ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXII (1981)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXII (1981)

MICHAEL BECKER (Aachen)

Inverse theorems for Favard operators in polynomial weight spaces

1. Introduction. The aim of this note is to establish a complete equivalence theorem concerning the rate of convergence for operators introduced by Favard (cf. [6]) in 1944, namely

$$\text{(1)}\qquad \begin{aligned} F_n^{\cdot} f(x) &:= \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) q_{k,n}^{\cdot}(x),\\ q_{k,n}^{\cdot}(x) &:= (\gamma\pi n)^{-1/2} \exp\left(-\frac{n}{\gamma}\left(\frac{k}{n}-x\right)^2\right), \end{aligned}$$

where $\gamma > 0$, $n \in \boldsymbol{N}$ (= the set of positive integers). The functions to be approximated are defined on the whole real axis $\boldsymbol{R}$ and are allowed to have polynomial growth at infinity. To this end we consider the spaces C_{2N} given via ($N \in \boldsymbol{P} := \boldsymbol{N} \cup \{0\}$)

$$w_0(x) := 1, \quad w_{2N}(x) := (1+x^{2N})^{-1} \quad (x \in \boldsymbol{R},\ N \in \boldsymbol{N}),$$

$$C_{2N} := \{f \in C(\boldsymbol{R});\ w_{2N} f \text{ uniformly continuous and bounded on } \boldsymbol{R}\},$$

$$\|f\|_{2N} := \|w_{2N} f\| := \sup_{x \in \boldsymbol{R}} |w_{2N}(x) f(x)|,$$

$C(\boldsymbol{R})$ being the set of all continuous functions on $\boldsymbol{R}$. Correspondingly, Lipschitz classes are given for $0 < \alpha \leqslant 2$ by

$$\mathrm{Lip}_{2,2N}\alpha := \{f \in C_{2N};\ \omega_{2,2N}(f;\ \delta) = O(\delta^\alpha),\ \delta \to 0+\},$$

where

$$\omega_{2,2N}(f;\ \delta) := \sup_{0<h\leqslant\delta} \|\Delta_h^2 f\|_{2N}, \quad \Delta_h^2 f(x) := f(x+h)-2f(x)+f(x-h).$$

Then we have the following result.

THEOREM 1. *Let $N \in \boldsymbol{P}$, $f \in C_{2N}$, $\alpha \in (0, 2]$. The following assertions are equivalent*:

$$\text{(2)}\qquad \|F_n^{\cdot} f - f\|_{2N} = O(n^{-\alpha/2}) \quad (n \to \infty),$$

$$\text{(3)}\qquad f \in \mathrm{Lip}_{2,2N}\alpha.$$

Whereas [6] is mainly concerned with (pure) convergence assertions, the equivalence for the saturation case $\alpha = 2$ was established in [4]. Moreover, the direct part (3) $\Rightarrow$ (2) for $0 < \alpha < 2$ follows from [1] (compare the remarks in Section 3 for some additional details). Thus there remains to prove the inverse part (2) $\Rightarrow$ (3) for the case of nonoptimal approximation. The proof in Section 3 rests mainly on a suitable representation and estimation of the second derivative of $F_n^\gamma f$, which will be derived in Section 2. Analogous results for Szász–Mirakjan and Baskakov operators have been established in [3]. The approximation of functions exhibiting exponential growth will be the subject of a further note (cf. [4], Section 5).

The material of this paper forms part of the author's doctoral dissertation. The author wishes to express his sincere gratitude to Professor R. J. Nessel, his dissertation adviser, for his constant encouragement and helpful remarks, and for his critical reading of the manuscript.

