ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXIX (1989)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXIX (1989)

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Semi-closure and related topics in Hashimoto topologies

Let $(X, \mathscr{T})$ be a topological space and let P be an ideal of subsets of X. For a set $A \subset X$ by $\mathrm{cl}_{\mathscr{T}} A$ and $\mathrm{int}_{\mathscr{T}} A$ we denote $\mathscr{T}$-closure and $\mathscr{T}$-interior of A, respectively. Furthermore,

$$D_P(A) = \{x \in X\colon\ U \cap A \notin P \text{ for each neighbourhood } U \text{ of } x\}.$$

Assume that P satisfies the following:

$$A \in P \Leftrightarrow A \cap D_P(A) = \emptyset \Leftrightarrow D_P(A) = \emptyset.$$

Then the operation $A \to A \cup D_P(A)$ is the closure operation, [5]; the topology defined by this way is denoted by $\mathscr{T}(P)$. A subset $A \subset X$ is $\mathscr{T}(P)$-closed ($\mathscr{T}(P)$-open) if and only if A is the union (the difference) of a $\mathscr{T}$-closed ($\mathscr{T}$-open) set and a set belonging to P, [5].

Moreover, if $D_P(X) = X$, then:

(a) a set $A \subset X$ is $\mathscr{T}(P)$-nowhere dense if and only if A is the union of a $\mathscr{T}$-nowhere dense set and a set belonging to P;

(b) $\mathrm{cl}_{\mathscr{T}(P)} W = \mathrm{cl}_{\mathscr{T}} W$ for every $\mathscr{T}(P)$-open set W, [5].

1. Remark. The condition $D_P(X) = X$ is equivalent to $\mathscr{T} \cap P = \{\emptyset\}$.

A subset A of a topological space $(X, \mathscr{T})$ is said to be *$\mathscr{T}$-semi-open* if there exists a $\mathscr{T}$-open set U satisfying $U \subset A \subset \mathrm{cl}_{\mathscr{T}} U$, [6], *$\mathscr{T}$-semi-closed* if the set $X \setminus A$ is $\mathscr{T}$-semi-open, [1].

The union (intersection) of all $\mathscr{T}$-semi-open ($\mathscr{T}$-semi-closed) sets contained in A (containing A) is called the *$\mathscr{T}$-semi-interior* (*$\mathscr{T}$-semi-closure*) of A and it is denoted as $s\text{-}\mathrm{int}_{\mathscr{T}} A$ ($s\text{-}\mathrm{cl}_{\mathscr{T}} A$), [1].

In the sequel we shall use the following properties:

(1) The union of any family of $\mathscr{T}$-semi-open sets is $\mathscr{T}$-semi-open, [6].

(2) If A is $\mathscr{T}$-semi-open and U is a $\mathscr{T}$-open set, then $A \cap U$ is $\mathscr{T}$-semi-open, [6].

(3) A point x belongs to $s\text{-cl}_{\mathcal{T}} A$ if and only if for each $\mathcal{T}$-semi-open set W containing x we have $W \cap A \neq \emptyset$.

(4) $\text{int}_{\mathcal{T}} \text{cl}_{\mathcal{T}} A \subset s\text{-cl}_{\mathcal{T}} A$ for each set $A \subset X$, [1].

(5) If A is $\mathcal{T}$-semi-open ($\mathcal{T}$-semi-closed), then the sets $\text{cl}_{\mathcal{T}} A$, $s\text{-cl}_{\mathcal{T}} A$, $\text{int}_{\mathcal{T}} A$, $s\text{-int}_{\mathcal{T}} A$ are $\mathcal{T}$-semi-open ($\mathcal{T}$-semi-closed), [2].

2. THEOREM. *Assume that* $\mathcal{T} \cap P = \{\emptyset\}$. *A set* $A \subset X$ *is* $\mathcal{T}(P)$*-semi-open* ($\mathcal{T}(P)$*-semi-closed*) *if and only if* $A = B \setminus H$ (*resp.* $A = B \cup H$), *where the set* B *is* $\mathcal{T}$*-semi-open* ($\mathcal{T}$*-semi-closed*) *and* $H \in P$.

