ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE
XLV (2) (2005), 225-235

Iwo Labuda

Boundaries of upper semicontinuous set valued maps

(*Pracę tę poświęcam pamięci Pawła Szeptyckiego*)

Abstract. Let x_0 be a q-point of a regular space X, Y a Hausdorff space whose relatively countably compact subsets are relatively compact and let $F : X \rightrightarrows Y$ be an upper semicontinuous set valued map. Then the active boundary $\operatorname{Frac} F(x_0)$ is the smallest compact kernel of F at x_0.

2000 Mathematics Subject Classification: 54C60.

Key words and phrases: Upper semicontinuous set valued map, active boundary, Choquet kernel, Vaĭnšteĭn-Choquet-Dolecki Theorem, compact filter base.

1. Introduction. Let Y be a topological space, $\mathcal{A}$, $\mathcal{B}$ families of its subsets. We write $\mathcal{A}\#\mathcal{B}$ and say that $\mathcal{A}$ *meshes with* $\mathcal{B}$ if, for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$, $A \cap B \neq \emptyset$.

Let $\mathcal{B}$ be a filter base. Following [10], we write $\mathcal{B} \rightsquigarrow A$ and say that $\mathcal{B}$ *aims at* A, if, for each neighborhood V of A there exists B in $\mathcal{B}$ such that $B \subset V$.

Let X be another topological space and let $F : X \rightrightarrows Y$ be a set-valued map. F is said to be *upper semicontinuous at* $x \in X$ (*usc at* x), if, for each open set V containing $F(x)$, there exists a neighborhood U of x such that $F(U) \subset V$. F is *upper semicontinuous* (usc) if it is upper semicontinuous at x for each $x \in X$.

Let now $x_0 \in X$ and $F : X \rightrightarrows Y$ be fixed. The *external part* or *map* (of F at x_0) is the map $E(\,.\,) := F(\,.\,) \setminus F(x_0)$. Let $\mathcal{U} = \mathcal{U}(x_0)$ be the filter of all neighborhoods of x_0 and let $\mathcal{N}$ be any filter base contained in $\mathcal{U}$. Then $E(\mathcal{N})$ is the image filter base of $\mathcal{N}$ by the external map, that is, $\{F(N) \setminus F(x_0) : N \in \mathcal{N}\}$. The *external filter*, i.e. the filter generated by this base, is still denoted by $E(\mathcal{N})$. We call it *external filter of F at x_0 relative to $\mathcal{N}$*. If $\mathcal{N} = \mathcal{U}$, we drop $\mathcal{U}$, and refer to it as the *external filter*. It may be degenerate, that is, may contain the empty set. However, if it does, x_0 is not interesting from our point of view and is discarded from further considerations.

A set $K \subset Y$ is said to be $\mathcal{N}$*-kernel of* F *at* x_0 if $E(\mathcal{N}) \rightsquigarrow K$. If $\mathcal{N} = \mathcal{U}$, we drop $\mathcal{N}$ and speak about the *kernel of* F *at* x_0. If, moreover, $K \subset F(\mathcal{N}^\bullet)$, where $\mathcal{N}^\bullet = \bigcap \mathcal{N}$, then we refer to K as *Choquet* $\mathcal{N}$*-kernel* (and Choquet kernel if $\mathcal{N} = \mathcal{U}$).

The *active* $\mathcal{N}$*-boundary* of F at x_0 is defined as the adherence of $E(\mathcal{N})$, that is,

$$\operatorname{Frac}_{\mathcal{N}} F(x_0) = \bigcap_{N \in \mathcal{N}} \overline{\{F(N) \setminus F(x_0)\}}.$$

Again, if $\mathcal{N} = \mathcal{U}(x_0)$, we drop the subscript and speak about the *active boundary.*

The name $\operatorname{Frac} F(x_0)$ originates from French '**fr**ontière **ac**tive'. The notion was introduced by Dolecki in order to prove that if X, Y are metric spaces and F is usc at x_0, then its active boundary is the smallest *compact Choquet kernel* (for F at x_0).

For historical reasons, we call theorems of this type *Vaĭnšteĭn-Choquet-Dolecki Theorems* (VCD-Theorems). Improving upon [8] and [11], some of the strongest VCD-Theorems in the case of a point x_0 *of countable character* were obtained in [13]. However, in [11] also a more general case of a so-called *q-point* was studied albeit with apparently different techniques and with less success. In the present paper both cases are covered in a unified approach and the results improve upon both [13] and [11].

