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On the asymptotic behavior of solutions to nonlinear differential equations of the second order

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Summary We study the asymptotic behavior of solutions to a nonlinear differential equation of the second order whose coefficient of nonlinearity is a bounded function for arbitrarily large values of x in R. We obtain certain sufficient conditions which guarantee boundedness of solutions, their convergence to zero as $x \to \infty$ and their unboundedness.

Keywords
asymptotic behavior; nonlinear differential equation; second order

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1. Introduction

Nonlinear differential equations of the second order can be derived from many fields, such as physics, mechanics, and engineering. An important question is whether these equations have bounded solutions, solutions convergent to zero as $x \to \infty$ or unbounded solutions. In recent years, especially boundedness of solutions to certain nonlinear differential equations of the second order has been widely discussed, notably by Ademola and Arawomo [1], Bucur [2], Constantin [3], Kiguradze [5], Kusano et al. [6], Lipovan [7], Mingareilli and Sadarangani [8], Mustafa [9], Saker [10], Tong [11], Trench [12], Tunç [13–17], Tunç and Tunç [18], Waltman [19], and Wong [20]. However, there exist only a few papers con-

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cerned with the convergence or unboundedness of solutions of the same type of equations (Ezeilo [4], Qarawani [21]).

Recently, Qarawani [21] considered the following non-linear differential equation of the second order

$$z'' + p(x)z' + q(x)z = h(x)\,|z|^{\alpha}\, e^{\left(\frac{\alpha-1}{2}\right)\int p(x)\,dx}\,\operatorname{sgn} z \tag{1}$$

where $\alpha \in (-1, 1) \setminus \{0\}$, $q \in C^{0}[x_0, \infty)$, $x_0 > 0$, $h, p \in C^{1}[x_0, \infty)$, $p > 0$, and $h(x)$ is bounded for all sufficiently large x in R. He discussed boundedness of solutions of Eq. (1), their convergence to zero as $x \to \infty$ as well as their unboundedness.

We may assume that $z > 0$, because if $z < 0$, simply set $z = -u$, $u > 0$. Therefore, instead of Eq. (1), we can consider the differential equation

$$z'' + p(x)z' + q(x)z = h(x)\,|z|^{\alpha}\, e^{\left(\frac{\alpha-1}{2}\right)\int p(x)\,dx}, \quad \alpha \in (-1, 1) \setminus \{0\}\,. \tag{2}$$

It is clear that if $z(x) = y(x)\exp(-\frac{1}{2}\int p(x)\,dx)$ then Eq. (2) reduces to the equation

$$y''(x) + y(x) = h(x)y^{\alpha}(x), \alpha \in (-1, 1) \setminus \{0\} \tag{3}$$

provided that

$$q(x) - \frac{1}{4}p^{2}(x) - \frac{1}{2}p'(x) = 1. \tag{4}$$

Qarawani [21] proved that if $h(x)$ is a continuously differentiable function that is bounded for all sufficiently large $x \in R$ and the integral $\int_{x_0}^{\infty} |h''(x)|\, dx$ is convergent, then any solution of Eq. (3) is bounded as $x \to \infty$. He also showed that if $h(x)$ satisfies the above conditions and $\int_{x_0}^{\infty} |h(x)|\, dx < \infty$, then for any solution $y(x)$ of Eq. (3), the asymptotic formula $y(x) = A\sin(x + w_0) + O\left(\int_{x_0}^{\infty} |h(x)|\, dx\right)$ holds. Finally, Qarawani proved that if $h(x)$ is a continuously differentiable function that is bounded for all sufficiently large $x \in R$ and $\int_{x_0}^{\infty} |h'(x)|\, dx = \infty$, then any solution of Eq. (3) is unbounded as $x \to \infty$.