Proof. Assume that A is a $\mathcal{T}(P)$-semi-open set; there exists a $\mathcal{T}(P)$-open set V such that $V \subset A \subset \text{cl}_{\mathcal{T}(P)} V$. Then $V = U \setminus H_1$, where U is $\mathcal{T}$-open and $H_1 \in P$. The set $H_2 = U \cap H_1$ belongs to P, so H_2 is $\mathcal{T}(P)$-closed. Thus we obtain $U \subset A \cup H_2 \subset \text{cl}_{\mathcal{T}(P)}(U \setminus H_1) \cup H_2 = \text{cl}_{\mathcal{T}(P)}((U \setminus H_1) \cup H_2) = \text{cl}_{\mathcal{T}(P)} U \subset \text{cl}_{\mathcal{T}} U$. Hence $A \cup H_2$ is a $\mathcal{T}$-semi-open set. Let us put $B = A \cup H_2$ and $H = H_1 \setminus A$. Then we have $A = B \setminus H$, the set B is $\mathcal{T}$-semi-open and $H \in P$.

Conversely, let $A = B \setminus H$, where B is a $\mathcal{T}$-semi-open set and $H \in P$. Then there exists a $\mathcal{T}$-open set $U \subset X$ satisfying $U \subset B \subset \text{cl}_{\mathcal{T}} U$. Since $\text{cl}_{\mathcal{T}} U = \text{cl}_{\mathcal{T}(P)} U = \text{cl}_{\mathcal{T}(P)}(U \setminus H)$, from the last inclusions it follows $U \setminus H \subset B \setminus H \subset \text{cl}_{\mathcal{T}} U = \text{cl}_{\mathcal{T}(P)}(U \setminus H)$. So we have shown that the set $A = B \setminus H$ is $\mathcal{T}(P)$-semi-open.

The conclusion concerning semi-closed sets is a consequence of the above and the suitable definitions.

Now we define

$$D_P^s(A) = \{x \in X\colon A \cap W \notin P \text{ for each } \mathcal{T}\text{-semi-open set } W \text{ containing } x\}.$$

3. THEOREM. *For any sets* $A, B \subset X$ *there holds:*

(a) $$D_P^s(A) \subset s\text{-cl}_{\mathcal{T}} A.$$

(b) $$D_P^s(A) \subset D_P(A).$$

(c) $$\textit{If} \quad A \subset B, \quad \textit{then} \quad D_P^s(A) \subset D_P^s(B).$$

(d) $$D_P^s(A) \cup D_P^s(B) \subset D_P^s(A \cup B).$$

(e) $$D_P^s(D_P^s(A)) \subset D_P^s(A).$$

(f) $$D_P^s(A) \quad \textit{is a } \mathcal{T}\textit{-semi-closed set.}$$

(g) $$\textit{If} \quad H \in P, \textit{ then} \quad D_P^s(A \setminus H) = D_P^s(A) = D_P^s(A \cup H).$$

Proof. Properties (a)–(c) are an immediate consequence of the definition of $D_P^s(A)$; (d) follows from (c).

Now, let $x \in D_P^s(D_P^s(A))$ and let W be a $\mathcal{T}$-semi-open set containing x. Then $W \cap D_P^s(A) \notin P$, so $W \cap D_P^s(A) \neq \emptyset$. Thus W is a $\mathcal{T}$-semi-open set containing some point $y \in D_P^s(A)$, which implies $W \cap A \notin P$. Hence we have $x \in D_P^s(A)$ and (e) is proved.

If $x \notin D_P^s(A)$, then for some $\mathscr{T}$-semi-open set W we have $x \in W$ and $W \cap A \in P$. This implies $W \subset X \backslash D_P^s(A)$, which means that $X \backslash D_P^s(A)$ is a $\mathscr{T}$-semi-open set. Thus $D_P^s(A)$ is $\mathscr{T}$-semi-closed.

Finally, we will prove (g). From (c) we have $D_P^s(A) \subset D_P^s(A \cup H)$. Let $x \notin D_P^s(A)$; then there exists a $\mathscr{T}$-semi-open set W such that $x \in W$ and $W \cap A \in P$. Thus $W \cap (A \cup H) \in P$ and $x \notin D_P^s(A \cup H)$. So we have shown $D_P^s(A \cup H) = D_P^s(A)$. Using this equality and the fact that $A \cap H \in P$ we obtain $D_P^s(A) = D_P^s((A \backslash H) \cup (A \cap H)) = D_P^s(A \backslash H)$, which finishes the proof.

4. THEOREM. *If $\mathscr{T} \cap P = \{\emptyset\}$, then $A \cup D_P^s(A) = s\text{-cl}_{\mathscr{T}(P)} A$ for each set $A \subset X$.*

Proof. For a point $x \notin s\text{-cl}_{\mathscr{T}(P)} A$ there exists a $\mathscr{T}(P)$-semi-open set W containing x such that $W \cap A = \emptyset$, so $W \cap A \in P$. According to Theorem 2, $W = U \backslash H$, where U is $\mathscr{T}$-semi-open and $H \in P$. Moreover, we have $x \in U$. Since $U \cap A = ((U \backslash H) \cap A) \cup (U \cap A \cap H) \in P$, so $x \notin D_P^s(A)$. Thus we have shown $D_P^s(A) \subset s\text{-cl}_{\mathscr{T}(P)} A$ and in the consequence $A \cup D_P^s(A) \subset s\text{-cl}_{\mathscr{T}(P)} A$.