Our topological reference is Kelley [12]. In particular, a topological space Y is *compact* if it satisfies the familiar Borel-Lebesgue axiom, and Y is *regular* if, for each point y in Y, its neighborhood filter has a base consisting of closed neighborhoods. Hausdorffness is not presupposed. Throughout the paper the word *space* refers to a topological space.

2. Compactness of filter bases. Let $\mathcal{B}$ be a filter base in a space Y. The *adherence* of $\mathcal{B}$ is defined by

$$adh\mathcal{B} = \bigcap \{\overline{B} : B \in \mathcal{B}\}.$$

Let $\mathcal{A}$ be a family of subsets of Y and let $\mathbb{D}$ denote a class of filters on Y. We say that $\mathcal{B}$ *is* $\mathbb{D}$*-compact at* $\mathcal{A}$, if

$$\mathcal{D} \in \mathbb{D}, \mathcal{D} \# \mathcal{B} \implies adh\mathcal{B} \# \mathcal{A}.$$

If $\mathcal{A} = \{A\}$, we speak about compactness at A. If $A = Y$, we often drop Y and speak about *compactness* provided no ambiguity about the space can arise (otherwise the term of *relative compactness* is used).

Let $\aleph$ be a cardinal. We denote by $\mathbb{F}_\aleph$ the class of all filters that admit a base of cardinality (strictly) less than $\aleph$. $\mathcal{B}$ is said to be $\aleph$*-compact at* $\mathcal{A}$, if it is $\mathbb{F}_\aleph$-compact at $\mathcal{A}$.

In the just introduced terminology the 'full' properties are obtained by dropping $\aleph$. Thus, $\mathbb{F}$ is the class of all filters (on Y) and, for instance, $\mathcal{B}$ is *compact at* A (that is, at the family $\mathcal{A}$ composed of one set A), if it is $\mathbb{F}$-compact at A. This, of course, happens, if $\mathcal{B}$ is $\aleph$-compact at A for each $\aleph$ or, for a fixed $\mathcal{B}$, whenever $\aleph$ is sufficiently large. For this reason 'full statements', that is, statements that are obtained 'by dropping $\aleph$', do not need to be formulated separately.

Dealing with cardinals, we use strict inequalities to capture the finitely compact case: we refer to $\aleph_0$-compactness as *finite compactness*. Finite compactness may seem a trivial notion. Yet, it is important because of the following observation [7]:

LEMMA 2.1 *$\mathcal{B}$ aims at A if and only if $\mathcal{B}$ is finitely compact at A.*

Hence, if $\mathcal{N}$ is any filter base contained in $\mathcal{U}(x_0)$, we have

LEMMA 2.2 *A set K is an $\mathcal{N}$-kernel of F at x_0 if and only if the external filter relative to $\mathcal{N}$, $E(\mathcal{N})$, is finitely compact at K.*

On the other hand, it is customary to refer to $\aleph_1$-compactness as *countable compactness*. Hence, for instance, $\mathcal{B}$ is *countably compact*, if any filter $\mathcal{F}$ having a countable base and meshing with $\mathcal{B}$ has a cluster point (in Y).

For the following theorem and other results about compactness of filter bases, see [14] (where the term 'compactoid' instead of 'compact' was used).

THEOREM 2.3 *Suppose $\mathcal{B} \rightsquigarrow A$. If A is $\aleph$-compact, then $\mathcal{B}$ is $\aleph$-compact at A.*

The *G_δ-topology* of Y is the one for which the original G_δ-subsets of Y are declared to be basic open sets. For instance, any member of the Borel σ-field generated by closed subsets of Y is G_δ-closed.

PROPOSITION 2.4 *Let Y be regular and $\mathcal{B}$ be countably compact at $\mathcal{A}$. Suppose $K \subset Y$ is G_δ-closed and $\mathcal{A} \rightsquigarrow K$. Then $adh\mathcal{B} \subset K$.*

PROOF Suppose there exists $y \in adh\mathcal{B} \setminus K$. As K is G_δ-closed, there exist open sets $\{G_n : n \in \mathbb{N}\}$ such that $y \in \bigcap G_n$ and $K \cap \bigcap G_n = \emptyset$. As Y is regular, we can find a decreasing sequence (H_n) of closed neighborhoods of y such that $H_n \subset G_n$ for $n \in \mathbb{N}$. Let $\mathcal{H}$ be the filter generated by (H_n). Then, $adh\mathcal{H} = \bigcap H_n$ is disjoint with K and its complement $V := Y \setminus adh\mathcal{H}$ is an open set containing K. Hence there exists $A \in \mathcal{A}$ such that $A \subset V$, i.e., A is disjoint with $adh\mathcal{H}$.

