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Almost continuous mappings

§ **1. Introduction.** Let f be a mapping of a Hausdorff topological space E into another Hausdorff topological space F. f is said to be almost continuous at $x \in E$, if for each neighborhood V of $f(x) \in F$, $\overline{f^{-1}(V)}$ is a neighborhood of x. f is almost continuous on E if it is so at each $x \in E$. Obviously a continuous mapping is almost continuous. But the converse is not true. For example, the real function defined by $f(x) = 1$ or 0 according as x is a rational or irrational number, is a discontinuous function. But it is almost continuous, as is easy to verify.

Further, the subset $\{(x, f(x)): x \in E\}$ of the product space $E \times F$ is said to be the graph of f. If f is continuous, the graph of f is a closed subset of $E \times F$, as is well known. Equally well known is the fact that the converse is not true.

It is known [3] that if the graph of a linear mapping on a topological vector space E into another topological vector space F is closed then it is not necessarily true that f is almost continuous. Nor is it necessarily true that the graph of an almost continuous linear mapping is closed [3].

The object of this paper is to study the problem as to when almost continuity alone or together with the closed graph of a mapping on a topological space into another topological space implies its continuity.

It is shown here that if f is an almost continuous, completely closed (§ 2) mapping of a topological space E onto a compact Hausdorff topological space F such that the graph of f is closed in $E \times F$, then f is continuous (Theorem 1). If, in particular, E and F are topological groups and f a homomorphism, then the above theorem has been established without the additional hypotheses that f be completely closed and be an onto mapping (Theorem 3). Finally in § 4, it has been shown that the set of points at which a real valued function on a Baire metric space is almost continuous, is everywhere dense (Theorem 4).

In the sequel, a topological space E, endowed with a topology u, will be denoted by E_u. If there are two topologies u and v on a set E, then $u \supset v$ (or $v \subset u$) means that u is finer than v (or v is coarser than u). Let A be a subset of E_u. Then $\bar{A}$ denotes the closure of A. If A and B are two subsets of E_u, then

$$A \sim B = \{x \in A: x \notin B\}.$$

§ 2. Topological spaces and almost continuity. Let f be a mapping of a topological space E_u onto another topological space F_v. First of all we define another topology $\overset{*}{v}$ on F which is coarser than v as follows:

Let $\{V_y\}$ denote a base of v-neighborhoods of $y \in F$. For each V_y in $\{V_y\}$, define

$$\overset{*}{V}_y = f(\overline{f^{-1}(V_y)}).$$

Since f is onto, there exists $x \in E$ such that $f(x) = y$ and hence $y \in \overset{*}{V}_y$ for each $\overset{*}{V}_y$ in $\{\overset{*}{V}_y\}$. Further, if $\overset{*}{V}_1$ and $\overset{*}{V}_2$ are in $\{\overset{*}{V}_y\}$ which are defined by V_1 and V_2 in $\{V_y\}$, then

$$\overset{*}{V}_1 \cap \overset{*}{V}_2 = f(\overline{f^{-1}(V_1)}) \cap f(\overline{f^{-1}(V_2)}) \supset f(\overline{f^{-1}(V_1 \cap V_2)}) \supset \overset{*}{V}_3,$$

where V_3 is in $\{V_y\}$ such that $V_1 \cap V_2 \supset V_3$. Hence by Exercise $B(a)$ ([5], p. 56), when stated in term of bases, there exists a topology $\overset{*}{v}$ (say) on F such that the family $\{\overset{*}{V}_y\}$ forms a base of $\overset{*}{v}$-neighborhoods of y for each $y \in F$. Since for each $y \in F$ and each $\overset{*}{V}_y$ in $\{\overset{*}{V}_y\}$, $\overset{*}{V}_y \supset V_y$, we have $\overset{*}{v} \subset v$.

The following lemma asserts that under certain conditions F, endowed with the topology $\overset{*}{v}$, is a T_1-space.

LEMMA 1. *Let f be a mapping of a topological space E_u onto another topological space F_v such that the graph of f is closed in $E \times F$. Then $F_{\overset{*}{v}}$ (where $\overset{*}{v}$ is defined above) is a T_1-space.*

Proof. It is sufficient to prove that each singleton $\{z\}$, $z \in F$, is a $\overset{*}{v}$-closed subset.