Now, let $x \notin A \cup D_P^s(A)$. Then $x \notin A$ and there exists a $\mathscr{T}$-semi-open set W containing x such that $A \cap W \in P$. Applying Theorem 2, we have that $W \backslash A = W \backslash (A \cap W)$ is a $\mathscr{T}(P)$-semi-open set, $x \in W \backslash A$ and $(W \backslash A) \cap A = \emptyset$. This implies that $x \notin s\text{-cl}_{\mathscr{T}(P)} A$, so $s\text{-cl}_{\mathscr{T}(P)} A \subset A \cup D_P^s(A)$ and the proof is completed.

5. COROLLARY. *Let $\mathscr{T} \cap P = \{\emptyset\}$. A set $A \subset X$ is $\mathscr{T}(P)$-semi-closed if and only if $D_P^s(A) \subset A$.*

6. COROLLARY. *Let P be the ideal of nowhere dense subsets of a topological space $(X, \mathscr{T})$. Then $A \cup D_P^s(A) = s\text{-cl}_{\mathscr{T}} A$ for every set $A \subset X$.*

Proof. Under our assumptions, classes of $\mathscr{T}$-semi-open sets and $\mathscr{T}(P)$-semi-open sets coincide, [7], Propositions 3 and 4. Therefore the conclusion simply follows from Theorem 4.

7. THEOREM. *If $\mathscr{T} \cap P = \{\emptyset\}$, then for each $\mathscr{T}(P)$-open set W we have $s\text{-cl}_{\mathscr{T}} W = s\text{-cl}_{\mathscr{T}(P)} W$.*

Proof. It follows from Theorem 2 that every $\mathscr{T}$-semi-closed set is $\mathscr{T}(P)$-semi-closed, so $s\text{-cl}_{\mathscr{T}(P)} W \subset s\text{-cl}_{\mathscr{T}} W$. Now, let $x \in s\text{-cl}_{\mathscr{T}} W$ and let U be a $\mathscr{T}$-semi-open set containing x. Then $U \cap W \neq \emptyset$ and, in consequence, $\text{int}_{\mathscr{T}(P)}(U \cap W) \neq \emptyset$. Thus $U \cap W \notin P$ and this implies $x \in D_P^s(W)$. Applying Theorem 4, we obtain $x \in s\text{-cl}_{\mathscr{T}(P)} W$ and in the next $s\text{-cl}_{\mathscr{T}} W \subset s\text{-cl}_{\mathscr{T}(P)} W$.

8. COROLLARY. *If $\mathscr{T} \cap P = \{\emptyset\}$, then for every $\mathscr{T}(P)$-closed set $A \subset X$ there holds $s\text{-int}_{\mathscr{T}} A = s\text{-int}_{\mathscr{T}(P)} A$.*

A topological space $(X, \mathscr{T})$ is said to be *extremally disconnected* if for every $\mathscr{T}$-open set U the closure $\mathrm{cl}_{\mathscr{T}} U$ is $\mathscr{T}$-open, [4], p. 452. The space $(X, \mathscr{T})$ is extremally dosconnected if and only if for every pair U, V of disjoint $\mathscr{T}$-open sets we have $\mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V = \emptyset$, [4], p. 452.

9. THEOREM. *Let $\mathscr{T} \cap P = \{\emptyset\}$. The topological space, $(X, \mathscr{T})$ is extremally disconnected if and only if $(X, \mathscr{T}(P))$ is extremally disconnected.*

Proof. Let $(X, \mathscr{T})$ be extremally disconnected and let W be a $\mathscr{T}(P)$-open set. Then $W = U \backslash H$, U is $\mathscr{T}$-open and $H \in P$; moreover, $\mathrm{cl}_{\mathscr{T}(P)} W = \mathrm{cl}_{\mathscr{T}} W = \mathrm{cl}_{\mathscr{T}} U$. Thus $\mathrm{cl}_{\mathscr{T}(P)} W$ is a $\mathscr{T}(P)$-open set and $(X, \mathscr{T}(P))$ is extremally disconnected.

