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On generalized Pell numbers and their graph representations

Abstract. In this paper we give a generalization of the Pell numbers and the Pell-Lucas numbers and next we apply this concept for their graph representations. We shall show that the generalized Pell numbers and the Pell-Lucas numbers are equal to the total number of k-independent sets in special graphs.

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1. Introduction. Consider simple, undirected graphs with the vertex set $V(G)$ and the edge set $E(G)$. By $d_G(x_i, x_j)$ we denote the distance between vertices x_i and x_j in G. Let $\mathbb{P}_n$ and $\mathbb{C}_n$ denote an n-vertex path and an n-vertex cycle, respectively. Let k be a fixed integer, $k \geq 2$. A subset $S \subseteq V(G)$ is a k-independent set of G if for each two distinct vertices $x, y \in S$, $d_G(x, y) \geq k$. In addition, a subset containing only one vertex and the empty set also are k-independent sets of G. Note that for $k = 2$ we obtain the definition of an independent set of the graph G in the classical sense. Let $NI_k(G)$ denote the number of all k-independent sets of the graph G and for $k = 2$, $NI_2(G) = NI(G)$. The parameter $NI(G)$ was study in a paper of Prodinger and Tichy, see [4] and this paper gave an impetus to the counting of independent sets in graphs. They called this parameter *the Fibonacci number of a graph* in view of the facts: $NI(\mathbb{P}_n) = F_{n+1}$ and $NI(\mathbb{C}_n) = L_n$, where the Fibonacci numbers F_n are defined recursively by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ and the Lucas numbers L_n are $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$. Independently Merrifield and Simmons introduced the number of independent sets (which they called σ-index) to the chemical literature, see [3]. They showed the correlation between σ-index and some physicochemical properties of a molecular graph. In the chemistry $NI(G)$ is named as the Merrifield-Simmons index. The Fibonacci numbers of graphs were investigate for example in [1], [2], [4]. In [9] more generalized concept was introduced, namely the generalized Fibonacci numbers of graph which gives the total number of k-independent sets of a graph G.

The k-independent sets, for $k \geq 2$ were studied in many papers, see for example in [5], [6], [9],[10].

The *Pell numbers* are defined by the recurrence relation $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$. The *Pell-Lucas numbers* (or the *companion Pell numbers*) are defined by the recurrence relation $Q_0 = Q_1 = 2$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$. The Pell-Lucas number can be also expressed by $Q_n = 2P_{n-1} + 2P_n$.

In this paper we give a generalization of the Pell numbers and the Pell-Lucas numbers. Firstly we apply this generalization to the counting of special families of subsets of the set of n integers. Next we give the graph interpretation of the generalized Pell numbers and the Pell-Lucas numbers. Note that some generalizations of the Pell numbers and Pell-Lucas numbers are known, see for example [8].

2. Main results. Let $X = \{1, 2, ..., n\}$, $n \geq 3$, be the set of n integers and let $\mathcal{X}$ be a family of subsets of X such that $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1 = \{\{i\}; i = 1, ..., n\}$ and $\mathcal{X}_2 = \{\{j, j+1\}; j = 2, ..., n-2\}$.

Let $k \geq 2$ be integer. Let $\mathcal{Y} \subset \mathcal{X}$ such that
(i). $|\mathcal{Y}| = t$, for fixed $t \geq 0$ and
(ii). for each $Y, Y' \in \mathcal{Y}$ there exist $i \in Y$ and $j \in Y'$ such that $|i - j| \geq k$.

By $p(n, k, t)$ we denote the number of all subfamilies $\mathcal{Y}$ having exactly t subsets and further let $P(n, k) = \sum_{t \geq 0} p(n, k, t)$.

Theorem 2.1 *Let $n \geq 3$, $k \geq 2$, $t \geq 0$ be integers. Then*
$p(n, k, 0) = 1$, $p(n, k, 1) = 2n - 3$.
For $t \geq 2$ and $n < (k-1)t - k + 3$, $p(n, k, t) = 0$.
For $t \geq 2$ and $(k-1)t - k + 3 \leq n \leq k + 2$ we have

$$p(n, k, t) = \begin{cases} 1 & for\ n = k+1\ and\ k \geq 2\ and\ t = 2 \\ 5 & for\ n = k+2\ and\ k \geq 2\ and\ t = 2 \\ 1 & for\ n = k+2\ and\ k = 2\ and\ t = 3. \end{cases}$$