Let $y \in F \sim \{z\}$, and suppose that there exists no $\overset{*}{v}$-neighborhood of y which is contained in $F \sim \{z\}$. This means that even for each member $\overset{*}{V}_y$ of the fundamental system $\{\overset{*}{V}_y\}$ of $\overset{*}{v}$-neighborhoods of y, $z \in \overset{*}{V}_y$. Let W be an arbitrary v-neighborhood of y. Then there exists a member V_y of the fundamental system $\{V_y\}$ of v-neighborhoods of y such that $y \in V_y \subset W$. But then, by assumption,

$$z \in \overset{*}{V}_y = f(\overline{f^{-1}(V_y)}).$$

This shows that $z = f(x)$, where $x \in \overline{f^{-1}(V_y)}$. Therefore for an arbitrary u-neighborhood U of x in E_u, there exists $x_1 \in U$ such that $f(x_1) \in V_y$. Since $V_y \subset W$, we have

$$(x_1, f(x_1)) \in G \cap (U \times W),$$

where G is the graph of f and $(U\times W)$ is a neighborhood of $(x, y)\in E\times F$. Since $U\times W$ is an arbitrary neighborhood of (x, y), $(x, y)\in \overline{G}= G$ because G is closed by hypothesis. Hence $y = f(x) = z\in F \sim \{z\}$ which is impossible. Therefore there exists a $\overset{*}{v}$-neighborhood P of y such that $y\in P\subset F \sim \{z\}$. Since y is arbitrary, it follows that $F\sim\{z\}$ is $\overset{*}{v}$-open and hence $\{z\}$ is $\overset{*}{v}$-closed.

Definition. A mapping f of a topological space onto another topological space is said to be *completely closed* if for each closed subset A of $E, f(A)$ is $\overset{*}{v}$-closed, where $\overset{*}{v}$ is defined on F as above.

Remark. Since $\overset{*}{v}\subset v$, every $\overset{*}{v}$-closed subset of F is also v-closed. Hence a completely closed mapping is closed in the usual sense, i.e. it maps u-closed subsets of E onto v-closed subsets of F.

Now we have the following:

Theorem 1. *Let f be an almost continuous, completely closed mapping of a topological space E_u onto a Hausdorff compact topological space F_v such that the graph of f is closed in $E\times F$. Then f is continuous.*

Proof. By Lemma 1, $F_{\overset{*}{v}}$ is a T_1-space because the graph of f is closed in $E\times F$ by hypothesis. First of all, by using complete closedness of f we show that $F_{\overset{*}{v}}$ is a Hausdorff space.

Let $y_1 \neq y_2, y_1, y_2\in F$. Then there exists a $\overset{*}{v}$-open neighborhood P of y_1 such that $y_2\notin P$, because $F_{\overset{*}{v}}$ is a T_1-space. Let $\overset{*}{V}_1 = f\big(\overline{f^{-1}(V_1)}\big)$ be a member of the base of $\overset{*}{v}$-neighborhoods of y_1 such that $y_1\in \overset{*}{V}_1\subset P$ and $y_2\notin \overset{*}{V}_1$. Since f is completely closed, $\overline{f^{-1}(V_1)}$ being u-closed in E_u implies $\overset{*}{V}_1$ is $\overset{*}{v}$-closed. Hence $F\sim \overset{*}{V}_1$ is a $\overset{*}{v}$-open $\overset{*}{v}$-neighborhood of y_2. Since $\overset{*}{V}_1\cap (F\sim \overset{*}{V}_1) = \emptyset$ (empty set), $y_1\in \overset{*}{V}_1$ and $y_2\in F\sim \overset{*}{V}_1$, it follows that $F_{\overset{*}{v}}$ is a Hausdorff space.

Since $\overset{*}{v}\subset v$ (see paragraphs preceding Lemma 1) and F_v is compact, it follows that $v = \overset{*}{v}$ (i.e. the topologies are equivalent) by a general theorem in topology ([5], p. 141, Theorem 8).

Now to show that $f\colon E_u\to F_v$ is continuous, it is sufficient to show that $f\colon E_u\to F_{\overset{*}{v}}$ is continuous. For this let $\overset{*}{V}_y$ be a member of the base of $\overset{*}{v}$-neighborhoods of y where $y = f(x), x\in E$. Since

$$f^{-1}(\overset{*}{V}_y) = f^{-1}\Big(f\big(\overline{f^{-1}(V)}\big)\Big)\supset \overline{f^{-1}(V_y)},$$

and since f being almost continuous implies that $\overline{f^{-1}(V_y)}$ is a neighborhood of x, it follows that $f^{-1}(\overset{*}{V}_y)$ is a neighborhood of x. This proves that $f\colon E_u \to F_{\overset{*}{v}}$ is continuous ([5], p. 86, Theorem 1).

