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Bochner representable operators on Köthe-Bochner spaces

Abstract. Let E be a Banach function space and X be a real Banach space. We study Bochner representable operators from a Köthe-Bochner space $E(X)$ to a Banach space Y. We consider the problem of compactness and weak compactness of Bochner representable operators from $E(X)$ (provided with the natural mixed topology) to Y.

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1. Introduction and preliminaries. The integral representation of linear operators on function spaces (in particular, Lebesgue spaces and Orlicz spaces) has been the object of much study (see [Ph], [DP], [G$_1$], [G$_2$], [DS], [D], [U$_1$], [U$_2$], [A$_1$], [A$_2$], [Na$_1$], [Na$_2$]). In particular, K. Andrews [A$_1$, Theorems 2 and 5], [A$_2$, Theorem 3] and V.G. Navodnov [Na$_2$, Corollary] obtained a Dunford-Pettis-Phillips type theorem for compact operators (resp. weakly compact operators) from the space $L^1(X)$ of Bochner integrable functions to a Banach space Y. Moreover, V.G. Navodnov ([Na$_2$], [Na$_1$]) has considered Bochner representable operators from a Köthe-Bochner space $E(X)$ to a Banach space Y.

In Section 2 we study the problem of compactness and weak compactness of Bochner representable operators from a Köthe-Bochner space $E(X)$ (provided with the natural mixed topology $\gamma_{E(X)}$) to a Banach space Y. The space $(E(X), \gamma_{E(X)})$ is a generalized DF-space, so we can apply the Grothendieck's DF techniques (see [G$_1$], [G$_2$], [Ru]).

Throughout the paper $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are real Banach spaces with the Banach duals X^* and Y^*. Let B_X and B_Y denote the closed unit balls in X and Y. Let $\mathcal{L}(X, Y)$ stand for the space of all bounded linear operators from X to Y provided with the uniform convergence norm $\|\cdot\|_{X\to Y}$. The *strong operator topology* (briefly SOT) is the topology on $\mathcal{L}(X,Y)$ defined by the family of seminorms $\{p_x : x \in X\}$, where $p_x(U) := \|U(x)\|_Y$ for $U \in \mathcal{L}(X,Y)$ (see [DS, p. 475–477]). Let $\mathbb{N}$ stand for the set of natural numbers.

Let (Ω, Σ, μ) be a complete finite measure space. Assume that $(E, \|\cdot\|_E)$ is a Banach function space and let E' stand for the Köthe dual of E. Then the associated norm $\|\cdot\|_{E'}$ on E' can be defined for $v \in E'$ by

$$\|v\|_{E'} = \sup\left\{\Big|\int_\Omega u(\omega)\,v(\omega)\,d\mu\Big| : \ u \in E, \quad \|u\|_E \le 1\right\}.$$

A Banach function space $(E, \|\cdot\|_E)$ is said to be *perfect*, if $E = E''$ and $\|u\|_{E''} = \|u\|_E$ for $u \in E$. It is well known that $(E, \|\cdot\|_E)$ is perfect if and only if $\|\cdot\|_E$ satisfies both the σ-Fatou property and the σ-Levy property (see [KA, Theorem 6.1.7]).

From now we will assume that $L^\infty \subset E \subset L^1$, where the inclusion maps are continuous. Moreover, we will assume that $\|\cdot\|_{E'}$ on E' is order continuous. Since $(L^1)' = L^\infty$, the space L^1 is excluded.

By $L^0(X)$ we denote the set of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$. Let $\mathcal{T}_0(X)$ stand for the topology on $L^0(X)$ of the F-norm $\|\cdot\|_{L^0(X)}$ that generates convergence in measure on sets of finite measure.