In this paper, instead of the condition (4) we discuss the results of Qarawani [21] under the following condition

$$q(x) - \frac{1}{4}p^{2}(x) - \frac{1}{2}p'(x) = \frac{1}{x^{2}}. \tag{5}$$

Condition (4) is a special case of our condition (5). Our aim is to improve the results established in [21], with condition (5) instead of condition (4). This paper is inspired by the results of Qarawani [21] and other results mentioned above. It develops and complements the work of Qarawani. The obtained results are useful for the study of the qualitative behavior of solutions to differential equations of higher order. It should be noted that the assumptions and results of this paper are different from those found in the literature.

2. Boundedness of solutions

2.1. Lemma. *Substitution* $z(x) = y(x)\exp(-\frac{1}{2}\int p(x)\,dx)$ *reduces Eq. (2) to*

$$x^2 y''(x) + y(x) = x^2 h(x) y^{\alpha}(x) \tag{6}$$

where $\alpha \in (-1,1)\setminus\{0\}$ *and* $q(x) - \frac{1}{4}p^2(x) - \frac{1}{2}p'(x) = \frac{1}{x^2}$.

Proof. Let $z(x) = y(x)\exp(-\frac{1}{2}\int p(x)\,dx)$. Then we have

$$z' = y'(x)\exp\left(-\frac{1}{2}\int p(x)\,dx\right) - \frac{1}{2}p(x)\exp(-\frac{1}{2}\int p(x)\,dx)y(x)$$

and

$$z'' = y''(x)\exp(-\frac{1}{2}\int p(x)\,dx) - y'(x)p(x)\exp(-\frac{1}{2}\int p(x)\,dx)$$
$$+ \frac{1}{4}p^2(x)\exp(-\frac{1}{2}\int p(x)\,dx)y(x) - \frac{1}{2}p'(x)\exp(-\frac{1}{2}\int p(x)\,dx)y(x).$$

Hence, substituting the expressions for $z(x), z'(x), z''(x)$ into Eq. (2) and dividing by

$$\exp\Big(-\frac{1}{2}\int p(x)\,dx\Big),$$

we get Eq. (6). □

2.2. Example. Consider the second-order differential equation

$$z''(x) + 4xz'(x) + \left(4x^2 + \frac{1}{x^2} + 2\right)z(x) = e^{-\left(\frac{x^2}{2}+x\right)} z^{\frac{1}{2}}(x). \tag{7}$$

From a comparison of Eq. (7) and Eq. (2) it follows that

$$p(x) = 4x, \quad q(x) = 4x^2 + \frac{1}{x^2} + 2, \quad \alpha = \frac{1}{2}.$$

Letting

$$z(x) = y(x)e^{\left(-\frac{1}{2}\int p(x)\,dx\right)} = y(x)e^{\left(-\frac{1}{2}\int 4x\,dx\right)} = y(x)e^{-x^2},$$

we get

$$z' = y'(x)e^{-x^2} - 2xy(x)e^{-x^2}$$

and

$$z'' = y''(x)e^{-x^2} - 4xy'(x)e^{-x^2} - 2y(x)e^{-x^2} + 4x^2y(x)e^{-x^2}.$$

Substituting $z(x), z'(x), z''(x)$ into Eq. (7), we obtain

$$y''(x)e^{-x^2} - 4xe^{-x^2}y''(x) - 2e^{-x^2}y(x) + 4x^2e^{-x^2}y(x) + 4xe^{-x^2}y''(x)$$
$$- 8x^2e^{-x^2}y(x) + 4x^2e^{-x^2}y(x) + \frac{y(x)e^{-x^2}}{x^2} + 2e^{-x^2}y(x) = e^{-(x^2+x)}y^{\frac{1}{2}}(x),$$

so

$$x^2y''(x) + y(x) = x^2e^{-x}y^{\frac{1}{2}}(x).$$

Note that condition $q(x) = \frac{1}{x^2} + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x)$ holds for the last equation.