§ 3. Topological groups and almost continuity. For topological groups ([1], p. 1), Lemma 1 can be improved as shown in the following:

THEOREM 2. *Let E_u be a topological group, F_v a Hausdorff topological group, and f a homomorphism of E_u into F_v such that the graph of f is closed in $E\times F$. Then $F_{\overset{*}{v}}$ is a Hausdorff topological group.*

Proof. Let $\{V\}$ denote a fundamental system of symmetric v-neighborhoods of the identity e' of F_v satisfying the following conditions:

(i) For each V in $\{V\}$ there exists a V_1 in $\{V\}$ such that $V_1^2 \subset V$.

(ii) For each V in $\{V\}$ and for any $a \in F$, there exists a V_2 in $\{V\}$ such that $aV_2a^{-1} \subset V$ or $V_2 \subset a^{-1}Va$.

(iii) $\bigcap V = \{e'\}$, where the intersection is taken over all the family $\{V\}$.

It is known [1] that in each Hausdorff topological group such a system $\{V\}$ of neighborhoods of e' exists. Conversely, given such a system in an abstract group, there exists a unique Hausdorff topology under which the abstract group is a topological group ([1], p. 4, Proposition 1). (Observe that (iii) is equivalent to the statement that the topological group is Hausdorff.)

Now first of all, we define another topology $\tilde{v}$ on F as follows:

For each V in $\{V\}$, define

$$\tilde{V} = \overline{f(\overline{f^{-1}(V)})}.$$

Since the direct and inverse images of symmetric sets under homomorphisms are symmetric, and since the closure of a symmetric set is also symmetric, it follows that each $\tilde{V}$ is symmetric. Clearly each $\tilde{V}$ contains e'. Now to show that condition (i) is satisfied by $\{\tilde{V}\}$, let V_1 be a member of $\{V\}$ such that $V_1^2 \subset V$. But then f being a homomorphism and F_v being a topological group,

$$\tilde{V} = \overline{f(\overline{f^{-1}(V)})} \supset \overline{f(\overline{f^{-1}(V_1)f^{-1}(V_1)})} \supset \overline{f(\overline{f^{-1}(V_1)})}\;\overline{f(\overline{f^{-1}(V_1)})} = \tilde{V}_1^2.$$

Furthermore, for each V in $\{V\}$ there exists a V_1 in $\{V\}$ such that $a^{-1}Va \supset V_1$ or $V \supset aV_1a^{-1}$ for any $a \in G$. Thus again by using the same arguments as above, we have

$$\tilde{V} = \overline{f(\overline{f^{-1}(V)})} \supset \overline{f(\overline{f^{-1}(aV_1a^{-1})})} \supset \overline{f(\overline{f^{-1}(a)f^{-1}(V_1)f^{-1}(a^{-1})})}$$
$$\supset \overline{f(\overline{f^{-1}(a)})}\;\overline{f(\overline{f^{-1}(V_1)})}\;\overline{f(\overline{f^{-1}(a^{-1})})} \supset a\tilde{V}_1a^{-1}.$$

This establishes condition (ii).

Now to show (iii), let $y \in \bigcap \tilde{V}$. Then $y \in \overline{f(\overline{f^{-1}(V)})}$ for each V in $\{V\}$. Let W be in $\{V\}$ such that $W^2 \subset V$. Then by assumption $y \in \tilde{W}$. Since $\tilde{W} = \overline{f(\overline{f^{-1}(W)})}$, there exists $x \in \overline{f^{-1}(W)}$ such that $f(x) \in yW$ which is a neighborhood of y because W is a neighborhood of e' in F. But in a topological group, the closure of any set is the intersection of products of that set with each member of the entire family of neighborhoods of its identity ([1], p.24). Thus $\overline{f^{-1}(W)} \subset Uf^{-1}(W)$ for each U in $\{U\}$ which is the total system of u-neighborhoods of the identity e in E_u. Hence $x \in Uf^{-1}(W)$. This proves that there exists $x_1 \in U$ such that $x_1^{-1}x \in f^{-1}(W)$. Since f is a homomorphism,