2. Basic estimates. In this section the results of Section 2 in [4] will be extended and improved in order to derive an estimation for $(F_n^\gamma f)''$. First recall ($x \in \boldsymbol{R}$, $n \in \boldsymbol{N}$, $r \in \boldsymbol{P}$)

$$F_n^\gamma 1(x) = \sum_{k=-\infty}^{\infty} q_{k,n}^\gamma(x) = 1 + 2\sum_{j=1}^{\infty} \exp(-n\gamma\pi^2 j^2)\cos(2\pi njx), \tag{4}$$

$$T_{n,r}^\gamma(x) := F_n^\gamma\big((t-x)^r;\, x\big) = \sum_{k=-\infty}^{\infty} \left(\frac{k}{n} - x\right)^r q_{k,n}^\gamma(x). \tag{5}$$

LEMMA 2. *For $\alpha \geqslant 0$, $n \in \boldsymbol{N}$ let*

$$S_\alpha := \sum_{j=1}^{\infty} j^\alpha \exp(-n\gamma\pi^2 j^2), \qquad A_\gamma := (e^{\gamma\pi^2} - 1)^{-1}.$$

There hold the following estimates:

$$S_\alpha \leqslant (\alpha/\gamma\pi^2 e)^{\alpha/2}(e^{n\gamma\pi^2} - 1)^{-1} n^{-\alpha/2} \leqslant A_\gamma(\alpha/\gamma\pi^2 e)^{\alpha/2} n^{-\alpha/2}, \tag{6}$$

$$S_\alpha \leqslant \begin{cases} \dfrac{e\alpha^{\alpha/2}\lambda^\lambda}{(\gamma\pi^2 e)^{\alpha/2+\lambda}}\, n^{-\alpha/2-\lambda} & (\lambda > 1), \\ A_\gamma(\alpha/\gamma\pi^2 e)^{\alpha/2} n^{-\alpha/2-\lambda} & (0 \leqslant \lambda \leqslant 1). \end{cases} \tag{7}$$

Proof. Defining $f_{n,\alpha}(x) := x^\alpha \exp\big(-n\gamma\pi^2 x(x-1)\big)$ we have

$$\begin{aligned} S_\alpha &= \sum_{j=1}^{\infty} j^\alpha \exp(-n\gamma\pi^2 j^2 + n\gamma\pi^2 j)\exp(-n\gamma\pi^2 j) \\ &\leqslant \max_{j \geqslant 2} f_{n,\alpha}(j)(e^{n\gamma\pi^2} - 1)^{-1}, \end{aligned} \tag{8}$$

so that (6) would follow from

$$\max_{x\geq 2} f_{n,\alpha}(x) \leq (\alpha/\gamma\pi^2 e)^{\alpha/2} n^{-\alpha/2}. \tag{9}$$

To prove (9), note that the function $f_{n,\alpha}$ attains its maximal value for $x_n := (1+(1+8\alpha/n\gamma\pi^2)^{1/2})/4$. For $n \geq n_0 := \alpha/6\gamma\pi^2$ we have $x_n \leq 2$, and therefore

$$\max_{x\geq 2} f_{n,\alpha}(x) = f_{n,\alpha}(2) = 2^\alpha \exp(-2n\gamma\pi^2).$$

Observing that $x^{\alpha/2}\exp(-2x\gamma\pi^2)$ is maximal for $x = \alpha/4\gamma\pi^2$, this delivers (9) for $n \geq n_0$. For $n < n_0$ we have $x_n > 2$, thus $x_n^2 - x_n > x_n^2/2$, so that

$$\max_{x\geq 2} f_{n,\alpha}(x) = f_{n,\alpha}(x_n) < x_n^\alpha \exp(-n\gamma\pi^2 x_n^2/2).$$

This implies (9) for $n < n_0$ since $x^\alpha \exp(-n\gamma\pi^2 x^2/2)$ attains its maximal value for $x = (\alpha/n\gamma\pi^2)^{1/2}$.

