, we nevertheless have it for $m|L_a^\varphi(E)$, separately for every $E \in \Sigma$ with σ-finite measure. Since, moreover, every $x \in L_a^\varphi$ is easily seen to have a support of σ-finite measure, we deduce that m is convex (hence β is locally convex) and $m \leqslant m_{\hat{\varphi}}$ on L_a^φ. Hence $\beta \subset \lambda_{\hat{\varphi}}|L_a^\varphi$, from which the assertions of Theorem 2 follow as before.

(We could arrive at essentially the same result by using the fact that every Musielak–Orlicz space is "isomodular" to the direct modular sum of Musielak–Orlicz spaces over finite measure spaces or, equivalently, "isomodular" to a Musielak–Orlicz space over the direct sum (in the sense of [3], p. 149) of a family of finite measure spaces, cf. [12], Theorem 5.1. And for such measure spaces the representation of m as m_ψ is available.)

References

[1] L. Drewnowski and M. Nawrocki, *On the Mackey topology of Orlicz sequence spaces*, Arch. Math. 39 (1982), 59–68.

[2] L. Drewnowski and W. Orlicz, *A note on modular spaces X*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys. 16 (1968), 809–814.

[3] D. H. Fremlin, *Topological Riesz spaces and measure theory*, Cambridge 1974.

[4] N. J. Kalton, *Compact and strictly singular operators on Orlicz spaces*, Israel J. Math. 26 (1977), 126–136.

[5] —, *Orlicz sequence spaces without local convexity*, Math. Proc. Cambridge Phil. Soc. 81 (1977), 253–277.

[6] —, *Quotients of F-spaces*, Glasgow Math. J. 19 (1978), 103–108.

[7] P. Kranz and W. Wnuk, *On the representation of Orlicz lattices*, Indagationes Math. 43 (1981), 375–383.

[8] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. 18 (1959), 49–65.

[9] M. Nawrocki, *On the Mackey topology of Musielak–Orlicz sequence spaces*, Comment. Math. 24 (1984), 101–107.

[10] P. Turpin, *Convexités dans les espaces vectoriels topologiques généraux*, Dissertationes Math. 131 (1976).

[11] W. Wnuk, *On a representation theorem for convex Orlicz lattices*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys. 28 (1980), 131–136.

[12] —, *Representations of Orlicz lattices*, Dissertationes Math. 235 (1984).

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