On the other hand, as $y \in adh\mathcal{B}$ and H_n's have nonempty interiors, $\mathcal{H}\#\mathcal{B}$. But $\mathcal{B}$ is countably compact at $\mathcal{A}$, so $\mathcal{H}$ has a cluster point in A for each $A \in \mathcal{A}$. A contradiction. ∎

Taking $\mathcal{A} = \{K\}$, we have

COROLLARY 2.5 *Let Y be regular and $\mathcal{B}$ be countably compact at K. If K is G_δ-closed, then $adh\mathcal{B} \subset K$.*

3. Countable compactness of external filters. Let $(x_n) = (x_n)_{n=1}^\infty$ be a sequence of points of a space X. Recall that a point $x_0 \in X$ is a *cluster* point of the *sequence* (x_n) if, for every neighborhood U of x_0 and for every $m \in \mathbb{N}$, there exists $n > m$ such that $x_n \in U$. On the other hand, x_0 is an *accumulation* point of a set A if for every U there exists $x \in A, x \neq x_0$ such that $x \in U$.

According to the previous section, a subset D of X is *countably compact at* $A \subset X$, if every sequence of points of D has a cluster point in A; if $A = X$, we drop X and call it *relatively countably compact.* Note that with this terminology, D is *countably compact* if it is countably compact at itself.

A point $x_0 \in X$ is called *q-point* if it admits a *q-sequence* $(Q_n)_{n=1}^{\infty}$, that is, a decreasing sequence of neighborhoods of x_0 having the following property: if $x_n \in Q_n$, $n = 1, 2 \ldots$, then the sequence (x_n) has a cluster point in X.

A space X is said to be *q-space* [15] if each of its points is a q-point. A point *of countable character* , i.e., one whose filter of neighborhoods is countably based, is obviously a q-point. Another example is provided by a point which admits a relatively countably compact neighborhood V. Somewhat more interesting examples of q-spaces are provided by Čech complete spaces.

The results of this paper can be seen as consequences of the following folklore lemma.

LEMMA 3.1 *Let $(x_n) \subset X$ be a sequence having x_0 as its cluster point and let $y_n \in F(x_n) \setminus F(x_0)$. Suppose F is upper semicontinuous at x_0. Then the set $\{y_n : n \in \mathbb{N}\}$ has an accumulation point belonging to $F(x_0)$.*

PROOF Denote by C the closure of the set $\{y_n : n \in \mathbb{N}\}$ and suppose C is disjoint with $F(x_0)$. Then $V = Y \setminus C$ is an open set containing $F(x_0)$ so, by the upper semicontinuity of F at x_0, for some $U \in \mathcal{U}(x_0)$, $F(U) \subset V$. Thus for arbitrarily large indices $k \in \mathbb{N}$, $x_k \in U$. Yet, for the corresponding y_k's ,

$$y_k \in F(x_k) \subset V \subset Y \setminus \{y_1, y_2, \ldots\},$$

a contradiction. Hence, there is $y \in C \cap F(x_0)$. In particular, $y \neq y_n$ for $n \in \mathbb{N}$. ■

REMARK 3.2 The lemma is also used – and attributed to [1] – in [3]. One can check that it actually is *not* stated in [1]. I call it folklore, because it must have been around all the time. Choquet, in order to discover his theorem, must have known it in some form. I knew it around 1985 when writing [13]. But only its 'limit point' version was used and, therefore, stated in that paper. It is given here in full, because it is crucial and its proof is easy.

If individual points do not have accumulation points, the accumulation point y exhibited in the proof above must also be a *cluster point of the sequence* (y_n). As we need the lemma in the latter form, from now on

(1) *Y is a range space which is assumed to be (at least) T_1*

Further, in the domain we assume

(2) *x_0 is a q-point in a regular space X*

A q-sequence at x_0 is denoted by $(Q_n : n \in \mathbb{N}))$ and $\mathcal{Q}$ is its q-filter i.e., the filter generated by the chosen q-sequence $(Q_n : n \in \mathbb{N}))$. $\mathcal{Q}^\bullet = \bigcap \mathcal{Q} = \bigcap Q_n$.

(3) F *is usc at each* x *in* $\mathcal{Q}^\bullet$.

$\mathcal{U} = \mathcal{U}(x_0)$ stands for the filter of neighborhoods of x_0 and $E(\mathcal{U}), E(\mathcal{Q})$ denote the image filters of $\mathcal{U}$ and $\mathcal{Q}$, respectively, by the external part E (of F at x_0).

