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Representation of simply ordered sets and the generalized continuum hypothesis

In this paper we consider conditions for representation of simply ordered sets by means of sequences made up of 0, 1 and ordered by the principle of first differences. Also (besides deriving some statements equivalent to the Generalized Continuum Hypothesis), we give a necessary and sufficient condition for such representation by sequences of type ω_λ under the assumption of the Generalized Continuum Hypothesis. Incidentally, Theorem 2 below which is a stronger version of Theorem 1 of Novotny [4] is proved in a simple and constructive way and without the use of Lemma 1 of Sierpiński [5].

First we introduce some definitions. Let (s_i) and (t_i) be two sequences of the same finite or transfinite type made up of 0 and 1. As usual we say that (s_i) is less than or equal to (t_i) according to the principle of *first differences* and we denote this by

(1) $$(s_i) \precsim (t_i)$$

if (s_i) is equal (identical) to (t_i) or if there exists an index j such that $s_j = 0$ and $t_j = 1$ and $s_i = t_i$ for every $i < j$.

Furthermore, we say that (s_i) is less than or equal to (t_i) according to the principle of *strong first differences* and we denote this by

(2) $$(s_i) \precsim^* (t_i)$$

if $s_i = 1$ implies $t_i = 1$ (or, $t_i = 0$ implies $s_i = 0$) for every index i.

Clearly, in view of (1) and (2), if $(s_i) \precsim^* (t_i)$ then $(s_i) \precsim (t_i)$.

As usual, if $(s_i) \neq (t_i)$ and $(s_i) \precsim (t_i)$ or $(s_i) \precsim^* (t_i)$ then we write $(s_i) \prec (t_i)$ or $(s_i) \prec^* (t_i)$ respectively.

Let D be a subset of a simply ordered set $(P, \leqslant)$ and let p and q be any two elements of P such that $p < q$. Then, as usual, we say D is *dense* in P if there exists an element d_1 of D such that

(3) $$p < d_1 < q.$$

Moreover, following Hausdorff [2], we say that D is *Hausdorff-dense* in P if there exist two elements d_2 and d_3 of D such that

$$(4) \qquad p \leqslant d_2 < d_3 \leqslant q.$$

Furthermore, we say that D is *weakly dense* in P if there exists an element d_4 of D such that

$$(5) \qquad p < d_4 \leqslant q.$$

Finally, we say that D is *very weakly dense* in P if there exists a d_5 of D such that

$$(6) \qquad p \leqslant d_5 \leqslant q.$$

Clearly, in view of (3), (4), (5), and (6), denseness implies Hausdorff-denseness which implies weak denseness which in turn implies very weak denseness.

THEOREM 1. *For every simply ordered set S and every ordinal number λ the following three statements are equivalent*:

(i) *S has a very weakly dense subset of power less than or equal to $\aleph_\lambda$.*

(ii) *S has a weakly dense subset of power less than or equal to $\aleph_\lambda$.*

(iii) *S has a Hausdorff dense subset of power less than or equal to $\aleph_\lambda$.*

Proof. Assume that S has a very weakly dense subset D of power less than or equal to $\aleph_\lambda$. Let E be the set formed by adding to D all the immediate successors and immediate predecessors (in P) of elements of D. Clearly $\overline{\overline{E}} \leqslant \aleph_\lambda$, and in view of (4) and (6) we see that E is Hausdorff dense in P. This shows that (i) implies (iii).

It is trivial that (iii) implies (ii) which implies (i), and this proves the theorem.

THEOREM 2. *Let $(P, \leqslant)$ be a simply ordered set with a very weakly dense subset of power less than or equal to $\aleph_\lambda$. Then $(P, \leqslant)$ is isomorphic to a set of sequences of type ω_λ made up of $0, 1$ and ordered by the principle of strong first differences $\precsim^*$.*

Proof. By Theorem 1 P has a weakly dense subset D of power less than or equal to $\aleph_\lambda$. Let $(d_i)_{i<v}$ be a well ordering of D with $v \leqslant \omega_\lambda$. Consider the mapping f from P onto a set S of sequences of 0 and 1 of type ω_λ such that $f(p) = (p_i)_{i<\omega_\lambda}$ for every $p \in P$, where

$$(7) \qquad p_i = \begin{cases} 1 & \text{if} \quad d_i \leqslant p \text{ and } i < v, \\ 0 & \text{otherwise.} \end{cases}$$

We show that f is an isomorphism between $(P, \leqslant)$ and $(S, \precsim^*)$.