2.3. Theorem. *Assume that $x^2h(x)$ is a continuously differentiable function that is bounded for all sufficiently large $x \in R$ and that $\int_{x_0}^{\infty}\left|x^2h'(x)\right| dx$ and $\int_{x_0}^{\infty} y'^2(x)\,dx$ are convergent. Then any solution of Eq. (6) is bounded as $x \to \infty$.*

Proof. By letting $x = e^t$, Eq. (6) reduces to

$$y''(t) - y'(t) + y(t) = e^{2t}h(e^t)y^{\alpha}(t). \tag{8}$$

Multiplying both sides of Eq. (8) by y' and integrating the result with respect to t from some positive t_0 to t, we get

$$\int_{t_0}^{t} y'(s)\,y''(s)\,ds - \int_{t_0}^{t} y'^2(s)ds + \int_{t_0}^{t} y(s)\,y'(s)\,ds = \int_{t_0}^{t} e^{2s}h(e^s)y^{\alpha}(s)y'(s)\,ds$$

$$y'^2(s)\,\big|_{t_0}^{t} + y^2(s)\,\big|_{t_0}^{t} = 2\int_{t_0}^{t} y'^2(s)ds + 2\int_{t_0}^{t} e^{2s}h(e^s)y^{\alpha}(s)y'(s)\,ds$$

$$y'^2(t) - y'^2(t_0) + y^2(t) - y^2(t_0) = 2\int_{t_0}^{t} y'^2(s)ds + 2\int_{t_0}^{t} e^{2s}h(e^s)y^{\alpha}(s)y'(s)\,ds.$$

Integrating the integral on the right-hand side by parts yields

$$\begin{aligned} y'^2(t) + y^2(t) &= y'^2(t_0) + y^2(t_0) + 2\int_{t_0}^{t} y'^2(s)ds - \frac{2e^{2t_0}h(e^{t_0})y^{\alpha+1}(t_0)}{\alpha+1} \\ &+ \frac{2e^{2t}h(e^t)y^{\alpha+1}(t)}{\alpha+1} - \frac{4}{\alpha+1}\int_{t_0}^{t} e^{2s}h(e^s)y^{\alpha+1}(s)ds - \frac{2}{\alpha+1}\int_{t_0}^{t} e^{3s}h'(e^s)y^{\alpha+1}(s)ds. \end{aligned}$$

Hence

$$\begin{aligned} y^2(t) &\leqslant y'^2(t) + y^2(t) \\ &\leqslant A_{t_0} + \frac{2\left|e^{2t}h(e^t)\right|\left|y^{\alpha+1}(t)\right|}{\alpha+1} + 2\int_{t_0}^{t}\left|y'^2(s)\right|ds + \frac{2}{\alpha+1}\int_{t_0}^{t}\left|e^{3s}h'(e^s)\right|\left|y^{\alpha+1}(s)\right|ds \end{aligned}$$

where $A_{t_0} \geqslant 0$ is an expression dependent only on t_0.

Let $M = \max_{t_0 \leqslant s \leqslant t}|y(s)|$. Without loss of generality we may assume that $M \geqslant a_0 > 0$, otherwise the theorem is proved. Since $e^{2t}h(e^t)$ is bounded, we have

$$\begin{aligned} M^2 &\leqslant A_{t_0} + \frac{2BM^{\alpha+1}}{\alpha+1} + 2\int_{t_0}^{t}\left|y'^2(s)\right|ds + \frac{2M^{\alpha+1}}{\alpha+1}\int_{t_0}^{t}\left|e^{3s}h'(e^s)\right|ds, \quad \alpha \in (-1,1)\setminus\{0\} \\ M^{1-\alpha} &\leqslant \frac{A_{t_0}}{M^{\alpha+1}} + \frac{2B}{\alpha+1} + \frac{2}{M^{\alpha+1}}\int_{t_0}^{t}\left|y'^2(s)\right|ds + \frac{2}{\alpha+1}\int_{t_0}^{t}\left|e^{3s}h'(e^s)\right|ds \\ M^{1-\alpha} &\leqslant \frac{A_{t_0}}{(a_0)^{\alpha+1}} + \frac{2B}{\alpha+1} + \frac{2}{(a_0)^{\alpha+1}}\int_{t_0}^{\infty}\left|y'^2(s)\right|ds + \frac{2}{\alpha+1}\int_{t_0}^{\infty}\left|e^{3s}h'(e^s)\right|ds. \end{aligned}$$

