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The density property in algebras of A-valued continuous functions

1. Introduction. Let F denote either the field R of reals or field C of complex numbers. For a topological space X and a locally convex space (a locally convex algebra) A over F let $C(X, A)$ denote the set of all continuous A-valued functions defined on X. The subset of $C(X, A)$ consisting of all bounded functions, i.e., of all functions for which $f(X)$ is a bounded set in A, we denote by $C^*(X, A)$. With respect to the pointwise addition and scalar multiplication of functions, $C(X, A)$ and $C^*(X, A)$ are linear spaces. In particular, when A is an algebra, $C(X, A)$ and $C^*(X, A)$ are also algebras with respect to the pointwise algebraic operations on functions.

Let Q denote the family of all continuous seminorms generating the topology for A and let K denote the set of all compact non-empty subsets of X. We endow the spaces $C(X, A)$ and $C^*(X, A)$ with locally convex topology defined respectively by seminorms $\{p_{k,q}\colon q \in Q, k \in K\}$ and $\{p_q\colon q \in Q\}$, where $p_{k,q}(f) = \sup\{q(f(x))\colon x \in k\}$ and $p_q(f) = \sup\{q(f(x))\colon x \in X\}$. Then $C(X, A)$ coincides with $C^*(X, A)$, if X is a compact space (cf. [11]).

Let now X be a locally compact space. We shall say that $f \in C^*(X, A)$ vanishes at infinity if for any given $\varepsilon > 0$ and $q \in Q$ there exists a compact set $k_{q,\varepsilon} \subset X$ such that $q(f(x)) < \varepsilon$ for each $x \notin k_{q,\varepsilon}$. The subset of $C^*(X, A)$ consisting of all functions which vanish at infinity we shall denote by $C_0(X, A)$ and endow with relative topology of $C^*(X, A)$.

Numerous generalizations of Stone–Weierstrass theorem for $C(X, A)$, $C^*(X, A)$ and $C_0(X, A)$ are known (cf. [22], p. 119). For a compact Hausdorff space X and a locally convex space A these generalizations are considered in [8], [11], [20] and [21]. Moreover, for completely regular space X and a B_0-algebra with involution A the generalization of Stone–Weierstrass theorem is considered in [26] and for a topological space X and a B^*-algebra A — in [2].

Generalizations of Stone–Weierstrass theorem for $C_0(X, A)$ are considered in [16] (when A is a C^*-algebra) in [6] (when A is a B^*-algebra) and with respect to the topology of $C_0(X, A)$ — in [8] and [25].

A generalization of Stone–Weierstrass theorem for $C(X, A)$ when X is not a compact Hausdorff space has been given without proof in [27].

In the present paper in Section 2 the Stone–Weierstrass theorem is generalized for $C(X, A)$ when X is a Hausdorff space and A is a locally convex algebra over F using the Nagata's generalization of Stone–Weierstrass theorem (cf. [17], p. 268). In Section 3 the generalizations of Stone–Weierstrass theorem for $C^*(X, A)$ is considered when either X is a pseudocompact space and A is a normed space (algebra) over F or X is a topological space and A is a finite dimensional normed space (algebra) over F. We prove a generalization of Holladay–Hausner's theorem [13], [15] using the Nel's generalization of Stone–Weierstrass theorem [19].

In Section 4 we consider generalizations of Stone–Weierstrass theorem for $C_0(X, A)$, when A is a locally convex algebra over F. A generalization of Kaplansky's theorem (cf. [16], p. 233) is proved.

The results mentioned above are applied to tensor products in Section 5. A generalization of Bourbaki's result (cf. [7], p. 315) for $C(X, A)$ and $C_0(X, A)$ when A is a locally convex space (algebra) over F, is proved. Moreover, we prove an analogue of Grothendieck's theorem (cf. [14], p. 128) for $C^*(X, A)$ when either X is a pseudocompact space and A is a Banach space over F or X is a topological space and A is a finite dimensional Banach space over F.

In Section 6 we find the conditions, when $C^*(X, A)$ is homomorphic with dense subalgebra of $C^*(Y, B)$ for topological spaces X and Y and Banach algebras A and B with unit. Moreover, it is proved that the algebras (1) $C^*(\beta X, A)$ and $C^*(X, A)$ are isometrically isomorphic if either X is a pseudocompact space and A be a Banach algebra with unit or X is a completely regular T_1-space and A is a finite dimensional Banach algebra with unit.