For the proof of (7) define $g_\lambda(x) := x^\lambda(\exp(\gamma\pi^2 x)-1)^{-1}$. We first show

$$\max_{x\geq 1} g_\lambda(x) \leq \begin{cases} e(\lambda/\gamma\pi^2 e)^\lambda & (\lambda > 1), \\ A_\gamma & (0 \leq \lambda \leq 1). \end{cases} \tag{10}$$

For g_λ to have a maximum at a point x_0 we must have that

$$\lambda = x_0\gamma\pi^2 \exp(x_0\gamma\pi^2)(\exp(x_0\gamma\pi^2)-1)^{-1}. \tag{11}$$

As $ye^y(e^y-1)^{-1} \geq 1$ is an increasing function, this only can happen for $\lambda \geq 1$. In this case by (11)

$$g_\lambda(x_0) = x_0^\lambda(\exp(x_0\gamma\pi^2)-1)^{-1} = \frac{\lambda}{\gamma\pi^2} x_0^{\lambda-1}\exp(-x_0\gamma\pi^2).$$

As $x^{\lambda-1}\exp(-x\gamma\pi^2)$ attains its maximal value for $x = (\lambda-1)/\gamma\pi^2$, there follows

$$\begin{aligned}\max_{x\geq 1} g_\lambda(x) &\leq g_\lambda(x_0) \leq \lambda(\gamma\pi^2)^{-\lambda}(\lambda-1)^{\lambda-1}e^{1-\lambda} \\ &= e(\lambda/\gamma\pi^2 e)^\lambda(1-1/\lambda)^{\lambda-1} \leq e(\lambda/\gamma\pi^2 e)^\lambda.\end{aligned}$$

This proves (10) for $\lambda > 1$, while for $\lambda \leq 1$ we note that g_λ is decreasing, thus $g_\lambda(x) \leq g_\lambda(1) = A_\gamma$. Now (7) is an easy consequence of (6), (10) in view of

$$S_\alpha \leq \max_{x\geq 1} g_\lambda(x)(\alpha/\gamma\pi^2 e)^{\alpha/2} n^{-\alpha/2-\lambda}. \ \square$$

Let us mention that Lemma 2 extends Lemma 2.2 in [4] by providing precise constants. From (7) we see that S_α can be majorized by an arbitrary power of $1/n$.

THEOREM 3. *For every* $n, r \in N$, $x \in \boldsymbol{R}$ *there holds*

$$T^{\gamma}_{n,r}(x) = \left(\frac{\gamma}{2n}\right)^{r} (F^{\gamma}_{n} 1)^{(r)}(x) + \sum_{j=1}^{[r/2]} c_{r,j} \left(\frac{\gamma}{2n}\right)^{j} T^{\gamma}_{n,r-2j}(x) \tag{12}$$

with constants

$$c_{r,j} = (-1)^{j+1} 2^{-j} \frac{r!}{j!\,(r-2j)!} \qquad (1 \leqslant j \leqslant [r/2]). \tag{13}$$

Proof. This is essentially Theorem 2.4 of [4] except that the summation in (12) is reversed and the constants are evaluated via (13). Indeed, using

$$\frac{d}{dx} T_{n,r}(x) = \frac{2n}{\gamma} T_{n,r+1}(x) + r T_{n,r-1}(x)$$

gives (in the following proofs we omit the index γ)

$$\begin{aligned} T_{n,r+1}(x) &= \frac{\gamma}{2n} \left[T'_{n,r}(x) + r T_{n,r-1}(x) \right] \\ &= \left(\frac{\gamma}{2n}\right)^{r+1} (F_n 1)^{(r+1)}(x) + (r + c_{r,1}) \left(\frac{\gamma}{2n}\right) T_{n,r-1}(x) + \\ &\quad + \sum_{j=2}^{[r/2]} \left(c_{r,j} - (r+2-2j)\, c_{r,j-1}\right) \left(\frac{\gamma}{2n}\right)^{j} T_{n,r+1-2j}(x) - \\ &\quad - (r - 2[r/2])\, c_{r,[r/2]} \left(\frac{\gamma}{2n}\right)^{[r/2]+1} T_{n,r-2[r/2]-1}(x). \end{aligned} \tag{14}$$

From this representation the theorem follows by induction. □

THEOREM 4. *For every* $r \in \boldsymbol{P}$, $n \in N$, $x \in \boldsymbol{R}$ *there holds*

$$|T^{\gamma}_{n,r}(x)| \leqslant 75 \max\{A_\gamma, 1\} (\gamma/e)^{r/2}\, r!\, n^{-r/2}. \tag{15}$$