Standing assumptions (1), (2) and (3) will not be repeated.

LEMMA 3.3 *Let* $N \in \mathcal{U}$ *and* $V_1 \supset V_2 \supset \dots$ *be a base of a filter* $\mathcal{V}$ *such that* $\mathcal{V}\#E(\mathcal{Q})$. *There exist* $(x_n, y_n) \in Q_n \times V_n,\ n \in \mathbb{N}$, *such that* $y_n \in F(x_n) \setminus F(x_0)$ *and the sequence* (y_n) *is countably compact at* $F(\mathcal{Q}^\bullet) \cap F(N)$.

PROOF We begin by choosing a closed neighborhood L_1 of x_0 such that $L_1 \subset Q_1 \cap N$. As $\mathcal{V}\#E(\mathcal{Q})$, we choose

$$y_1 \in V_1 \cap F(L_1) \setminus F(x_0)$$

and $x_1 \in L_1$ so that $y_1 \in F(x_1)$.

Then we choose a closed neighborhood L_2 of x_0 with

$$L_2 \subset L_1 \cap Q_2 \cap \{x \in X : F(x) \subset Y \setminus \{y_1\}\}.$$

As $\mathcal{V}\#E(\mathcal{Q})$, we choose $y_2 \in V_2 \cap F(L_2) \setminus F(x_0)$ and $x_2 \in L_2$ such that $y_2 \in F(x_2) \setminus F(x_0)$ etc...

Suppose that, for some $n \in \mathbb{N}$ and $i = 1, 2, \dots n$, we have already selected $L_i,\ y_i \in V_i \cap F(L_i) \setminus F(x_0)$ and $x_i \in L_i$ such that $y_i \in F(x_i) \setminus F(x_0)$. Using the upper semi-continuity of F at x_0, the fact that y_i's do not belong to $F(x_0)$, and the regularity of X, we choose a closed neighborhood L_{n+1} of x_0 contained in

$$Q_{n+1} \cap L_n \cap \{x : F(x) \subset Y \setminus \{y_1, y_2, \dots y_n\}\}.$$

As $\mathcal{V}\#E(\mathcal{Q})$, we further choose

$$y_{n+1} \in V_{n+1} \cap F(L_{n+1}) \setminus F(x_0)$$

and $x_{n+1} \in L_{n+1}$ so that $y_{n+1} \in F(x_{n+1})$.

Hence, we have inductively defined the required sequence $((x_n, y_n))$. As (Q_n) is a q-sequence, (x_n) must have a cluster point, say ξ. We note that $\xi \in \bigcap L_n \subset \mathcal{Q}^\bullet$. By the choice of (L_n), for each $n \in \mathbb{N}$,

$$\xi \in L_{n+1} \subset \{x : F(x) \cap \{y_1, y_2, \dots y_n\} = \emptyset\}$$

and so

$$F(\xi) \cap \{y_1, y_2, \dots\} = \emptyset.$$

Hence $y_n \in F(x_n) \setminus F(\xi)$ and, by our selection process, (y_n) is a sequence of distinct points. By Lemma 3.1, the sequence (y_n) must have a cluster point in $F(\xi)$, call it η. It is clear that η belongs to $(adhE(\mathcal{Q})$ and) $F(\mathcal{Q}^\bullet) \cap F(N)$.

The set $\{y_n : n \in \mathbb{N}\}$ obtained in the selection process is actually *countably compact at* $F(\mathcal{Q}^\bullet)$. Indeed, let (y_i) be an extracted sequence of distinct points.

Although (y_i) must not necessarily be a subsequence of the sequence (y_n), by passing to a further subsequence, we can assume it is. Setting $(y_i) = (y_{n_i})$, we observe that for the corresponding subsequence (x_{n_i}) of (x_n) one has $x_{n_i} \in Q_i$. Therefore, as (Q_i) is a q-sequence, x_{n_i} will still have a cluster point, say $\zeta \in \mathcal{Q}^\bullet$. By the same argument as above $y_{n_i} \in F(x_{n_i}) \setminus F(\zeta)$ has a cluster point in $F(\zeta)$. ∎

REMARK 3.4 Comparing the *statements* of the lemma above and the one of Proposition 3.2 in [3], they appear to be totally different from each other. Yet their *proofs* are using essentially the same argument. This similarity is not quite accidental. Both proofs are 'cleaned up' versions of the proof of yet another apparently quite different statement: Lemma 9 of [11]. Especially the language of games used in [3] seems to be a clever description of the procedure which makes all these proofs work, see also final comments at the end of this paper.