2. The dense subspaces and subalgebras of $C(X, A)$. Let X be a topological space and A a locally convex algebra over F. By f_a we shall denote the constant functions with value $a \in A$, i.e., $f_a(x) \equiv a$ on X, and by αf_a – the functions $x \to \alpha(x)a$, where $\alpha \in C(X, F)$ and $a \in A$. We have

PROPOSITION 1. *Let X be a topological space and A be a locally convex space over F. If $\mathfrak{A}$ is a linear subspace of $C(X, A)$ which for every $\alpha \in C(X, R)$ and $a \in A$ contains αf_a, then $\mathfrak{A}$ is dense in $C(X, A)$.*

Proof. Let $f \in C(X, A)$, $q \in Q$, $k \in K$ and $\varepsilon > 0$. It is sufficient to show that there exists $g \in \mathfrak{A}$ such that

$$p_{k,q}(g-f) < \varepsilon.$$

(1) By βX we denote the Stone–Čech compactification of space X.

Let $O(a, \varepsilon)$ denote the ε-neighborhood of $a \in A$ defined by q. Then the sets $\{O(a, \varepsilon): a \in f(k)\}$ cover $f(k)$. As $f(k)$ is compact, there exists a finite cover $\{O(a_l, \varepsilon),\ a_l \in f(k),\ l = 1, 2, \ldots, n\}$ of $f(k)$. In view of this, there exist $\mu_1, \mu_2, \ldots, \mu_n \in C(f(k), [0, 1])$ such that $\mu_r(a) = 0$ if $a \notin O(a_r, \varepsilon)$ for each $r = 1, 2, \ldots, n$ and $\sum_{r=1}^{n} \mu_r(a) \equiv 1$ on $f(k)$ (cf. [18]). As every locally convex space is completely regular [14], $f(X)$ is also a completely regular space. Hence μ_r has an extension $\bar{\mu}_r \in C(f(X), R)$ for each $r = 1, 2, \ldots, n$ (cf. [10], p. 43).

Let now $h_r = (\bar{\mu}_r \circ f) f_{a_r}$ for each $r = 1, 2, \ldots, n$. As $a_r \in A$ and $\bar{\mu}_r \circ f \in C(X, R)$ for each $r = 1, 2, \ldots, n$ then $\sum_{r=1}^{n} h_r \in \mathfrak{A}$ by our assumption. Moreover, for each $x \in k$ we have

$$\begin{aligned} q\Big[\Big(\sum_{r=1}^{n} h_r - f\Big)(x)\Big] &= q\Big[\sum_{r=1}^{n} (\bar{\mu}_r \circ f)(x)\big(a_r - f(x)\big)\Big] \\ &\leqslant \sum_{r=1}^{n} (\mu_r \circ f)(x)\, q\big(a_r - f(x)\big) \\ &< \varepsilon \sum_{r=1}^{n} \mu_r\big(f(x)\big) = \varepsilon. \end{aligned}$$

Consequently,

$$p_{k,q}\Big(\sum_{r=1}^{n} h_r - f\Big) < \varepsilon.$$

This completes the proof.

In [12], p. 28, Proposition 1 has been proved for compact space X.

THEOREM 1. *Let X be a Hausdorff space and A be a locally convex algebra over R with unit e_A. If $\mathfrak{A}$ is a subalgebra of $C(X, A)$ such that*

1° *all A-valued constant functions belong to $\mathfrak{A}$ and*

2° *for every pair x, y of distinct points of X there is a function $a_{xy} \in C(X, R)$ which separates the points x and y (i.e., $a_{xy}(x) \neq a_{xy}(y)$) and such that $a_{xy} f_{e_A} \in \mathfrak{A}$, then $\mathfrak{A}$ is dense in $C(X, A)$.*

Proof. Let $e = f_{e_A}$. It is clear that

$$\mathfrak{A}_0 = \{a \in C(X, R): ae \in \mathfrak{A}\}$$

is a subalgebra of $C(X, R)$ which contains a unit and separates the points of X. Hence, by Nagata's generalization of Stone–Weierstrass theorem (cf. [17], p. 286), $\mathfrak{A}_0$ is dense in $C(X, R)$.