$$\big(f(x_1)\big)^{-1}f(x) \in f\big(f^{-1}(W)\big) = W.$$

Or $f(x_1) \in f(x)\,W$, since W is symmetric. Therefore

$$f(x_1) \in yW^2 \subset yV,$$

because $f(x) \in yW$. This shows that

$$\big(x_1, f(x_1)\big) \in G \cap (U \times yV),$$

where G is the graph of f. Since $U \times yV$ is an arbitrary neighborhood of (e, y) in $E \times F$, $(e, y) \in \bar{G} = G$ because G is closed by hypothesis. Hence $f(e) = y$. But a homomorphism always maps the identity into identity, $f(e) = e' = y$. In other words, $\bigcap \tilde{V} = \{e'\}$. Hence $F_{\tilde{v}}$ is a Hausdorff topological group ([1], p. 4, Proposition 1).

Now, as before, let $\overset{*}{v}$ be the topology on F which has the family $\{\overset{*}{V}\}$, $\overset{*}{V} = f\big(\overline{f^{-1}(V)}\big)$, where V runs over a base of v-neighborhoods of e' in F, as a base of $\overset{*}{v}$-neighborhoods of e' in F. Using the same arguments as those used for $\tilde{v}$, it can be shown that F, endowed with $\overset{*}{v}$, is a topological group. Since for each $\tilde{V}$,

$$\tilde{V} = \overline{f\big(\overline{f^{-1}(V)}\big)} \supset f\big(\overline{f^{-1}(V)}\big) = \overset{*}{V}$$

which is a $\overset{*}{v}$-neighborhood of e' in F, it follows that $\tilde{v} \subset \overset{*}{v}$. Since $\tilde{v}$ is Hausdorff as proved above, $\overset{*}{v}$ is also Hausdorff. This completes the proof.

Remark. One observes that the following relations hold between the three topologies v, $\overset{*}{v}$ and $\tilde{v}$ on F, viz: $v \supset \overset{*}{v} \supset \tilde{v}$.

For topological groups, Theorem 1 can be proved under less restricted conditions, as is the case in the following:

THEOREM 3. *Let E_u be a topological group and F_v a compact Hausdorff topological group. Let f be an almost continuous homomorphism of E_u into F_v, the graph of which is closed in $E \times F$. Then f is continuous.*

Proof. By the above remark we have $v \supset \overset{*}{v} \supset \tilde{v}$ on F. since F_v. is compact and $F_{\tilde{v}}$ Hausdorff, by a general theorem in topology we have $v = \overset{*}{v} = \tilde{v}$. Thus for any neighborhood $\overset{*}{V}$ of e' in F,

$$f^{-1}(\overset{*}{V}) = f^{-1}\left(\overline{f\left(\overline{f^{-1}(V)}\right)}\right) \supset \overline{f^{-1}(V)}$$

shows that $f^{-1}(\overset{*}{V})$ is a neighborhood of e in E_u since $\overline{f^{-1}(V)}$ is a neighborhood of e in E_u because f is almost continuous. Now the continuity of a homomorphism of a topological group into another topological group is equivalent to its continuity at the identity. Hence f is continuous.

Remark. The topology $\tilde{v}$ was discussed by A. Robertson and W. Robertson [6] in connection with a closed graph theorem for topological vector spaces. By using almost similar arguments, some closed graph theorems for topological groups (see [4]) and some theorems for linear topological spaces (see [3]) have been proved by the author.

§ 4. Real line and almost continuity.

THEOREM 4. *Let E_u be a metric Baire space and f a real valued function on E_u. Then the set of points of almost continuity in E_u is dense everywhere in E.*

Proof. By Bradford and Goffman's theorem [2], there exists a residual subset H of E such that the restriction $f|H$ is continuous in the relative topology on H. Since E_u is a Baire space, H must be dense everywhere in E.

Now we show that for each $x \in H$, f is almost continuous at x. Let V be an open neighborhood of the real number $f(x)$. Then $f^{-1}(V) \cap H$ is an open neighborhood of x in the relative topology on H. That means, there exists a u-open neighborhood U of x in E such that

$$f^{-1}(V) \cap H = U \cap H.$$

Hence we have

$$\overline{f^{-1}(V)} \supset \overline{f^{-1}(V) \cap H} = \overline{U \cap H} \supset U,$$

since H is dense in E. This proves that f is almost continuous at $x \in H$.

Since the real line is a metric Baire space, the following corollary follows immediately from the above theorem.

COROLLARY 1. *Let f be any real valued function on the real line R. Then the set of points at which f is almost continuous, is everywhere dense in R.*

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