Recall that a space Y is called *feebly compact* if any filter on Y which admits a countable base consisting of open sets, has a cluster point. As a consequence of the lemma, we deduce the following *VCD-Theorem 1*:

THEOREM 3.5 *$E(\mathcal{Q})$ is countably compact at $\operatorname{Frac}_{\mathcal{Q}} F(x_0) \cap F(\mathcal{Q}^\bullet)$. In particular, $\operatorname{Frac}_{\mathcal{Q}} F(x_0)$ is a feebly compact $\mathcal{Q}$-kernel of F at x_0. If $F(x_0)$ is G_δ-closed and Y is regular, then $\operatorname{Frac}_{\mathcal{Q}} F(x_0)$ (and so $\operatorname{Frac} F(x_0)$ as well) is contained in $F(x_0)$. $\operatorname{Frac}_{\mathcal{Q}} F(x_0)$ is the smallest G_δ-closed set at which $E(\mathcal{Q})$ is countably compact.*

PROOF The first sentence follows from Lemma 3.2. In view of Lemma 2.1, it is trivial that $\operatorname{Frac}_{\mathcal{Q}} F(x_0) \cap F(\mathcal{Q}^\bullet)$ is a $\mathcal{Q}$-kernel and so, a larger set $\operatorname{Frac}_{\mathcal{Q}} F(x_0)$ is so too. The fact that it is feebly compact, although nontrivial, is a general fact about adherences of countably compact filter bases (see [14]). For the last part observe that it follows from Lemma 3.2 that $E(\mathcal{Q})$ is countably compact at $F(\mathcal{U})$ and apply Proposition 2.4. and Corollary 2.5. ∎

As the regularity of Y was needed for the first time, it may be a proper moment to mention that in regular spaces the language of kernels and the one of compact filters are equivalent: by a classical result of Vaughan, a filter base is compact if and only if it aims at its adherence which is compact ([17], [14]).

4. Compactness of kernels. It will be convenient to write $y_n \curvearrowright y$ to denote that the set $\{y_n : n \in \mathbb{N}\}$ is relatively countably compact and y is its cluster point. We identify a sequence (y_n) with its elementary filter, that is, the filter generated by the tails $\{y_i : i \geq n\}, n = 1, 2, \dots$ and write $(y_n) \geq \mathcal{F}$ whenever the elementary filter of (y_n) is finer than the filter $\mathcal{F}$.

Now consider the following cluster sets

$$K = K(x_0) = \{y \in Y : \exists (x_n) \geq \mathcal{Q} \;\&\; y_n \in F(x_n) \setminus F(x_0), y_n \curvearrowright y\}$$

$$L = L(x_0) = K \cap F(\mathcal{Q}^\bullet)$$

and

$$M = M(x_0) = K \cap F(x_0)$$

Lemma 4.1 *The cluster set L is a Choquet $\mathcal{Q}$-kernel of F at x_0.*

We now give a few precisions on our terminology and introduce a few conditions on the range space Y. Let $\mathcal{B}$ be a family of subsets of Y.

A set B in Y is said to be *sequentially closed* if limits of sequences from B must belong to B. Y is said to be *sequential relative to* $\mathcal{B}$ if sequentially closed sets in $\mathcal{B}$ are closed.

Similarly, B is *countably closed* if cluster points of sequences from B must belong to B. Y is *semisequential relative to* $\mathcal{B}$ if countably closed sets in $\mathcal{B}$ are closed.

A set B in Y has *sequentially determined closure* if for each $y \in \overline{B}$ there exists a sequence $(y_n) \subset B$ such that $y_n \to y$. Following [7], Y is said to have the *Fréchet property relative to* $\mathcal{B}$ if each $B \in \mathcal{B}$ has sequentially determined closure.

Let us say that Y satisfies the $\aleph$-condition $(\mathcal{C})$ relative to $\mathcal{B}$, if, for every relatively countably compact set $B \in \mathcal{B}$, its closure $\overline{B}$ is $\aleph$-compact. According to our terminological conventions, $\aleph$ should be dropped if it can be arbitrary. Thus Y satisfies the condition $(\mathcal{C})$ relative to all subsets if the *closures of relatively countably compact subsets of* Y *are compact*. For simplicity's sake, as no ambiguity can arise in this paper, we will say that Y *satisfies the condition* $(\mathcal{C})$ if it has the italicized property.

Similarly, Y satisfies the countable condition $(\mathcal{C})$ relative to countable subsets, if for each relatively countably compact set $\{y_n : n \in \mathbb{N}\} \subset Y$, its closure $\overline{\{y_n : n \in \mathbb{N}\}}$ is countably compact. We refer to this condition as *countable condition* $(\mathcal{C})$ or *condition* $(\mathcal{C}_\sigma)$.