Conversely, let $(X, \mathscr{T}(P))$ be extremally disconnected and let U, V be $\mathscr{T}$-open sets such that $\mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V \neq \emptyset$. Since $\mathrm{cl}_{\mathscr{T}} U = \mathrm{cl}_{\mathscr{T}(P)} U$ and $\mathrm{cl}_{\mathscr{T}} V = \mathrm{cl}_{\mathscr{T}(P)} V$, we obtain $\mathrm{cl}_{\mathscr{T}(P)} U \cap \mathrm{cl}_{\mathscr{T}(P)} V \neq \emptyset$ and from the assumption, $U \cap V \neq \emptyset$. Hence $(X, \mathscr{T})$ is extremally disconnected.

10. LEMMA. *A topological space $(X, \mathscr{T})$ is extremally disconnected if and only if for each $\mathscr{T}$-semi-open set $A \subset X$ there holds $\mathrm{cl}_{\mathscr{T}} A = s\text{-}\mathrm{cl}_{\mathscr{T}} A$.*

Proof. Assume that $(X, \mathscr{T})$ is extremally disconnected. Let $x \in \mathrm{cl}_{\mathscr{T}} A$ and let U be a $\mathscr{T}$-semi-open set containing x. Since $\mathrm{cl}_{\mathscr{T}} U = \mathrm{cl}_{\mathscr{T}} \mathrm{int}_{\mathscr{T}} U$, the set $\mathrm{cl}_{\mathscr{T}} U$ is a neighbourhood of x. Hence we have $\mathrm{cl}_{\mathscr{T}} U \cap A \neq \emptyset$. The set $\mathrm{cl}_{\mathscr{T}} U \backslash U$ is nowhere dense, $\mathrm{cl}_{\mathscr{T}} U \cap A$ is $\mathscr{T}$-semi-open, so $U \cap A \neq \emptyset$. Thus $x \in s\text{-}\mathrm{cl}_{\mathscr{T}} A$ and in the consequence $\mathrm{cl}_{\mathscr{T}} A \subset s\text{-}\mathrm{cl}_{\mathscr{T}} A$. This implies $\mathrm{cl}_{\mathscr{T}} A = s\text{-}\mathrm{cl}_{\mathscr{T}} A$.

Now, let $\mathrm{cl}_{\mathscr{T}} A = s\text{-}\mathrm{cl}_{\mathscr{T}} A$ for every $\mathscr{T}$-semi-open set $A \subset X$. For each $\mathscr{T}$-open set $V \subset X$ we have $V \subset \mathrm{int}_{\mathscr{T}} \mathrm{cl}_{\mathscr{T}} V$. Since $\mathrm{int}_{\mathscr{T}} \mathrm{cl}_{\mathscr{T}} V$ is $\mathscr{T}$-semi-closed, we obtain the inclusion $s\text{-}\mathrm{cl}_{\mathscr{T}} V \subset \mathrm{int}_{\mathscr{T}} \mathrm{cl}_{\mathscr{T}} V$. Hence $\mathrm{cl}_{\mathscr{T}} V = s\text{-}\mathrm{cl}_{\mathscr{T}} V \subset \mathrm{int}_{\mathscr{T}} \mathrm{cl}_{\mathscr{T}} V$ and from this it follows that $\mathrm{cl}_{\mathscr{T}} V$ is a $\mathscr{T}$-open set, which completes the proof.

11. THEOREM. *Let $\mathscr{T} \cap P \neq \{\emptyset\}$. The topological space $(X, \mathscr{T})$ is extremally disconnected if and only if $D_P(A) = D_P^s(A)$ for every $\mathscr{T}$-semi-open set $A \subset X$.*

Proof. Let $(X, \mathscr{T})$ be an extremally disconnected space and let A be a $\mathscr{T}$-semi-open subset of X. Evidently, it sufficies to show that $D_P(A) \subset D_P^s(A)$. Let us take a point $x \in D_P(A)$ and a $\mathscr{T}$-semi-open set W containing x. Then $\mathrm{cl}_{\mathscr{T}} W$ is a $\mathscr{T}$-neighbourhood of x and $\mathrm{cl}_{\mathscr{T}} W \cap A \notin P$. Hence $\emptyset \neq \mathrm{cl}_{\mathscr{T}} W \cap A = (W \cap A) \cup (\mathrm{cl}_{\mathscr{T}} W \backslash W) \cap A$. Since $\mathrm{cl}_{\mathscr{T}} W \cap A$ is $\mathscr{T}$-semi-open and $\mathrm{cl}_{\mathscr{T}} W \backslash W$ is a $\mathscr{T}$-nowhere dense set, we obtain $W \cap A \neq \emptyset$. Moreover, $W \cap A$ is a $\mathscr{T}$-semi-open set, so $W \cap A \notin P$. Thus $x \in D_P^s(A)$ and the inclusion $D_P(A) \subset D_P^s(A)$ is shown. Conversely, the equality $D_P(A) = D_P^s(A)$ implies $\mathrm{cl}_{\mathscr{T}(P)} A = s\text{-}\mathrm{cl}_{\mathscr{T}(P)} A$ for each $\mathscr{T}$-semi-open set $A \subset X$. Let B be a $\mathscr{T}(P)$-semi-open set. Then