Proof. By (4) and Lemma 2 one has

$$|T_{n,0}(x)| = |F_n 1(x)| \leqslant 1 + 2S_0 \leqslant 1 + 2A_\gamma \leqslant 2 \max\{A_\gamma, 1\},$$

$$|T_{n,1}(x)| = \frac{\gamma}{2n} |(F_n 1)'(x)| \leqslant 2\pi\gamma S_1 \leqslant 2A_\gamma \left(\frac{\gamma}{e}\right)^{1/2} n^{-1/2}.$$

This gives (15) for $r = 0, 1$. Now let us proceed via induction, i.e suppose (15) to be valid for $0 \leqslant r < s$. Then one has

$$(16)\quad \sum_{j=1}^{[s/2]} |c_{s,j}| \left(\frac{\gamma}{2n}\right)^j |T_{n,s-2j}(x)|$$

$$\leqslant 75 \max\{A_{\cdot}, 1\} \sum_{j=1}^{[s/2]} \frac{s!}{j!(s-2j)!} \left(\frac{\gamma}{4n}\right)^j \left(\frac{\gamma}{e}\right)^{s/2-j} (s-2j)!\, n^{-s/2+j}$$

$$= 75 \max\{A_{\cdot}, 1\} \left(\frac{\gamma}{e}\right)^{s/2} s!\, n^{-s/2} \sum_{j=1}^{[s/2]} \frac{1}{j!} \left(\frac{e}{4}\right)^j$$

$$\leqslant 75 \max\{A_{\cdot}, 1\} \left(\frac{\gamma}{e}\right)^{s/2} s!\, n^{-s/2} (\exp(e/4)-1)$$

$$\leqslant 73 \max\{A_{\cdot}, 1\} \left(\frac{\gamma}{e}\right)^{s/2} s!\, n^{-s/2}.$$

Furthermore, (4) and Lemma 2 deliver

$$(17)\quad \left(\frac{\gamma}{2n}\right)^s |(F_n 1)^{(s)}(x)| \leqslant 2(\pi\gamma)^s S_s \leqslant 2A_{\cdot} \left(\frac{\gamma s}{e}\right)^{s/2} n^{-s/2}.$$

In view of $s^{s/2} \leqslant s!$ estimates (16), (17) together with Theorem 3 imply (15) for $r = s$.

The next theorem provides the desired estimations for $(F_n f)''$ (constants $M_{\cdot,N}$ may have different values at each occurrence).

THEOREM 5. *For $\gamma > 0$, $N \in P$ there exist constants $M_{\cdot,N}$ and $n_0(\gamma) \in N$ such that for all $f \in C_{2N}$*

$$(18)\quad \|(F_n f)''\|_{2N} \leqslant n M_{\cdot,N} \|f\|_{2N} \quad (n \in N),$$

$$(19)\quad \left\|(F_n f)''(x) - n^2 \sum_{k=-\infty}^{\infty} \Delta_{1/n}^2 f\left(\frac{k}{n}\right) q_{k,n}(x)\right\|_{2N} \leqslant M_{\cdot,N} \|f\|_{2N} \quad (n \geqslant n_0).$$

Proof. By straightforward computations we obtain

$$(20)\quad (F_n f)''(x) = \left(\frac{2n}{\gamma}\right)^2 \sum_{k=-\infty}^{\infty} \left[\left(\frac{k}{n} - x\right)^2 - \frac{\gamma}{2n}\right] f\left(\frac{k}{n}\right) q_{k,n}(x),$$

$$q_{k+1,n}(x) - 2q_{k,n}(x) + q_{k-1,n}(x)$$

$$= q_{k,n}(x)\left\{\exp\left(\frac{2}{\gamma}\left(\frac{k}{n} - x\right) - \frac{1}{\gamma n}\right) - 2 + \exp\left(-\frac{2}{\gamma}\left(\frac{k}{n} - x\right) - \frac{1}{\gamma n}\right)\right\}$$