Proposition 4.2 *The cluster set K is countably closed. If Y satisfies the condition $(\mathcal{C}_\sigma)$, then K is countably closed countably compact.*

Proof Let (k_i) be a sequence of distinct points in K and let k be its cluster point. For each $i \in \mathbb{N}$, we find a sequence $(x^i_j : j \in \mathbb{N})$ finer than $\mathcal{Q}$ and a sequence $(y^i_j : j \in \mathbb{N})$ such that $y^i_j \in F(x^i_j) \setminus F(x_0)$ with $x^i_j \curvearrowright k^i$. As $(x^i_j)_{j=1}^\infty \geq \mathcal{Q}$, we can find j_i such that $\{x^i_j\}_{j=j_i}^\infty \subset Q_i$ for $i = 1, 2, \ldots$ To simplify notation, we can assume that $(x^i_j : j \in \mathbb{N})$ had been chosen that way from the very beginning. Let (x_k) and (y_k) be the sequences obtained by reordering the double sequences (x^i_j) and (y^i_j), respectively, into a single sequence, taking together the elements for which $i+j$ has a common value and ordering these groups in increasing order of $i+j$. Then $(x_k) \geq \mathcal{Q}$ and $y_k \in F(x_k) \setminus F(x_0)$.

We need to show that the single sequence (y_k) so defined must be relatively countably compact. Let $\{(\eta_n : n \in \mathbb{N}\}$ be an infinite subset of the set $\{y_k : k \in \mathbb{N}\}$. By passing to a further subsequence, we may assume that (η_n) is actually a subsequence of the original sequence (y_k). By the very definition of reordering we have used, the elementary filter generated by (η_n) is meshing with $E(\mathcal{Q})$. It must therefore have a cluster point because $E(\mathcal{Q})$ is countably compact.

As it is clear that k is a cluster point of (y_k), we have shown that $k \in K$. This proves the first statement. Now observe that $\{k_i : i \in \mathbb{N}\} \subset \overline{\{y_n : n \in \mathbb{N}\}}$ and, if the condition $(\mathcal{C}_\sigma)$ is satisfied, the existence of a cluster point k is guaranteed. Hence the fact that $k \in K$ shows the countable compactness of K. ∎

It will be visible from the proof of the next theorem, *VCD-Theorem 2*, that it is crucial to reach the (full) compactness of a kernel. For that, unfortunately, another condition on the space Y is needed. Let us call Y *equicompact* if it satisfies the condition $(\mathcal{C})$ relative to the family of countably closed subsets. That is, we impose that *countably closed countably compact subsets are relatively compact in* Y.

THEOREM 4.3 *Let* Y *be an equicompact Hausdorff space satisfying* $(\mathcal{C}_\sigma)$. *Then* $\overline{K} = \operatorname{Frac}_{\mathcal{Q}} F(x_0)$ *is compact. In particular,* $E(\mathcal{Q})$ *and* $E(\mathcal{U})$ *are compact and* $\operatorname{Frac} F(x_0)$ *is the smallest compact kernel of* F *at* x_0.

PROOF By Proposition 4.2 and the assumption on Y, $\overline{K}$ is now compact. It is clear that $\overline{K} \subset adhE(\mathcal{Q})$. As Y is Hausdorff, the compact set $\overline{K}$ is the intersection of its closed neighborhoods. Let $V(\overline{K}) = V$ be such a closed neighborhood. There exists n so large that $E(Q_n) \subset V$ and, therefore $adhE(\mathcal{Q}) \subset V$. It follows that $\operatorname{Frac}_{\mathcal{Q}} F(x_0) \subset \overline{K}$ and so we have the equality. By Theorem 2.3 (with $\aleph$ dropped), $E(\mathcal{Q})$ is compact. Hence $E(\mathcal{U})$ is also compact (necessarily – at its adherence) and therefore $\operatorname{Frac} F(x_0)$ is not empty. Further, as a closed subset of a compact set, $\operatorname{Frac} F(x_0)$ is compact.