Let p and q be elements of P with $p \leqslant q$ and let $f(p) = (p_i)_{i<\omega_\lambda}$ and $f(q) = (q_i)_{i<\omega_\lambda}$. Then for every d_i of D we see that $d_i \leqslant p$ implies $d_i \leqslant q$

and consequently, in view of (7) and (2) we derive $f(p) \precsim^* f(q)$. Thus f is order-preserving.

Since $(S, \precsim^*)$ is simply ordered, it remains to show that f is one-to-one.

Let p and q be elements of P such that

$$f(p) = (p_i)_{i<\omega_\lambda} = (q_i)_{i<\omega_\lambda} = f(q).$$

Then, in view of (7), for every $d_i \in D$ we have

$$(d_i \leqslant p) \leftrightarrow (p_i = 1) \leftrightarrow (q_i = 1) \leftrightarrow (d_i \leqslant q). \tag{8}$$

Now, without loss of generality, if we assume $p < q$ then since D is weakly dense in P there exists a d_j in D such that $p < d_j \leqslant q$. But this, letting $i = j$ in (8), is a contradiction. Hence $p = q$, showing that f is one-to-one. Thus the theorem is proved. We observe that the construction given in (7) resembles that given in [3].

In what follows we let, as usual, $\aleph_0 = \lim\limits_{i<\lambda} 2^{\aleph_i}$ for $\lambda = 0$.

THEOREM 3. *Let S be a set of sequences of type ω_λ made up of $0, 1$ and ordered by the principle of first differences $\precsim$. Then S has a Hausdorff-dense subset D of power less than or equal to $\aleph_\tau$, where*

$$\aleph_\tau = \lim_{i<\lambda} 2^{\aleph_i}.$$

Proof. Let T_{ω_λ} be the set of all sequences of type ω_λ made up of $0, 1$ and such that for every element $(t_i)_{i<\omega_\lambda}$ of T_{ω_λ} there exists an index j with $t_j = 1$ and $t_i = 0$ for every $i > j$. Then clearly,

$$\overline{\overline{T_{\omega_\lambda}}} = \lim_{i<\omega_\lambda} 2^i = \lim_{i<\lambda} 2^{\aleph_i} = \aleph_\tau. \tag{9}$$

Next, let A be the set of all elements of S which have an immediate successor in S and B be the set of all elements of S which have an immediate predecessor in S. Let $b \in B$ be the immediate successor (in S) of $a \in A$ and let $a = (a_i)_{i<\omega_\lambda}$ and $b = (b_i)_{i<\omega_\lambda}$. Then $a \prec b$ and hence there exists an index j such that $a_j = 0$, $b_j = 1$ and $a_i = b_i$ for every $i < j$. Consider the element $t = (t_i)_{i<\omega_\lambda}$ of T_{ω_λ} where $t_i = a_i = b_i$ for every $i < j$, $t_i = 1$ for $i = j$ and $t_i = 0$ for every $i > j$. Clearly

$$a \prec t \precsim b.$$

Since b is the immediate successor (in S) of a it follows that we may associate with every $a \in A$, or with every $b \in B$ an element t of T_{ω_λ} as above, in a one-to-one manner. This, in view of (9), shows that

$$\overline{\overline{A}} \leqslant \overline{\overline{T_{\omega_\lambda}}} = \aleph_\tau \quad \text{and} \quad \overline{\overline{B}} \leqslant \overline{\overline{T_{\omega_\lambda}}} = \aleph_\tau. \tag{10}$$

Now, let $t = (t_i)_{i<\omega_\lambda}$ be an element of T_{ω_λ} and let j be the index such that $t_j = 1$ and $t_i = 0$ for every $i > j$. For each such t let $S(t)$ be the set (possibly empty) of all elements $(s_i)_{i<\omega_\lambda}$ of S such that