Since the integrals $\int_{t_0}^{\infty}\left|e^{3s}h'(e^s)\right| ds$ and $\int_{t_0}^{\infty}\left|y'^2(s)\right| ds$ are convergent,

$$|y(t)| \leqslant M \leqslant (C + 2D + 2E + 2F)^{\frac{1}{1-\alpha}}, \quad \alpha \in (-1, 1) \setminus \{0\}.$$

Therefore, $y(t)$ is bounded for $t \to \infty$. Hence $y(x)$ is also bounded as $x \to \infty$. □

We give an example illustrating the theorem.

2.4. Example. Consider the second-order differential equation

$$x^2 y''(x) + y(x) = \frac{\sqrt{y}}{x}.$$

We will show that all its solutions are bounded for $x \to \infty$. It is clear that

$$\alpha = \frac{1}{2}, \quad h(x) = \frac{1}{x^3}, \quad \left|x^2 h(x)\right| = \left|\frac{1}{x}\right| \leqslant 1 \text{ for } |x| \geqslant 1,$$
$$\int_1^{\infty}\left|x^2 h'(x)\right| dx = \int_1^{\infty}\left|\frac{-3}{x^2}\right| dx = \int_1^{\infty}\frac{3}{x^2}\, dx \quad \text{converges to 3.}$$

Applying to the above differential equation the same approach as in the proof of the theorem, we get

$$\begin{aligned} M^{\frac{1}{2}} &\leqslant \frac{A_{t_0}}{M^{\frac{3}{2}}} + \frac{4B}{3} + \frac{2}{M^{\frac{3}{2}}}\int_{t_0}^{t}\left|y'^2(s)\right| ds + \frac{4}{3}\int_{t_0}^{t}\left|e^{3s}h'(e^s)\right| ds. \\ &\leqslant \frac{A_{t_0}}{(a_0)^{\frac{3}{2}}} + \frac{4B}{3} + \frac{2C}{(a_0)^{\frac{3}{2}}} + \frac{4}{3}\int_{t_0}^{\infty}\left|e^{3s}h'(e^s)\right| ds \end{aligned}$$

Since the integral $\int_{t_0}^{\infty}\left|e^{3s}h'(e^s)\right| ds$ converges,

$$|y(t)| \leqslant M \leqslant (D + 4)^2 \quad \text{as} \quad t \to \infty.$$

Therefore, $y(t)$ is bounded for $t \to \infty$, hence $y(x)$ is also bounded as $x \to \infty$.

3. Unboundedness of solutions

3.1. Theorem. *Suppose that $x^2h(x)$ is a continuously differentiable function that is bounded for $x \in [x_0, \infty)$ and assume that $\int_{x_0}^{\infty} \left|x^2h'(x)\right| dx = \infty$, $\int_{x_0}^{\infty} y'^2(x)\,dx$ is convergent and $|y(x)| \geqslant a$ for some $a > 0$ and all $x \in [x_0, \infty)$. Then any solution of Eq. (2) is unbounded as $x \to \infty$.*

Proof. Since the function $x^2h(x)$ is bounded, there exists a positive constant L such that $\left|x^2h(x)\right| \leqslant L$ for all sufficiently large x. Suppose that the solution $y(t)$ of Eq. (8) is bounded for sufficiently large t. Hence there exists a positive constant M such that $M = \max_{t_0 \leqslant s \leqslant t} |y(s)|$ for all sufficiently large t.