Finally, note that the argument just used to show the inclusion of $\operatorname{Frac}_{\mathcal{Q}} F(x_0)$ in $\overline{K}$, repeated for an arbitrary compact kernel of F at x_0, shows the minimality property of $\operatorname{Frac} F(x_0)$. ∎

REMARK 4.4 (1) Other combinations of conditions can be used to reach our goal. For instance, if $(\mathcal{C}_\sigma)$ is satisfied and Y is semisequential relative to relatively countably compact sets, then K is closed and countably compact. Spaces in which closed countably compact subsets coincide with the compact ones are often called *isocompact*. Though the conjunction of the just mentioned semisequentiality property and isocompactness is more than equicompactness, we gain the equality of K and $\operatorname{Frac}_{\mathcal{Q}} F(x_0)$ in that case. If Y is regular, the latter equality holds without the assumption of isocompactness of Y.

(2) The condition $(\mathcal{C})$ appeared already in [13] (under the name of 'countably determined compactness'). Since it simplifies the statement of the theorem, we used it in the abstract.

5. Choquet kernels. A condition on the value $F(x_0)$ assuring $\operatorname{Frac} F(x_0)$ to be a Choquet kernel was given in 3.5. We now turn to conditions on Y.

We define the following cluster set

$$C = C(x_0) = \{y \in F(\mathcal{Q}^\bullet) : \exists\, (x_n) \geq \mathcal{Q}\ \&\ y_n \in F(x_n) \setminus F(x_0), y_n \to y\}.$$

PROPOSITION 5.1 *Let* Y *be a Hausdorff space having the Fréchet property relative to its countable relatively countably compact sets. Then* $C = L = K$ *and* K *is the smallest Choquet* Q*-kernel of* F *at* x_0.

Proof We first show $K \subset C$. Indeed, if $y \in K$, then by the Fréchet property there exists a subsequence y_{n_k} converging to y. But then the corresponding subsequence (x_{n_k}) is still finer than $\mathcal{Q}$ and, as y is the unique cluster point of y_{n_k}, Lemma 3.1 implies that $y \in F(\mathcal{Q}^\bullet)$. This shows that $y \in C$. Hence $K = C$.

Let D be a Choquet $\mathcal{Q}$-kernel of F at x_0. We need to show that $C \subset D$. Consider the map E_D defined by $E_D(x) = E(x)$ for $x \neq \mathcal{Q}^\bullet$ and $E_D(x) = D$ otherwise. The fact that D is a $\mathcal{Q}$-kernel means that $E_D(\mathcal{Q}) \rightsquigarrow D$. If $y \in C \setminus D$, then there exist $(x_n) \geq \mathcal{Q}$ and the corresponding $y_n \in E(x_n)$ converging to y. Since $y_n \in E(x_n)$, by Lemma 3.1 again, the unique cluster point y of (y_n) must be in D contradicting the choice of y. ■

Thus, we are prompted to say that Y satisfies the condition $(\mathcal{A}_\sigma)$ if it satisfies $(\mathcal{C}_\sigma)$ and, moreover,

$$y \in \overline{\{y_n\}} \implies \exists (y_{n_k}),\ y_{n_k} \to y.$$

It seems fitting to use the name *countably angelic* for Hausdorff spaces satisfying $(\mathcal{A}_\sigma)$. The following *VCD-Theorem 3* can now be stated for such spaces:

Theorem 5.2 *Let Y be countably angelic. Then the cluster set $C(x_0)$ is countably closed, countably compact and sequentially compact. Moreover, it is the smallest Choquet $\mathcal{Q}$-kernel of F at x_0.*

Remark 5.3 Of course, a proper 'isocompactness' condition will assure compactness. But assume again $(\mathcal{C}_\sigma)$ only, together with the semisequentiality property considered in Remark 4.4 (1). Then, if Y is regular or Hausdorff isocompact, $\operatorname{Frac} F(x_0) = K$, but $K = L$ is not reached. For more on the minimality of kernels see [7].

6. Points of countable character. If the filter of all neighborhoods $\mathcal{U}(x_0)$ admits a countable base, this base can be taken as a q-sequence. In the proof of Lemma 3.2 the cluster points ξ, ζ can be taken equal to x_0 and therefore the lemma and, consequently, all the results above *remain valid under the assumption that F is usc at the single point x_0 and without any separation property on X*. As $\mathcal{Q} = \mathcal{U}$, statements that refer to $\operatorname{Frac}_{\mathcal{Q}} F(x_0)$ and $\mathcal{Q}$-kernels become statements about $\operatorname{Frac} F(x_0)$ and kernels.

There is one small detail concerning separation properties of X which needs to be taken care of. Of course, if X is T_1, then $\bigcap \mathcal{U}(x_0) = x_0$ and so our definition of Choquet kernel coincides with the usual one that requires the kernel to be a subset of $F(x_0)$. But even if X is arbitrary, the needed equality holds.