For $f \in L^0(X)$ let $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Then the space

$$E(X) = \{f \in L^0(X) : \widetilde{f} \in E\}$$

equipped with the norm $\|f\|_{E(X)} := \|\widetilde{f}\|_E$ is a Banach space and is usually called a *Köthe-Bochner space* (see [CM], [L] for more details). We will denote by $\mathcal{T}_E(X)$ the topology of the norm $\|\cdot\|_{E(X)}$. Recall that the algebraic tensor product $E \otimes X$ is the subspace of $E(X)$ spanned by the functions of the form $u \otimes x$, $(u \otimes x)(\omega) = u(\omega)x$, where $u \in E$, $x \in X$ and $\omega \in \Omega$. For $r > 0$ denote

$$B_{E(X)}(r) = \{f \in E(X) : \|f\|_{E(X)} \le r\}.$$

Let τ be a linear topology on $E(X)$. A linear operator $T : E(X) \to Y$ is said to be $(\tau, \|\cdot\|_Y)$-*compact* (resp. $(\tau, \|\cdot\|_Y)$-*weakly compact*) if there exists a neighbourhood U of 0 for τ such that $T(U)$ is relatively norm compact (resp. relatively weakly compact) in Y. By $Bd(E(X), \tau)$ we will denote the collection of all τ-bounded subsets of $E(X)$.

2. Bochner representable operators on Köthe-Bochner spaces. We start by recalling terminology concerning Bochner representable operators $T : E(X) \to Y$ (see [A$_1$], [A$_2$], [Na$_1$], [Na$_2$] for more details).

A function $\mathcal{K} : \Omega \to \mathcal{L}(X,Y)$ is said to be SOT-*measurable* if for every $x \in X$ the function $\mathcal{K}_x : \Omega \ni \omega \mapsto \mathcal{K}(\omega)(x) \in Y$ is strongly Σ-measurable. We say that two SOT-measurable functions $\mathcal{K}_1$ and $\mathcal{K}_2$ are SOT-*equivalent* (briefly, $\mathcal{K}_1 \approx \mathcal{K}_2$) if $\mathcal{K}_1(\omega)(x) = \mathcal{K}_2(\omega)(x)$ for all $x \in X$ and μ-almost all $\omega \in \Omega$ (see [Na_1], [Na_2]).

Recall that a bounded linear operator $T : E(X) \to Y$ is said to be *Bochner representable* if there exists a SOT-measurable function $\mathcal{K} : \Omega \to \mathcal{L}(X,Y)$ (called the *representing kernel* for T) such that for each $f \in E(X)$ the function $\langle f, \mathcal{K}\rangle : \Omega \ni \omega \mapsto \langle f(\omega), \mathcal{K}(\omega)\rangle \in Y$ is Bochner integrable and

$$T(f) = \int_\Omega \langle f(\omega),\, \mathcal{K}(\omega)\rangle \, d\mu \quad \text{for all} \;\; f \in E(X).$$

The following theorem will be of importance (see [Na_1, Theorem 1]).

Theorem 2.1 *For a SOT-measurable function $\mathcal{K} : \Omega \to \mathcal{L}(X,Y)$ the following statements are eqiuvalent:*

(i) *$\mathcal{K}$ is the representing kernel for a Bochner representable operator $T \colon E(X) \to Y$.*

(ii) *There exists a SOT-measurable function $\mathcal{K}_0 : \Omega \to \mathcal{L}(X,Y)$ such that $\mathcal{K}_0 \approx \mathcal{K}$ and the μ-equivalence class of the function $\|\mathcal{K}_0(\cdot)\|_{X\to Y}$ belongs to E'.*

Recall that the *mixed topology* $\gamma[\mathcal{T}_E(X), \mathcal{T}_0(X) \restriction_{E(X)}]$ (briefly $\gamma_{E(X)}$) on $E(X)$ is the finest Hausdorff locally convex topology on $E(X)$ which agrees with $\mathcal{T}_0(X)$ on $\|\cdot\|_{E(X)}$-bounded subsets of $E(X)$ (see [C, Chap. III], [N_2], [F, §3] for more details). Then

$$\mathcal{T}_0(X) \restriction_{E(X)} \subset \gamma_{E(X)} \subset \mathcal{T}_E(X).$$