$B = A \setminus H$, where A is $\mathcal{T}$-semi-open and $H \in P$. From [5] and Theorem 3 it follows that $D_P(B) = D_P(A)$ and $D^s_P(B) = D^s_P(A)$. Using these equalities and the assumption we have $\mathrm{cl}_{\mathcal{T}(P)} B = s\text{-}\mathrm{cl}_{\mathcal{T}(P)} B$. Now, applying Lemma 10 and Theorem 9 we obtain that $(X, \mathcal{T})$ is extremally disconnected.

A one-to-one map f of a topological space $(X, \mathcal{T}_1)$ onto a topological space $(Y, \mathcal{T}_2)$ is said to be a *semihomeomorphism* if for every $\mathcal{T}_1$-semi-open set $A \subset X$ and $\mathcal{T}_2$-semi-open set $B \subset Y$ the sets $f(A)$ and $f^{-1}(B)$ are $\mathcal{T}_2$- and $\mathcal{T}_1$-semi-open, respectively, [3]. A one-to-one map f of $(X, \mathcal{T}_1)$ onto $(Y, \mathcal{T}_2)$ is a semihomeomorphism if and only if $f(s\text{-}\mathrm{cl}_{\mathcal{T}_1} A) = s\text{-}\mathrm{cl}_{\mathcal{T}_2} f(A)$ for each set $A \subset X$, [3]. Moreover, if $f\colon X \to Y$ is a semihomeomorphism, then a set $A \subset X$ is nowhere dense if and only if $f(A)$ is nowhere dense, [3].

12. THEOREM. *Let f be a one-to-one map of a topological space $(X, \mathcal{T}_1)$ onto $(Y, \mathcal{T}_2)$ and let P_1, P_2 be ideals of nowhere dense sets in X and Y, respectively. Then the following conditions are equivalent:*

(a) *f is a semihomeomorphism,*

(b) *$f(D^s_{P_1}(A)) = D^s_{P_2}(f(A))$ for each set $A \subset X$,*

(c) *$D^s_{P_1}(f^{-1}(B)) = f^{-1}(D^s_{P_2}(B))$ for each set $B \subset Y$.*

Proof. Assume that f is a semihomeomorphism. Let $A \subset X$, $x \in D^s_{P_1}(A)$ and let V be a $\mathcal{T}_2$-semi-open set containing $f(x)$. Then $f^{-1}(V)$ is a $\mathcal{T}_1$-semi-open set containing x and $f^{-1}(V) \cap A \notin P_1$. The last implies $V \cap f(A) \notin P_2$, so $f(x) \in D^s_{P_2}(f(A))$. Thus we have shown $f(D^s_{P_1}(A)) \subset D^s_{P_2}(f(A))$. The proof of the inverse inclusion is analogous; hence (a)$\Rightarrow$(b).

Now let (b) be satisfied. Applying Corollary 6 for every subset $A \subset X$ we obtain

$$s\text{-}\mathrm{cl}_{\mathcal{T}_2} f(A) = f(A) \cup D^s_{P_2}(f(A)) = f(A) \cup f(D^s_{P_1}(A)) = f(A \cup D^s_{P_1}(A)) = f(s\text{-}\mathrm{cl}_{\mathcal{T}_1} A),$$

which finishes the proof of the equivalence (a)$\Leftrightarrow$(b).

The equivalence (a)$\Leftrightarrow$(c) follows from (a)$\Leftrightarrow$(b) used for the semihomeomorphism f^{-1}.

References

[1] S. G. Crossley, S. K. Hildebrand, *Semi-closure*, Texas J. Sci. 22 (1971), 99–112.

[2] —, —, *Semi-closed sets and semi-continuity in topological spaces*, Texas J. Sci. 22 (1971), 123–126.

[3] —, —, *Semi-topological properties*, Fund. Math. 74 (1972), 233–253.

[4] R. Engelking, *General topology*, Warszawa 1977.

[5] H. Hashimoto, *On the *-topology and its application*, Fund. Math. 91 (1976), 5–10.

[6] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly 70 (1963), 36–41.

[7] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math. 15 (1965), 961–970.