Let now β be an arbitrary function in $C(X, R)$. For any given $q \in Q$, $k \in K$ and $\varepsilon > 0$ there exists $a \in \mathfrak{A}_0$ such that

$$p_k(a - \beta) < \frac{\varepsilon}{q(e_A)}$$

(here $p_k(\alpha) = \sup_{x \in k} |\alpha(x)|$ for every $\alpha \in C(X, R)$ and $k \in X$). Since

$$p_{k,q}(\alpha e - \alpha\beta) = p_k(\alpha - \beta) q(e_A) < \varepsilon,$$

βe belongs to the closure $\mathrm{cl}\,\mathfrak{A}$ of $\mathfrak{A}$ with respect to the topology of $C(X, A)$. As $\mathrm{cl}\,\mathfrak{A}$ satisfies the condition of Proposition 1, then $\mathrm{cl}\,\mathfrak{A} = C(X, A)$.

THEOREM 1′. *Let X be a Hausdorff space and A be a locally convex algebra over C with unit e_A. If $\mathfrak{A}$ is a subalgebra of $C(X, A)$ such that*

1° *all A-valued constant functions belong to $\mathfrak{A}$,*

2° *for every pair x, y of distinct points of X there is an $a_{xy} \in C(X, C)$ separating the points x and y and such that $a_{xy} e_A \in \mathfrak{A}$,*

3° *if $a \in C(X, C)$ and $a e_A \in \mathfrak{A}$, then $\bar{a} e_A \in \mathfrak{A}$ (where $\bar{a}$ is the complex-conjugate of the function a),*

then $\mathfrak{A}$ is dense in $C(X, A)$.

Proof. As in Theorem 1, $\mathfrak{A}_0 = \{a \in C(X, C) \colon ae \in \mathfrak{A}\}$ is a subalgebra of $C(X, C)$. Let $\mathfrak{A}_1$ denote the subalgebra of all real-valued functions of $\mathfrak{A}_0$. Since $\mathfrak{A}_1$ contains the unit and by our assumption separates the points of X, then $\mathfrak{A}_1$ is dense in $C(X, R)$ as above. Now $\beta e_A \in \mathrm{cl}\,\mathfrak{A}$ for every $\beta \in C(X, R)$ and by Proposition 1, $\mathrm{cl}\,\mathfrak{A} = C(X, A)$.

3. The dense subspaces and subalgebras of $C^*(X, A)$. Let X be a compact Hausdorff space and A be a locally convex algebra over F. Then Theorem 1 and Theorem 1′ are true also for $C^*(X, A)$. Moreover, we have

PROPOSITION 2. *Let one of the following conditions hold:*

(a) *X is a pseudocompact space and A is a normed space over F;*

(b) *X is a topological space and A is a finite dimensional normed space over F.*

If $\mathfrak{A}$ is a linear subspace of $C^*(X, A)$ which for any $a \in C^*(X, R)$ and $a \in A$ contains af_a, then $\mathfrak{A}$ is dense in $C^*(X, A)$.

Proof. In case (a) $f(X)$ is a compact subset of A (cf. [24], Theorem 2.3) for every $f \in C^*(X, A)$ and in case (b)

$$\{a \in A \colon \|a\|_A \leqslant \|f\|_{C^*(X,A)}\}$$

is a compact subset of A for every $f \in C^*(X, A)$. Therefore, in the same way as in the proof of Proposition 1, we prove that there exists $g \in \mathfrak{A}$ such that

$$\|g - f\|_{C^*(X,A)} \leqslant \varepsilon.$$

Consequently, $\mathfrak{A}$ is dense in $C^*(X, A)$.

THEOREM 2. *Let X be a pseudocompact space* (2) *and A be a normed algebra over F with unit e. If $\mathfrak{A}$ is a subalgebra of $C^*(X, A)$ containing all A-valued constant functions and for any $a \in C^*(X, R)$ the function ae, then $\mathfrak{A}$ is dense in $C^*(X, A)$.*

Proof. It is obvious by Proposition 2.