Since $B_{E(X)}(1)$ is closed in $(E(X), \mathcal{T}_0(X) \restriction_{E(X)})$ (see [KA, Lemma 4.3.4]), by [W, Theorem 2.4.1] we get

$$Bd\,(E(X), \gamma_{E(X)}) = Bd\,(E(X), \|\cdot\|_{E(X)}). \tag{1}$$

This means that $(E(X), \gamma_{E(X)})$ is a generalized DF-space (see [Ru]). Hence using the Grothendieck classical results (see [Ru, p. 429], [G_1, Corollary 1 of Theorem 11], [G_2, Chap. IV, 4.3, Corollary 1 of Theorem 2]) we get:

Theorem 2.2 *Let $T : E(X) \to Y$ be a $(\gamma_{E(X)}, \|\cdot\|_Y)$-continuous linear operator which transforms $\gamma_{E(X)}$-bounded sets into relatively norm compact (resp. relatively weakly compact) sets in Y. Then T is $(\gamma_{E(X)}, \|\cdot\|_Y)$-compact (resp. $((\gamma_{E(X)}, \|\cdot\|_Y)$-weakly compact).*

A linear operator $T : E(X) \to Y$ is called $(\gamma, \|\cdot\|_Y)$-*linear* if $\|T(f_n)\|_Y \to 0$ whenever $\|f_n\|_{L^0(X)} \to 0$ and $\sup_n \|f_n\|_{E(X)} < \infty$ (see [W]). It is known that a linear operator $T : E(X) \to Y$ is $(\gamma, \|\cdot\|_Y)$-linear if and only if T is $(\gamma_{E(X)}, \|\cdot\|_Y)$-continuous (see [W, Theoem 2.6.1(iii)]).

PROPOSITION 2.3 *Assume that* $(E, \|\cdot\|_E)$ *is a perfect Banach function space such that the associated norm* $\|\cdot\|_{E'}$ *on* E' *is order continuous. Then every Bochner representable operator* $T : E(X) \to Y$ *is* $(\gamma_{E(X)}, \|\cdot\|_Y)$*-continuous.*

PROOF Assume that $T : E(X) \to Y$ is a Bochner representable operator. Then, by Theorem 2.1, there exists a SOT-measurable function $\mathcal{K}_0 : \Omega \to \mathcal{L}(X, Y)$ such that the μ-equivalence class $v_0 = [\, \| \mathcal{K}_0(\cdot) \|_{X\to Y} \,]$ belongs to E' and

$$T(f) = \int_\Omega \langle f(\omega), \mathcal{K}_0(\omega) \rangle \, d\mu \quad \text{for all} \quad f \in E(X).$$

Hence for $f \in E(X)$ we get

$$\begin{aligned} \|T(f)\|_Y &= \Big\| \int_\Omega \langle f(\omega), \mathcal{K}_0(\omega) \rangle \, d\mu \Big\|_Y \\ &\le \int_\Omega \| \langle f(\omega), \mathcal{K}_0(\omega) \rangle \|_Y \, d\mu \\ &\le \int_\Omega \| f(\omega) \|_X \cdot \| \mathcal{K}_0(\omega) \|_{X\to Y} \, d\mu \, . \end{aligned}$$

Putting

$$\varphi_{v_0}(u) = \int_\Omega u(\omega)\, v_0(\omega) \, d\mu \quad \text{for} \quad u \in E,$$

we see that φ_{v_0} is a γ-linear functional on E (see [N$_1$, Theorem 3.1]). It follows that T is a $(\gamma, \|\cdot\|_Y)$-linear operator, i.e., T is $(\gamma_{E(X)}, \|\cdot\|_Y)$-continuous. ■

Before stating our main result we require a preliminary definition (see [A$_1$, p. 258]). A SOT-measurable function $\mathcal{K} : \Omega \to \mathcal{L}(X, Y)$ is said to have its essential range in the *uniformly norm compact operators* (resp. *uniformly weakly compact operators*) if there exists a relatively norm compact (resp. relatively weakly compact) set C in Y such that $\langle x, \mathcal{K}(\omega) \rangle \in C$ for μ-almost all $\omega \in \Omega$ and all $x \in B_X$.