Lemma 6.1 *If F is usc at x_0, then $F(\mathcal{U}^\bullet) = F(x_0)$.*

Proof Suppose not and let $x \in \bigcap \mathcal{U}(x_0)$. Let $y \in F(x) \setminus F(x_0)$. Then the constant sequence $x, x, x, \ldots$ has x_0 as its cluster point. But then $y, y, \ldots$ must have a cluster point in $F(x_0)$, i.e., $y \in F(x_0)$; a contradiction. ■

Thus, $L = M$ where, if x_0 is of countable character, we can write

$$M = \{y \in F(x_0) : \exists x_n \to x_0 \ \&\ y_n \in F(x_n) \setminus F(x_0), y_n \curvearrowright y\}.$$

To conclude, the *unity between the case of a q-point and that of a point of countable character is fully achieved.* Compare the introduction to [7], where the 'Michael quest' and the 'Choquet quest' are discussed.

As a final comment, let me point out that the denomination 'Choquet kernel' reflects the somewhat convoluted history of the subject of Vaĭnšteĭn-Choquet-Dolecki Theorems. Choquet was actually the first to assert (without proof) the existence of a compact 'kernel' for a usc map F between metric spaces X, Y in [4]. A little earlier, Vaĭnšteĭn ([16], Theorem 1) had a special case of the theorem (with proof) for a closed continuous function $f : Y \to X$. It took thirty years before Dolecki introduced the notion of active boundary into formal existence, provided a proof of the theorem of Choquet and made the connection between the results of Vaĭnšteĭn and Choquet ([5], [6], [9], [8]). To keep the confusion alive and well, the authors of [2] repeated the results obtained earlier in [13]. They take a different interesting approach in [3] using topological games. This extends the class of spaces which can serve as domain spaces. On the other hand, their condition on the range space Y is stronger than our conditions here and no discussion of Choquet kernels is offered.

References

[1] G. Beer, *Topologies on Closed and Closed Convex Sets,* Kluwer Academic Publishers 1993.

[2] J. Cao, W. Moors and I. Reilly, *On the Choquet-Dolecki Theorem,* J.Math.Anal.Appl. **234** (1999), 1-5.

[3] J. Cao, W. Moors and I. Reilly, *Topological properties defined by games and their applications,* Topology Appl. **123** (2002), 47-55.

[4] G. Choquet, *Convergences,* Ann.Univ.Grenoble **23** (1948), 57-112.

[5] S. Dolecki, *Remarks on semicontinuity,* Bull.Acad.Polon.Sci.,Sér.Sci.Math. **25** (1977), 863-867.

[6] S. Dolecki, *Semicontinuity in constrained optimization II,* Control Cybernet. **7(2)** (1978), 5-16.

[7] S. Dolecki, *Active boundaries of upper semicontinuous and compactoid relations; closed and inductively perfect maps,* Rostock Math. Kolloq. **54** (2000), 51-68.

[8] S. Dolecki and A. Lechicki, *On structure of upper semicontinuity,* J.Math.Anal.Appl. **88** (1982), 547-554.

[9] S. Dolecki and S. Rolewicz, *Metric characterization of upper semicontinuity,* J.Math. Anal.Appl. **69** (1979), 146-152.

[10] L. Drewnowski, I. Labuda, *On minimal upper semicontinuous compact-valued maps,* Rocky Mountain J. Math. **20** (1990), 1-16.

[11] R. Hansell, J. Jayne, I. Labuda and C. A. Rogers, *Boundaries of and selectors for upper semicontinuous multi-valued maps,* Math. Z. **189** (1985), 297-318.

[12] J. Kelley, *General Topology,* Van Nostrand 1955.

[13] I. Labuda, *On a theorem of Choquet and Dolecki,* J.Math.Anal.Appl. **126** (1987), 1-10.

[14] I. Labuda, *Compactoidness,* Rocky Mountain J. Math. **36** (2006).

[15] E. Michael, *A note on closed maps and compact sets,* Israel J. Math. **2** (1964), 173-176.

[16] I. A. Vaĭnšteĭn, *On closed mappings of metric spaces,* Dokl.Akad.Nauk SSSR **57** (1947), 319-321 (Russian).

[17] J. E. Vaughan, *Total nets and filters*, Topology (Proc. Ninth Annual Spring Topology Conf., Memphis State Univ., Memphis, Tenn., 1975), Lecture Notes in Pure and Appl. Math., Vol. 24, pp. 259–265, Dekker, New York 1976.

Iwo Labuda
Department of Mathematics, The University of Mississippi
University, MS 38677, USA
E-mail: mmlabuda@olemiss.edu

(Received: 30.03.05)