Holladay [15] and Hausner [13] have generalized the Stone–Weierstrass theorem for $C^*(X, A)$ (assuming that X is a compact Hausdorff space) when, respectively, A is the skew field of real quaternions and the real Cayley–Dickson's type algebra of dimension 2^n ($n > 1$) or real Clifford algebra of dimension 2^n (n even).

Using Proposition 2, we shall prove a generalization of Holladay–Hausner's result.

THEOREM 3. *Let X be a topological space and A be a finite dimensional normed algebra over R with unit e. If $\mathfrak{A}$ is a subalgebra of $C^*(X, A)$ such that*

1° *all A-valued constant functions belong to $\mathfrak{A}$,*

2° *for every pair z_1, z_2 of disjoint zero-sets in X there exists an $a \in C^*(X, R)$ such that $\mathrm{cl}\, a(z_1) \cap \mathrm{cl}\, a(z_2) = \emptyset$ and $ae \in \mathfrak{A}$,*

then $\mathfrak{A}$ is dense in $C^*(X, A)$.

Proof. It is clear that $\mathfrak{A}_0 = \{a \in C^*(X, R)\colon ae \in \mathfrak{A}\}$ is a subalgebra of $C^*(X, R)$ which by our assumption contains the unit and separates the disjoint non-empty zero-sets of X. Hence, by Nel's generalization of Stone–Weierstrass theorem (cf. [19], p. 229), $\mathfrak{A}_0$ is dense in $C^*(X, R)$. Now it is easy to show in the same way as in the proof of Theorem 1 that βe belongs to the closure $\mathrm{cl}\,\mathfrak{A}$ of with respect to the topology of $C^*(X, A)$ for every $\beta \in C^*(X, R)$. Consequently, $\mathrm{cl}\,\mathfrak{A}$ satisfies the condition of Proposition 2(b) and $\mathrm{cl}\,\mathfrak{A} = C^*(X, A)$.

4. The dense subspaces and subalgebras of $C_0(X, A)$. In this section we shall generalize the Stone–Weierstrass theorem for $C_0(X, A)$, when A is a locally convex algebra over F.

PROPOSITION 3. *Let X be a locally compact Hausdorff space and A be a locally convex space over F. If $\mathfrak{A}$ is a linear subspace of $C_0(X, A)$ and $\mathfrak{A}$ contains all the functions af_a, where $a \in C_0(X, R)$, $a \in A$, then $\mathfrak{A}$ is dense in $C_0(X, A)$.*

Proof. Let $f \in C_0(X, A)$, $q \in Q$ and $\varepsilon > 0$. Let $k_{q,\varepsilon}$ denote the compact subset of X such that $q(f(x)) < \varepsilon$ for all $x \notin k_{q,\varepsilon}$, and v_x denote

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(2) The space X is called *pseudocompact* if it is T_1-space and every complex-valued continuous function defined on X is bounded.

a neighborhood of $x \in k_{q,\varepsilon}$ which closure is compact set in X. Then $\{U_{q,\varepsilon}(x)\colon x \in k_{q,\varepsilon}\}$, where

$$U_{q,\varepsilon}(x) = v_x \cap \{x' \in X\colon\ q\big(f(x') - f(x)\big) < \varepsilon\},$$

is an open cover of $k_{q,\varepsilon}$, which contains a finite cover $\{U_{q,\varepsilon}(x_k)\colon x_k \in k_{q,\varepsilon},\ k = 1, 2, \ldots, n\}$.

Let X_∞ denote the one-point compactification of X. As for each $k = 1, 2, \ldots, n$ the sets $U_{q,\varepsilon}(x_k)$ and $X \setminus k_{q,\varepsilon}$ are open in X_∞ and cover X_∞, there exist continuous real-valued positive functions $\mu_1, \mu_2, \ldots, \mu_{n+1}$ on X_∞ such that $p_k(x) = 0$ if $x \notin U_{q,\varepsilon}(x_k)$ for each $k = 1, 2, \ldots, n$ and $\mu_{n+1}(x) = 0$ if $x \in k_{q,\varepsilon}$. Moreover, $\sum\limits_{k=1}^{n+1} \mu_k(x) \equiv 1$ on X_∞ and μ_k vanishes outside of the compact set $\operatorname{cl} U_{q,\varepsilon}(x_k)$ for each $k = 1, 2, \ldots, n$. Consequently, $\mu_k \in C_0(X, R)$ for each $k = 1, 2, \ldots, n$ and by our assumption