First we recall the well known Dunford-Pettis-Phillips type theorem for compact operators (resp. weakly compact operators) from the space $L^1(X)$ to a Banach space Y (see [Na$_2$, Corollary]).

THEOREM 2.4 *For a bounded linear operator* $T : L^1(X) \to Y$ *the following statements are equivalent:*

(i) T *is* $(\|\cdot\|_{L^1(X)}, \|\cdot\|_Y)$*-compact* (*resp.* $(\|\cdot\|_{L^1(X)}, \|\cdot\|_Y)$*-weakly compact*).

(ii) T *is Bochner representable operator with the representing kernel* $\mathcal{K}$ *having its essential range in the uniformly norm compact operators* (*resp. uniformly weakly compact operators).*

Now we are in position to extend and strengthen the implication (ii)$\Longrightarrow$(i) of Theorem 2.4 for the case of linear operators from a Köthe-Bochner space $E(X)$ to a Banach space Y.

Theorem 2.5 *Assume that* $(E, \|\cdot\|_E)$ *is a perfect Banach function space such that the associated norm* $\|\cdot\|_{E'}$ *on* E' *is order continuous. Let* $T : E(X) \to Y$ *be a Bochner representable operator with the representing kernel* $\mathcal{K}$ *having its essential range in the uniformly norm compact operators (resp. uniformly weakly compact operators). Then* T *is* $(\gamma_{E(X)}, \|\cdot\|_Y)$*-compact (resp.* $(\gamma_{E(X)}, \|\cdot\|_Y)$*-weakly compact).*

Proof In view of Proposition 2.3 T is $(\gamma_{E(X)}, \|\cdot\|_Y)$-continuous. Since $Bd\,(E(X), \gamma_{E(X)}) = Bd\,(E(X), \|\cdot\|_{E(X)})$ (see (1)), by Theorem 2.2 it is enough to show that for every $r > 0$, the set $T(B_{E(X)}(r))$ is relatively norm compact (resp. relatively weakly compact) in Y. Indeed, let $r > 0$. Since the inclusion map $(E, \|\cdot\|_E) \hookrightarrow (L^1, \|\cdot\|_{L^1})$ is supposed to be continuous, there exists $r_0 > 0$ such that $B_{E(X)}(r_0) \subset B_{L^1(X)}(\frac{1}{2})$.

Moreover, by our assumption there exists a relatively norm compact (resp. relatively weakly compact) set C in Y such that $\langle x, \mathcal{K}(\omega)\rangle \in C$ for μ-almost all $\omega \in \Omega$ and all $x \in B_X$. Now we shall show that $T(f) \in \overline{\mathrm{conv}}\, C$ for all $f \in B_{E(X)}(r_0)$, where $\overline{\mathrm{conv}}\, C$ stands for the norm closed (=weakly closed) convex hull of C in Y. Indeed, let $f \in B_{E(X)}(r_0)$. Then by [DS, p. 117] there exists a sequence (s_n) of X-valued Σ-simple functions such that $\|s_n(\omega)\|_X \le 2\,\|f(\omega)\|_X$ μ-a.e. on Ω and $s_n \to f$ in μ-measure. Then $\|s_n - f\|_{L^0(X)} \longrightarrow 0$ and $\sup_n \|s_n\|_{E(X)} \le 2\,\|f\|_{E(X)} \le 2\,r_0$. Hence $\|T(s_n) - T(f)\|_Y \longrightarrow 0$, because T is $(\gamma, \|\cdot\|_Y)$-linear.