Multiplying both sides of Eq. (8) by y' and integrating the result with respect to t from some positive t_0 to t, we get

$$0 \leqslant y'^2(t) + y^2(t) = y'^2(t_0) + y^2(t_0) + 2\int_{t_0}^{t} y'^2(s)ds - \frac{2e^{2t_0}h(e^{t_0})y^{\alpha+1}(t_0)}{\alpha+1}$$
$$+ \frac{2e^{2t}h(e^{t})y^{\alpha+1}(t)}{\alpha+1} - \frac{4}{\alpha+1}\int_{t_0}^{t} e^{2s}h(e^{s})y^{\alpha+1}(s)ds - \frac{2}{\alpha+1}\int_{t_0}^{t} e^{3s}h'(e^{s})y^{\alpha+1}(s)ds.$$

It follows that

$$y'^2(t_0) + y^2(t_0) + 2\int_{t_0}^{t} y'^2(s)ds - \frac{2e^{2t_0}h(e^{t_0})y^{\alpha+1}(t_0)}{\alpha+1} + \frac{2e^{2t}h(e^{t})y^{\alpha+1}(t)}{\alpha+1}$$
$$- \frac{4}{\alpha+1}\int_{t_0}^{t} e^{2s}h(e^{s})y^{\alpha+1}(s)ds \geqslant \frac{2}{\alpha+1}\int_{t_0}^{t} e^{3s}h'(e^{s})y^{\alpha+1}(s)ds \geqslant \int_{t_0}^{t} e^{3s}h'(e^{s})y^{\alpha+1}(s)ds,$$

so

$$\left|y'^2(t_0) + y^2(t_0)\right| + \left|\frac{2e^{2t_0}h(e^{t_0})y^{\alpha+1}(t_0)}{\alpha+1}\right| + \left|\frac{2e^{2t}h(e^{t})y^{\alpha+1}(t)}{\alpha+1}\right| + 2\int_{t_0}^{t} \left|y'^2(s)\right| ds$$
$$\geqslant \left|\int_{t_0}^{t} e^{3s}h'(e^{s})y^{\alpha+1}(s)ds\right|.$$

Applying the mean value theorem to the integral on the right-hand side, we obtain

$$A_{t_0} + \frac{2LM^{\alpha+1}}{\alpha+1} + 2F \geqslant \left|\int_{t_0}^{t} e^{3s}h'(e^{s})y^{\alpha+1}(s)ds\right| = \left|y^{\alpha+1}(s^*)\right| \left|\int_{t_0}^{t} e^{3s}h'(e^{s})ds\right|$$

where $s^* \in [t_0, t]$. Since $\left|\int_{t_0}^{\infty} e^{3s}h'(e^{s})ds\right| = \infty$, sending t to ∞ in the last inequality shows that $M = \infty$. The contradiction proves that $y(t)$ is unbounded as $t \to \infty$. We conclude that the solution $y(x)$ is unbounded as $x \to \infty$. □

3.2. Example. Consider the differential equation of the second order

$$x^2y''(x) + y(x) = \sqrt{y}.$$

We will show that all the solutions are unbounded for $x \to \infty$. Here

$$\alpha = \frac{1}{2}, \quad h(x) = \frac{1}{x^2}, \quad \left|x^2 h(x)\right| = 1 \quad \text{for } |x| \geqslant 1$$

and

$$\int_1^\infty \left|x^2 h'(x)\right| dx = 2\int_1^\infty \frac{dx}{x} = \infty.$$

Applying the same approach as in the proof of Theorem 3.1, we get

$$A_{t_0} + \frac{2LM^{\alpha+1}}{\alpha+1} + 2F \geqslant \left|\int t_0{}^t e^{3s} h'(e^s) y^{\alpha+1}(s)\, ds\right| = \left|y^{\alpha+1}(s^*)\right| \left|\int_{t_0}^t e^{3s} h'(e^s)\, ds\right|.$$

Since $\int_{t_0}^\infty \left|e^{3s} h'(e^s)\right| ds = \infty$, we have $M = \infty$. The contradiction proves that the solution $y(x)$ is unbounded as $x \to \infty$.

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