$$\sum_{k=1}^{n} h_k \in \mathfrak{A},$$

where $h_k = \mu_k f a_k$ and $a_k = f(x_k)$ for each $k = 1, 2, \ldots, n$. As

$$\begin{aligned} q\Big(\sum_{k=1}^{n} \mu_k(x) - f(x)\Big) &= q\Big[\sum_{k=1}^{n} \mu_k(x)\big(f(x_k) - f(x)\big) - \mu_{n+1}(x) f(x)\Big] \\ &\leqslant \sum_{k=1}^{n} \mu_k(x)\, q\big(f(x_k) - f(x)\big) + \mu_{n+1}(x)\, q\big(f(x)\big) \\ &< \varepsilon \sum_{k=1}^{n+1} \mu_k(x) = \varepsilon \end{aligned}$$

for each $x \in X$, then

$$p_q\Big(\sum_{k=1}^{n} h_k - f\Big) \leqslant \varepsilon.$$

Consequently, $\mathfrak{A}$ is dense in $C_0(X, A)$.

THEOREM 4. *Let X be a locally compact Hausdorff space and A be a locally convex algebra over R with unit e. If $\mathfrak{A}$ is a subalgebra of $C_0(X, A)$ such that*

1° *for every $x \in X$ there is an $a_x \in C(X, R)$ with $a_x(x) \neq 0$ and $a_x e \in \mathfrak{A}$,*

2° *for each pair x, y of distinct points of X there exists an $a_{xy} \in C_0(X, R)$ separating the points x and y and such that $a_{xy} e \in \mathfrak{A}$,*

3° *for every $a \in A$, $a \in C_0(X, R)$, the condition $ae \in \mathfrak{A}$ implies $af_a \in \mathfrak{A}$, then $\mathfrak{A}$ is dense in $C_0(X, A)$.*

Proof. It is clear that $\mathfrak{A}_0 = \{a \in C_0(X, R)\colon ae \in \mathfrak{A}\}$ is a subalgebra of $C_0(X, R)$, which satisfies the conditions of Stone–Weierstrass theorem for $C_0(X, R)$. Hence $\mathfrak{A}_0$ is dense in $C_0(X, R)$.

Let $\beta \in C_0(X, R)$. Then for any given $q \in Q$, $a \in A$ and $\varepsilon > 0$ there exists a $\alpha \in \mathfrak{A}_0$ such that

$$p_x(\alpha - \beta) < \frac{\varepsilon}{q(a)}$$

as

$$p_q(\alpha f_a - \beta f_a) = p_x(\alpha - \beta) q(a) < \varepsilon$$

then by condition (c) βf_a belongs to the closure of $\mathfrak{A}$ with respect to the topology of $C_0(X, A)$. Consequently, by Proposition 3, $\mathfrak{A}$ is dense in $C_0(X, A)$.

In [16], p. 233 (cf. also [18], p. 406), Kaplansky has generalized the Stone–Weierstrass theorem for $C_0(X, A)$ when A is a C^*-algebra. The following theorem is a generalization of Kaplansky's result:

THEOREM 4'. *Let X be a locally compact Hausdorff space and A be a locally convex algebra over C with unit. If $\mathfrak{A}$ is a subalgebra of $C_0(X, A)$ such that*

1° *for every $x \in X$ there is an $\alpha_x \in C(X, R)$ with $\alpha_x(x) \neq 0$ and $\alpha_x e \in \mathfrak{A}$.*

2° *for each pair x, y of distinct points of X there exists an $\alpha_{xy} \in C_0(X, C)$ separating x and y and such that $\alpha_{xy} e \in \mathfrak{A}$,*

3° *for every $a \in A$, $\alpha \in C_0(X, C)$, the condition $\alpha e \in \mathfrak{A}$ implies $\alpha f_a \in \mathfrak{A}$, then $\mathfrak{A}$ is dense in $C_0(X, A)$.*

Proof. In the same way as in Theorem 1', by the Stone–Weierstrass theorem for $C_0(X, R)$ and Proposition 3, we show that $\mathfrak{A}$ is dense in $C_0(X, A)$.