(11) $$s_i = t_i \quad \text{for every } i < j.$$

Let C be a set which contains one element from $S(t)$ for every $t \in T_{\omega_\lambda}$ such that $S(t) \neq \emptyset$. Then clearly,

(12) $$\overline{\overline{C}} \leqslant \overline{\overline{T_{\omega_\lambda}}} = \aleph_\tau.$$

Finally, let

$$D = A \cup B \cup C.$$

It is immediate that $D \subset S$ and that, in view of (10) and (12) we have

(13) $$\overline{\overline{D}} \leqslant \aleph_\tau.$$

We show that D is Hausdorff-dense in S.

Let $p = (p_i)_{i<\omega_\lambda}$ and $q = (q_i)_{i<\omega_\lambda}$ be two elements of S with $p \precnsim q$. If q is the immediate successor (in S) of p then $p \in A$ and $q \in B$ and hence $p \precsim p \precnsim q \precsim q$ with p and q elements of D, as desired.

On the other hand, if q is not the immediate successor (in S) of p then there exists an $s = (s_i)_{i<\omega_\lambda}$ in S such that $p \precnsim s \precnsim q$. Hence there exists an index j such that $p_j = 0$, $s_j = 1$ and $p_i = s_i$ for every $i < j$; moreover, there exists an index k such that $s_k = 0$, $q_k = 1$ and $s_i = q_i$ for every $i < k$. Let $m = \max\{j, k\}+1$ and let $t = (t_i)_{i<\omega_\lambda}$ be an element of T_{ω_λ} with $t_i = s_i$ for every $i < m$, $t_i = 1$ for $i = m$ and $t_i = 0$ for every $i > m$. Since $t_i = s_i$ for every $i < m$, in view of (11), it follows that $s \in S(t)$ and $S(t) \neq \emptyset$. Thus, since $C \subset D$ there exists an element $d = (d_i)_{i<\omega_\lambda}$ of D such that $d_i = t_i = s_i$ for every $i < m$. But this implies that

$$p \precnsim d \precnsim q$$

with $d \in D$. Now, if q is the immediate successor (in S) of d then $q \in B \subset D$ and we have $p \precnsim d \precnsim q \precsim q$, as desired. On the other hand, if q is not the immediate successor (in S) of d then by the above construction we can find an e such that $p \precnsim d \precnsim e \precnsim q$ with d and e elements of D, as desired. Thus, D is Hausdorff-dense in S and in view of (13) the theorem is proved.

Corollary 1. *Every simply ordered set which is isomorphic to a set of sequences of type ω_λ made up of $0, 1$ and ordered by the principle of first differences has a Hausdorff-dense subset of power less than or equal to $\aleph_\tau$ where*

$$\aleph_\tau = \lim_{i<\lambda} 2^{\aleph_i}.$$

Let us recall [1] that

$$\aleph_\lambda \leqslant \lim_{i<\lambda} 2^{\aleph_i} \quad \text{for every ordinal number } \lambda. \tag{14}$$

Also, it is easy to show that the Generalized Continuum Hypothesis (which asserts that $2^{\aleph_i} = \aleph_{i+1}$ for every ordinal i) is equivalent to

$$\aleph_\lambda = \lim_{i<\lambda} 2^{\aleph_i} \quad \text{for every ordinal number } \lambda. \tag{15}$$

THEOREM 4. *Under the assumption of the Generalized Continuum Hypothesis, for every ordinal number λ, a simply ordered set is isomorphic to a set of sequences of type ω_λ made up of* 0, 1 *and ordered by the principle of first differences if and only if it has a very weakly dense subset of power less than or equal to $\aleph_\lambda$.*

Proof. If a simply ordered set $(P, \leqslant)$ is isomorphic to a set of sequences of type ω_λ made up of 0, 1 and ordered by $\precsim$ then by Corollary 1 it has a Hausdorff-dense subset D of power less than or equal to $\lim_{i<\lambda} 2^{\aleph_i}$ which, in view of (15), implies $\overline{\overline{D}} \leqslant \aleph_\lambda$. Consequently, $(P, \leqslant)$ has a very weakly dense subset of power less than or equal to $\aleph_\lambda$. On the other hand, if $(P, \leqslant)$ has a very weakly dense subset of power less than or equal to $\aleph_\lambda$ then by Theorem 2 it is isomorphic to a set of sequences of type ω_λ made up of 0, 1 and ordered by $\precsim^*$ and a fortiori by $\precsim$. Thus, the theorem is proved.