$$= q_{k,n}(x)\left\{\left(\frac{2}{\gamma}\right)^2 \left[\left(\frac{k}{n} - x\right)^2 - \frac{\gamma}{2n}\right] + (\gamma n)^{-2} + \right.$$

$$\left. + \sum_{j=3}^{\infty} \frac{(-1/\gamma)^j}{j!} \sum_{i=0}^{[j/2]} \binom{j}{2i} 2^{2i+1} \left(\frac{k}{n} - x\right)^{2i} n^{2i-j}\right\}.$$

These identities deliver

$$n^2 \sum_{k=-\infty}^{\infty} \Delta_{1/n}^2 f\left(\frac{k}{n}\right) q_{k,n}(x) - (F_n f)''(x)$$

$$= \gamma^{-2} F_n f(x) + 2n^2 \sum_{j=3}^{\infty} \frac{(-1/\gamma)^j}{j!} \sum_{i=0}^{[j/2]} \binom{j}{2i} 4^i n^{2i-j} F_n\left((t-x)^{2i} f(t);\ x\right),$$

so that

$$(21) \qquad \left\| n^2 \sum_{k=-\infty}^{\infty} \Delta_{1/n}^2 f\left(\frac{k}{n}\right) q_{k,n}(x) - (F_n f)''(x) \right\|_{2N}$$

$$\leqslant \gamma^{-2} \|F_n f\|_{2N} + 2\|f\|_{2N} S_{\gamma,N}(n),$$

$$S_{\gamma,N}(n) := \sum_{j=3}^{\infty} \frac{n^2}{j!} \gamma^{-j} \sum_{i=0}^{[j/2]} \binom{j}{2i} 4^i n^{2i-j} \left\| F_n\left((t-x)^{2i}(1+t^{2N});\ x\right) \right\|_{2N}.$$

By Theorem 4 one has

$$w_{2N}(x) \left| \sum_{k=-\infty}^{\infty} \left(\frac{k}{n} - x\right)^{2i} \left(1 + \left(\frac{k}{n}\right)^{2N}\right) q_{k,n}(x) \right| \leqslant \sum_{m=0}^{2N} \binom{2N}{m} |T_{n,2i+m}(x)|$$

$$\leqslant M_\gamma \sum_{m=0}^{2N} \binom{2N}{m} \left(\frac{\gamma}{e}\right)^{i+m/2} (2i+m)!\, n^{-i-m/2}$$

$$\leqslant M_\gamma (2i+2N)!\, (\gamma/ne)^i \sum_{m=0}^{2N} \binom{2N}{m} (\gamma/ne)^{m/2}.$$

Hence

$$(22) \qquad \|F_n((t-x)^{2i}(1+t^{2N});\ x)\|_{2N} \leqslant M_{\gamma,N} (2i+2N)!\, (\gamma/ne)^i.$$

In particular, there follows

$$(23) \qquad \|F_n f\|_{2N} \leqslant \|F_n(1+t^{2N};\ x)\|_{2N} \|f\|_{2N} \leqslant M_{\gamma,N} \|f\|_{2N}.$$

Now (18) is an easy consequence of (20), (22); indeed, one has

$$\|(F_n f)''\|_{2N} \leqslant \|f\|_{2N} \left\{ \left(\frac{2n}{\gamma}\right)^2 \|F_n((t-x)^2(1+t^{2N});\ x)\|_{2N} + \right.$$

$$\left. + \frac{2n}{\gamma} \|F_n(1+t^{2N};\ x)\|_{2N} \right\} \leqslant n M_{\gamma,N} \|f\|_{2N}.$$