5. Applications to tensor products. Let X be a topological space, A be a linear topological space (a topological algebra) over F and let $\mathfrak{A}$ and $\mathfrak{B}$ be linear subspaces (subalgebras) of $C(X, F)$ and A, respectively. Denote by $\mathfrak{A} \otimes A$ the algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{B}$ and by π the mapping

$$\sum_{r=1}^{n} \alpha_r \otimes b_r \to \sum_{r=1}^{n} \alpha_r f_{b_r},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $b_1, b_2, \ldots, b_n \in \mathfrak{B}$. It is clear that π is a linear injection (respectively, an isomorphism) from $\mathfrak{A} \otimes \mathfrak{B}$ into $C(X, A)$. We have

PROPOSITION 4. *Let X be a Hausdorff space and A be a locally convex space (a locally convex algebra) over F. If $\mathfrak{A}$ and $\mathfrak{B}$ are linear dense subspaces (dense subalgebras) of $C(X, F)$ and A respectively, then π is a linear injection (an isomorphism) from $\mathfrak{A} \otimes A$ into a dense subspace (a dense subalgebra) of $C(X, A)$.*

Proof. It is sufficient to show that $\pi(\mathfrak{A}\otimes\mathfrak{B})$ is dense in $C(X, A)$. Let $f \in C(X, A)$, $q \in Q$, $k \in K$ and $\varepsilon > 0$. Proving Proposition 1, we have shown that there exist $\mu_1, \mu_2, \ldots, \mu_n \in C(X, F)$ and $a_1, a_2, \ldots, a_n \in A$ such that

$$p_{k,q}\Big(f - \sum_{r=1}^{n} \mu_r f_{a_r}\Big) < \varepsilon. \tag{1}$$

(If A is an algebra, it is also true.) In view of our hypothesis, for each $k = 1, 2, \ldots, n$ there exists $\alpha_k \in \mathfrak{A}$ such that

$$p_k(\alpha_k - \mu_k) < \frac{\varepsilon}{\sum\limits_{r=1}^{n} q(a_r)} \tag{2}$$

and an $b_k \in \mathfrak{B}$ such that

$$q(b_k - a_k) < \frac{\varepsilon}{\sum\limits_{k=1}^{n} p_k(\alpha_k)}. \tag{3}$$

As

$$\begin{aligned} f(x) - \sum_{k=1}^{n} \alpha_k(x) b_k \\ &= f(x) - \sum_{k=1}^{n} \mu_k(x) a_k + \sum_{k=1}^{n} \big(\mu_k(x) - \alpha_k(x)\big) a_k + \sum_{k=1}^{n} \alpha_k(x)(a_k - b_k) \end{aligned}$$

for each $x \in X$, then

$$\begin{aligned} q\Big(f(x) - \sum_{k=1}^{n} \alpha_k(x) b_k\Big) \\ &\leqslant p_{k,q}\Big(f - \sum_{k=1}^{n} \mu_k f_{a_k}\Big) + \sum_{k=1}^{n} p_k(\alpha_k - \mu_k) q(a_k) + \sum_{k=1}^{n} p_k(\alpha_k) q(a_k - b_k) \end{aligned}$$

for each $x \in k$. Now by (1), (2) and (3)

$$p_{k,q}\Big(f - \sum_{k=1}^{n} \alpha_k f_{b_k}\Big) < 3\varepsilon.$$

Consequently, $\pi(\mathfrak{A}\otimes\mathfrak{B})$ is dense in $C(X, A)$.

In the case, when $\mathfrak{A} = C(X, F)$ and $\mathfrak{B} = A$ Proposition 4 is known (cf. [9], p. 206).

COROLLARY 1. *Let X be a compact Hausdorff space and A be a locally convex space (a locally convex algebra) over F. If $\mathfrak{A}$ and $\mathfrak{B}$ are linear dense subspaces (dense subalgebras) of $C^*(X, F)$ and A respectively, then π is a linear injection (an isomorphism) from $\mathfrak{A}\otimes\mathfrak{B}$ onto a dense subspace (a dense subalgebra) of $C^*(X, A)$.*

In the case when $A = \mathfrak{B}$ is a normed space over R, Corollary 1 is known (cf. [7], p. 315). Moreover, when A is a Banach algebra $\mathfrak{A} = C^*(X, C)$ and $\mathfrak{B} = A$, Corollary 1 has been proved in [12] and for pseudocompact space X in [3]. When compact space X has a finite covering dimension, then the last result has been generalized in [23] for linear topological space A.