Since $s_n = \sum_{i=1}^{k_n} \mathbb{1}_{A_{n,i}} \otimes x_{n,i}$ and $s_n \in B_{E(X)}(2\,r_0)$ for $n \in \mathbb{N}$, we get $s_n \in B_{L^1(X)}(1)$, i.e., $\sum_{i=1}^{k_n} \|x_{n,i}\|_X \mu(A_{n,i}) \le 1$ for $n \in \mathbb{N}$. Hence, using [DU, Corollary 2.2.8, p. 48] we get

$$\begin{aligned} T(s_n) &= \sum_{i=1}^{k_n} T\,(\mathbb{1}_{A_{n,i}} \otimes x_{n,i}) \\ &= \sum_{i=1}^{k_n} \int_{A_{n,i}} \langle x_{n,i},\, \mathcal{K}(\omega)\rangle\, d\mu \\ &= \sum_{i=1}^{k_n} \|x_{n,i}\|_X \int_{A_{n,i}} \Big\langle \frac{x_{n,i}}{\|x_{n,i}\|_X}, \mathcal{K}(\omega)\Big\rangle\, d\mu \\ &\in \Big(\sum_{i=1}^{k_n} \|x_{n,i}\|_X\, \mu(A_{n,i})\Big)\, \overline{\mathrm{conv}}\, C \subset \overline{\mathrm{conv}}\, C. \end{aligned}$$

Hence $T(f) \in \overline{\mathrm{conv}}\, C$, that is, $T(B_{E(X)}(r_0))$ is relatively norm compact (resp. relatively weakly compact) in Y, because $\overline{\mathrm{conv}}\, C$ is norm compact in Y by Mazur's theorem (resp. $\overline{\mathrm{conv}}\, C$ is weakly compact in Y by Krein-Šmulian's theorem). Since $T(B_{E(X)}(r)) \subset \frac{r}{r_0}\, \overline{\mathrm{conv}}\, C$, we obtain that $T(B_{E(X)}(r))$ is relatively norm compact (resp. relatively weakly compact) in Y, as desired. ■

Now we will consider Bochner representable operators on the space $L^\infty(X)$. By a Young function we mean here a continuous convex mapping $\Phi : [0, \infty) \to [0, \infty)$ that vanishes only at 0 and $\Phi(t)/t \to 0$ as $t \to 0$ and $\Phi(t)/t \to \infty$ as $t \to \infty$.

The *Orlicz-Bochner space* $L^\Phi(X) := \{f \in L^0(X) : \int_\Omega \Phi(\lambda\|f(\omega)\|_X) d\mu < \infty$ for some $\lambda > 0\}$ is a Banach space under the norm $\|f\|_{L^\Phi(X)} := \inf\{\lambda > 0 : \int_\Omega \Phi(\|f(\omega)\|_X/\lambda)\, d\mu \leq 1\}$ (see [RR] for more details).

We will need the following characterization of the mixed topology $\gamma_{L^\infty(X)}$ on $L^\infty(X)$ (see [N_2, Theorem 4.5]).

THEOREM 2.6 *Then mixed topology $\gamma_{L^\infty(X)}$ on $L^\infty(X)$ is generated by the family of norms $\|\cdot\|_{L^\Phi(X)}$, where Φ runs over the family of all Young functions.*

As a consequence of Theorems 2.5 and 2.6 we get:

COROLLARY 2.7 *Let $T : L^\infty(X) \to Y$ be a Bochner representable operator and assume that the representing kernel $\mathcal{K}$ for T has its range in the uniformly norm compact operators (resp. uniformly weakly compact operators). Then there exists a Young function Φ such that the set*

$$\left\{ \int_\Omega \langle f(\omega),\, \mathcal{K}(\omega)\rangle\, d\mu :\ f \in L^\infty(X),\ \|f\|_{L^\Phi(X)} \leq 1 \right\}$$

is relatively norm compact (resp. relatively weakly compact) in Y.

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