Therefore in view of (21), (23) there remains to show that

$$S_{\gamma,N}(n) \leqslant M_{\gamma,N} \qquad (n \geqslant n_0(\gamma)).$$

To this end, using (22) one has

$$S_{\gamma,N}(n) \leqslant \sum_{j=3}^{\infty} \frac{\gamma^{-j}}{j!} \sum_{i=0}^{[j/2]} \binom{j}{2i} M_{\gamma,N}(2i+2N)!\,(4\gamma/e)^i n^{2+i-j}$$
$$\leqslant M_{\gamma,N} \sum_{j=3}^{\infty} \frac{(j+2N)!}{j!} \gamma^{-j} n^{2+[j/2]-j} \sum_{i=0}^{[j/2]} \binom{j}{2i} (4\gamma/e)^i$$
$$\leqslant M_{\gamma,N} \sum_{j=3}^{\infty} \frac{(j+2N)!}{j!} [(1+2\sqrt{\gamma/e})^{1/2}/\gamma]^j n^{2+[j/2]-j}.$$

Noting $n^{2+[j/2]-j} = \min\{1, n^{2-j/2}\} \leqslant n^{-2j/5}$ for $j \geqslant 20$, one obtains

$$S_{\gamma,N}(n) \leqslant M_{\gamma,N} + M_{\gamma,N} \sum_{j=20}^{\infty} \frac{(j+2N)!}{j!} [(1+2\sqrt{\gamma/e})^{1/2}/\gamma n^{2/5}]^j \leqslant M_{\gamma,N}$$

for $n \geqslant n_0(\gamma)$ (where n_0 is chosen such that the term in square brackets is smaller than 1). This completes the proof of (19). $\square$

3. Proof of the inverse theorem. To establish the equivalence theorem we have to prove the inverse part for the case of non-optimal approximation.

THEOREM 6. *Let $N \in \boldsymbol{P}$, $f \in C_{2N}$, $\alpha \in (0, 2)$. Then the rate of approximation*

(24) $$\|F_n^{\cdot} f - f\|_{2N} = O(n^{-\alpha/2}) \qquad (n \to \infty)$$

implies $f \in \mathrm{Lip}_{2,2N}\,\alpha$.

Proof. Let $f \in C_{2N}$ satisfy (24). Following the elementary argument of [5], p. 694–696, it is sufficient to show that

(25) $$\omega_{2,2N}(f;\, h) \leqslant M[\delta^\alpha + h^2 + (h/\delta)^2\,\omega_{2,2N}(f;\, \delta)].$$

From (24) there follows for all sufficiently large n and $h \in (0, 1]$

(26) $$\|\Delta_h^2 f\|_{2N} \leqslant \|\Delta_h^2 [f - F_n f]\|_{2N} + \|\Delta_h^2 [F_n f]\|_{2N}$$
$$\leqslant M_N \|f - F_n f\|_{2N} + \Big\| \int\!\!\int_{-h/2}^{h/2} (F_n f)''(x+s+t)\,ds\,dt \Big\|_{2N}$$
$$\leqslant M_N n^{-\alpha/2} + M_N h^2 \,\|(F_n f)''\|_{2N}.$$

Introducing the Steklov means

$$f_\delta(x) := \delta^{-2} \int\!\!\int_{-\delta/2}^{\delta/2} f(x+s+t)\,ds\,dt,$$

in view of the inequality

$$w_{2N}(x)/w_{2N}(x \pm \delta) \leqslant (1+\delta)^{2N} \leqslant 4^N \qquad (0 < \delta \leqslant 1)$$

one has that (cf. [2], p. 13)

$$\|f - f_\delta\|_{2N} \leqslant 4^N \omega_{2,2N}(f;\, \delta), \qquad \|f_\delta''\|_{2N} \leqslant 4^N \delta^{-2}\,\omega_{2,2N}(f;\, \delta).$$