PROPOSITION 5. *Let X be a locally compact Hausdorff space and A be a locally convex space (a locally convex algebra) over F. If $\mathfrak{A}_0$ and $\mathfrak{B}$ are linear dense subspaces (dense subalgebras) of $C_0(X, F)$ and A, respectively, then π is a linear injection (an isomorphism) from $\mathfrak{A}\otimes\mathfrak{B}$ onto a dense subspace (a dense subalgebra) of $C_0(X, A)$.*

Proof is analogous to the proof of Proposition 4.

Let now A be a Banach space (a Banach algebra). By $C^*(X, F)\hat{\hat{\otimes}} A$ we denote the completion of algebraic tensor product $C^*(X, F)\otimes A$ with respect to the weakest tensor product norm $\| \ \|^{\hat{}}$ (cf. [23], p. 355). It is known, that

$$\Big\| \sum_{k=1}^{n} a_k \otimes a_k \Big\|^{\hat{\hat{}}} = \Big\| \pi\Big(\sum_{k=1}^{n} a_k \otimes a_k \Big) \Big\|_{C^*(X, A)}$$

for every element of $C^*(X, F)\otimes A$. Therefore π is a linear isometry (an isometric isomorphism). Since $C^*(X, A)$ is a Banach space, then π has a linear isometric (an isomorphically isometric) extension $\hat{\hat{\pi}}$ from $C^*(X, F)\hat{\hat{\otimes}} A$ onto the closure of $\pi\big(C^*(X, F)\otimes A\big)$ with respect to the topology of $C^*(X, A)$. We shall prove the following analogue of Grothendieck's theorem [11], p. 128.

THEOREM 5. *Let one of the following conditions hold:*

(a) *X is a pseudocompact space and A is a Banach space (Banach algebra) over F;*

(b) *X is a topological space and A is a finite dimensional Banach space (Banach algebra) over F.*

Then $C^*(X, F)\hat{\hat{\otimes}} A$ and $C^*(X, A)$ are linearly isometric spaces (isomorphic and isometric algebras).

Proof. According to the preceding arguments, we must show only that $\pi\big(C^*(X, F)\otimes A\big)$ is dense in $C^*(X, A)$. As $af_a = \pi(a\otimes a) \in \pi\big(C^*(X, F)\otimes A\big)$ for every $a \in C^*(X, R)$ and $a \in A$, then, by Proposition 2, $\pi\big(C^*(X, F)\otimes A\big)$ is dense in $C^*(X, A)$.

6. Homomorphisms from $C^*(X, A)$ onto a dense subalgebra of $C^*(Y, B)$. Let X and Y be two topological spaces, A and B be two Banach algebras, $\varphi\colon A\to B$ be a continuous homomorphism and $\psi\colon Y\to X$ be a continuous mapping. Denote by $F_{\varphi,\psi}$ the mapping $f\to\varphi\circ f\circ\psi$ for every

$f \in C^*(X, A)$. As $F_{\varphi,\psi}(f) \in C^*(Y, B)$ for each $f \in C^*(X, A)$, then $F_{\varphi,\psi}$ is a homomorphism with

$$\ker F_{\varphi,\psi} = \{f \in C^*(X, A)\colon f(x) \in \ker\varphi,\ x \in \psi(Y)\}.$$

The properties of $F_{\varphi,\psi}$ are considered in [5]. In this section as an application of Theorem 2, we have

PROPOSITION 6. *Let X be a topological space, Y be a pseudocompact space, A and B be two Banach algebras with unit, $\varphi\colon A \to B$ be a continuous homomorphism with $\varphi(A)$ dense in B and $\psi\colon Y \to X$ be a continuous mapping. If every $a \in C^*(\psi(Y), R)$ has an extension $\bar{a} \in C^*(X, R)$, then $F_{\varphi,\psi}(C^*(X, A))$ is dense in $C^*(Y, B)$.*

Proof. Let $b \in B$. For any $\varepsilon > 0$ there exists an $a \in A$ such that $\|b - \varphi(a)\|_B < \varepsilon$. As $f_a \in C^*(X, A)$ and

$$\|f_b - F_{\varphi,\psi}(f_a)\|_{C^*(Y,B)} = \|b - \varphi(a)\|_B,$$

then f_b for every $b \in B$ belongs to the closure $\mathrm{cl}\, F_{\varphi,\psi}(C^*(X, A))$ of $F_{\varphi,\psi}(C^*(X, A))$ with respect to the topology of $C^*(Y, B)$.