REMARK 1. *In view of Theorems* 2 *and* 4, *under the assumption of the Generalized Continuum Hypothesis, for every ordinal number λ, a simply ordered set is isomorphic to a set of sequences of type ω_λ made up of* 0, 1 *and ordered by the principle of first differences if and only if it is isomorphic to a set of sequences of the same type and kind and ordered by the principle of strong first differences.*

THEOREM 5. *The Generalized Continuum Hypothesis is equivalent to the statement: for every ordinal number λ if a simply ordered set is isomorphic to a set of sequences of type ω_λ made up of* 0, 1 *and ordered by the principle of first differences then it has a Hausdorff-dense subset of power less than or equal to $\aleph_\lambda$.*

Proof. Clearly, in view of Theorems 1 and 4, the Generalized Continuum Hypothesis implies the statement mentioned in the theorem.

Next, assume the statement mentioned in the theorem. Consider the set S_{ω_λ} of all sequences of type ω_λ made up of 0, 1 and ordered by the principle of first differences. Obviously, S_{ω_λ} is a simply ordered set of the kind described by the statement mentioned in the theorem. As in the proof of Theorem 2, let T_{ω_λ} be the set of all elements $(t_i)_{i<\omega_\lambda}$ such that for each $(t_i)_{i<\omega_\lambda}$ there exists an index j with $t_j = 1$ and $t_i = 0$ for every $i < j$. With each such $(t_i)_{i<\omega_\lambda}$ associate an element $(u_i)_{i<\omega_\lambda}$ of S_{ω_λ} such that $u_i = t_i$ for every $i < j$ and $u_i = 0$ for $i = j$ and $u_i = 1$ for every $i < j$. Clearly, $(u_i)_{i<\omega_\lambda}$ is the immediate predecessor (in S_{ω_λ}) of the corresponding $(t_i)_{i<\omega_\lambda}$. Thus, every element of T_{ω_λ} has an immediate predecessor (in S_{ω_λ}). Consequently, T_{ω_λ} is a subset of every Hausdorff-dense subset of S_{ω_λ}. But then in view of (9), (14) and the conclusion of the statement mentioned in the theorem

$$\aleph_\lambda \leqslant \lim_{i<\lambda} 2^{\aleph_i} \leqslant \aleph_\lambda \quad \text{for every ordinal number } \lambda$$

which, in view of (15) implies the Generalized Continuum Hypothesis, as desired.

THEOREM 6. *The Generalized Continuum Hypothesis is equivalent to the statement: for every ordinal number λ if a simply ordered set has a very weakly dense subset of power less than or equal to $\aleph_\tau = \lim_{i<\lambda} 2^{\aleph_i}$ then it is isomorphic to a set of sequences of type ω_λ made up of $0, 1$ and ordered by the principle of strong first differences.*

Proof. Clearly, in view of (15), Theorem 4, and Remark 1, the Generalized Continuum Hypothesis implies the statement mentioned in the theorem.

Next, assume the statement mentioned in the theorem. Consider a well-ordered set $(W, \leqslant)$ of power $\aleph_\tau$. Obviously, W is weakly dense in itself. Thus, $(W, \leqslant)$ is a simply ordered set of the kind described by the statement mentioned in the theorem and hence $(W, \leqslant)$ is isomorphic to a subset V of the set S_{ω_λ} of all sequences of type ω_λ made up of $0, 1$ and where V is ordered by the principle of strong first differences. However, in view of Lemma 2 of Sierpiński [5] and the conclusion of the statement mentioned in the theorem and (14) we have

$$\aleph_\lambda \leqslant \lim_{i<\lambda} 2^{\aleph_i} = \overline{\overline{W}} \leqslant \aleph_\lambda \quad \text{for every ordinal number } \lambda$$

which, in view of (15), implies the Generalized Continuum Hypothesis, as desired.

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