Together with Theorem 5 this delivers for $n \geqslant n_0$

$$(27)\quad \|(F_n f)''\|_{2N} \leqslant \|(F_n[f-f_\delta])''\|_{2N} + \\ + \|n^2 \sum_{k=-\infty}^{\infty} \int\int_{-1/2n}^{1/2n} f_\delta''(k/n+s+t)\,ds\,dt\,q_{k,n}(x)\|_{2N} + \\ + \|(F_n f_\delta)''(x) - n^2 \sum_{k=-\infty}^{\infty} \Delta_{1/n}^2 f_\delta(k/n)\,q_{k,n}(x)\|_{2N} \\ \leqslant nM_{\gamma,N}\|f-f_\delta\|_{2N} + M_{\gamma,N}(\|f_\delta''\|_{2N} + \|f_\delta\|_{2N}) \\ \leqslant M_{\gamma,N}[(n+\delta^{-2})\,\omega_{2,2N}(f;\,\delta) + \|f\|_{2N}].$$

By choosing n such that $(n+1)^{-1/2} \leqslant \delta < n^{-1/2}$ we obtain (25) from (26), (27) for $0 < h \leqslant 1$, $0 < \delta \leqslant \delta_0$. This proves the assertion.

Let us add two remarks about the remaining parts of Theorem 1.

Remark 3.1. The saturation result, i.e. the equivalence of (2), (3) for $\alpha = 2$, follows from [4], where the spaces

$$X_N := \{f \in C(\mathbf{R});\ w_{2N}(x) f(x) = o(1),\ |x| \to \infty\}$$

were considered. All arguments, however, carry over immediately. The same holds true for the direct parts for $0 < \alpha < 2$ in [1], which treats the particular case $\gamma = 1$. To consider $\gamma \neq 1$, however, means just modified constants.

Remark 3.2. Remark 3.1 also holds true for the case $N = 0$, i.e. for functions uniformly continuous and bounded on $\mathbf{R}$. Let us only mention Lemma 4.4 in [4], which states that (setting $\varphi(y) := (\alpha\gamma y/\lambda)+1$)

$$f_j(x) := \frac{1}{\lambda}\int_0^{\infty} (\varphi(y))^{-1/2}\{x/\varphi(y)\}^j \exp(-y-\alpha x^2/\varphi(y))\,dy$$

satisfies the equation

$$\lambda f_j(x) - (\gamma/4) f_j''(x) = x^j \exp(-\alpha x^2) \qquad (j \in \{0, 1\}).$$

For $N = 0$ it remains to show that f_0, f_1 belong to $D_0(B) := \{f \in X_0;\ f', f'' \in X_0\}$. To this end, note that $h(y) := -y/2 - \alpha x^2/\varphi(y)$ attains its maximal value at $y_0 := -\lambda/\alpha\gamma + |x|\sqrt{2\lambda/\gamma}$, so that

$$g(x) := \exp(\lambda/2\alpha\gamma - |x|\sqrt{2\lambda/\gamma}) \geqslant \exp(h(y))$$

for $y \geqslant 0$. This implies that the integrals ($m \in \mathbf{N}$, $\beta > 0$)

$$\frac{1}{\lambda}\int_0^{\infty} x^m \varphi(y)^{-\beta} \exp(-y/2+h(y))\,dy \leqslant x^m g(x)\lambda^{-1}\int_0^{\infty} \exp(-y/2)\,dy$$

vanish for $|x| \to \infty$, and therefore $f_0, f_1 \in D_0(B)$, since f_0, f_1 and their derivatives are (sums of) such integrals.

References

[1] M. Becker, *Über den Satz von Trotter mit Anwendungen auf die Approximationstheorie*, Forsch.-ber. Nordrhein-Westfalen No. 2577, Opladen 1976, p. 1–36.

[2] —, *Umkehrsätze für positive lineare Operatoren*, Dissertation, Aachen 1977.

[3] —, *Global approximation theorems for Szász–Mirakjan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J. 27 (1978), p. 127–142.

[4] —, P. L. Butzer and R. J. Nessel, *Saturation for Favard operators in weighted function spaces*, Studia Math. 59 (1976), p. 33–47.

[5] H. Berens and G. Lorentz, *Inverse theorems for Bernstein polynomials*, Indiana Univ. Math. J. 21 (1972), p. 693–708.

[6] J. Favard, *Sur les multiplicateurs d'interpolation*, J. Math. Pures Appl. 23 (1944), p. 219–247.

Received September 1, 1978