Let $\beta \in C^*(Y, R)$. Then by our assumption $\beta \circ \psi^{-1} \in C^*(\psi(Y), R)$ has an extension $\overline{\beta \circ \psi^{-1}} \in C^*(X, R)$. As $\overline{\beta \circ \psi^{-1}} f e_A \in C^*(X, A)$ and

$$F_{\varphi,\psi}(\overline{\beta \circ \psi^{-1}} f e_A)(y) = \overline{\beta \circ \psi^{-1}}(\psi(y)) f e_B = \beta f e_B(y)$$

for each $y \in Y$, then $\beta f e_B \in \mathrm{cl}\, F_{\varphi,\psi}(C^*(X, A))$ for each $\beta \in C^*(Y, R)$. Consequently, by Theorem 2,

$$\mathrm{cl}\, F_{\varphi,\psi}(C^*(X, A)) = C^*(Y, B).$$

When either X is a completely regular space and Y is a compact Hausdorff space or X is a metric space and Y is a pseudocompact space, the conditions of Proposition 6 are fulfilled (cf. [10], p. 43).

COROLLARY 2. *Let X be a pseudocompact space, A and B be two Banach algebras with unit and $\varphi\colon A \to B$ be a continuous injective homomorphism. Then $C^*(X, A)$ is homomorphic with a dense subalgebra of $C^*(X, B)$.*

Proof is obvious by Proposition 6.

Let $\mathrm{rad}\, A$ denote the radical of A, let ε denote the identity mapping on X and let ϱ be the canonical homomorphism from A onto $A/\mathrm{rad}\, A$. It is clear that $\ker F_{\varrho,\varepsilon} = C^*(X, \mathrm{rad}\, A)$. Moreover, $\mathrm{rad}\, C^*(X, A) = C^*(X, \mathrm{rad}\, A)$ if X is a pseudocompact space (cf. [4]). So, by Corollary 2, we have

COROLLARY 3. *Let X be a pseudocompact space and A be a Banach algebra with unit. Then $C^*(X, A)$ is homomorphic and $C^*(X, A)/\mathrm{rad}\, C^*(X, A)$ is isomorphic with a dense subalgebra of $C^*(X, A/\mathrm{rad}\, A)$.*

In the same way as in Proposition 6, by Proposition 5, we have

PROPOSITION 6'. *Let X and Y be two topological spaces, A be a finite dimensional Banach algebra with unit and ψ: $Y \to X$ be a homomorphism. If every $a \in C^*(\psi(Y), R)$ has an extension $\bar{a} \in C^*(X, R)$, then $F_{\varphi,\psi}(C^*(X, A))$ is dense in $C^*(Y, A)$.*

Let now X be a completely regular T_1-space. It is well known that there exists a homeomorphism ψ: $X \to \beta X$ and every $a \in C^*(\psi(X), R)$ has an extension $a \in C^*(X, R)$. Let φ denote the identity map on Banach algebra A with unit. It is proved in [5] that $F_{\varphi,\psi}$ is an isometric isomorphism from $C^*(\beta X, A)$ onto a closed subalgebra of $C^*(X, A)$. In view of this, by Proposition 6 and Proposition 6', we have

COROLLARY 4. *Let one of the following conditions hold:*

(a) *X is a pseudocompact space and A is a Banach algebra with unit.*

(b) *X is a completely regular T_1-space and A is a finite dimensional Banach algebra with unit.*

Then the algebras $C^*(\beta X, A)$ and $C^*(X, A)$ are isometrically isomorphic.

When A is a B-algebra, Corollary 4 is known (cf. [1], Corollary 13).

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