For $t \geq 2$ and $n \geq k + 3$ we have

$$p(n, k, t) = p(n - k + 1, k, t - 1) + p(n - 1, k, t) + p(n - k, k, t - 1).$$

Proof For $t = 0, 1$ the initial conditions are obvious. Let $t \geq 2$. Let $\mathcal{X} \supset \mathcal{Y}_0 = \{\{1\}, \{(t-2)k - (t-3) + k\}, \{\{ik - (i-1), ik - (i-2)\}; i = 1, ..., t-2\}\}$. Since $(t-2)k - (t-3) + k = (k-1)t - k + 3$, hence to construct a family $\mathcal{Y}_0$ we need $n \geq (k-1)t - k + 3$. Otherwise if $n < (k-1)t - k + 3$, then it is easy to observe that there does not exist any family $\mathcal{Y}$ satisfying conditions (i) and (ii), so $p(n, k, t) = 0$. For $n = k+1$ and $n = k+2$ we can find that $p(k+1, k, 2) = 1$ and $p(k+2, k, 2) = 5$. Because for $t = 3$ we have $3(k-1) - k + 3 = 2k = k + 2$ if and only if $k = 2$, so it immediately follows that $p(k+2, k, 3) = 1$

Assume now that $t \geq 2$ and $n \geq k + 3$. Let $\mathcal{Y} \subset \mathcal{X}$ be a subfamily satisfying conditions (i) and (ii). We recall that $\mathcal{Y}$ has exactly t subsets such that for each $Y, Y' \in \mathcal{Y}$ there are $a \in Y$ and $b \in Y'$ such that $|a - b| \geq k$. Let $p_{\{n\}}(n, k, t)$ (respectively: $p_{-\{n\}}(n, k, t)$) be the number of all t-element subfamilies $\mathcal{Y}$ such that $\{n\} \in \mathcal{Y}$ (respectively: $\{n\} \notin \mathcal{Y}$). Then $p(n, k, t) = p_{\{n\}}(n, k, t) + p_{-\{n\}}(n, k, t)$. Two cases occur now:

(1). $\{n\} \in \mathcal{Y}$.
Then the definition of the family $\mathcal{Y}$ implies that $\{n-i\} \notin \mathcal{Y}$, for $i = 1, ..., k-1$ and $\{n-j, n-j+1\} \notin \mathcal{Y}$, for $j = 2, ..., k-1$. Let $\mathcal{X}^* \subset \mathcal{X}$ such that $\mathcal{X}^* = \mathcal{X}_1^* \cup \mathcal{X}_2^*$, where $\mathcal{X}_1^* = \mathcal{X}_1 \setminus \{\{n-i\}; i = 0, 1, ..., k-1\}$, $\mathcal{X}_2^* = \mathcal{X}_2 \setminus \{\{n-j, n-j+1\}; j = 2, ..., k-1\}$. In the other words $\mathcal{X}_1^* = \{\{r\}; r = 1, ..., n-k\}$ and $\mathcal{X}_2^* = \{\{s, s+1\}; s = 2, ..., n-k\}$. Clearly $\mathcal{Y} = \mathcal{Y}^* \cup \{n\}$, where $\mathcal{Y}^* \subset \mathcal{X}^*$, $\mathcal{Y}^*$ contains exactly $(t-1)$ subsets and for every $Y, Y' \in \mathcal{Y}^*$ there are $a \in Y$ and $b \in Y'$ such that $|a-b| \geq k$. Since in the family $\mathcal{X}^*$ the integer $n-k+1$ belongs only to the subset $\{n-k, n-k+1\} \in \mathcal{X}_2^*$, hence the number of considered subfamilies in $\mathcal{X}^*$ is the same as in the family $\mathcal{X}_1^* \cup \{n-k+1\} \cup \mathcal{X}_2^* \setminus \{n-k, n-k+1\}$. This implies that $p_{\{n\}}(n, k, t) = p(n-k+1, k, t-1)$.
(2). $\{n\} \notin \mathcal{Y}$.
We distinguish the following possibilities
(2.1). $\{n-1\} \notin \mathcal{Y}$.
Then $\mathcal{Y} \subseteq \mathcal{X} \setminus \{\{n\}, \{n-1\}\} = \{\{i\}; i = 1, ..., n-2\} \cup \mathcal{X}_2$. Since in the family $\{\{i\}; i = 1, ..., n-2\} \cup \mathcal{X}_2$ the integer $n-1$ belongs only to the subset $\{n-2, n-1\} \in \mathcal{X}_2$, so we can find the number of subfamilies $\mathcal{Y}$ of $(\mathcal{X}_1^* \setminus \{n\}) \cup (\mathcal{X}_2^* \setminus \{n-2, n-1\})$. Then there are exactly $p(n-1, k, t)$ subfamilies $\mathcal{Y}$ in this case.
(2.2). $\{n-1\} \in \mathcal{Y}$.
Evidently $\{n-i\} \notin \mathcal{Y}$ and $\{n-i, n-i+1\} \notin \mathcal{Y}$, for $i = 2, ..., k$. Proving analogously as in case (1) we obtain $p(n-k, k, t-1)$ subfamilies $\mathcal{Y}$, such that $\{n-1\} \in \mathcal{Y}$.
Consequently from the above possibilities we have that $p_{-\{n\}}(n, k, t) = p(n-1, k, t) + p(n-k, k, t-1)$

Finally from the above cases $p(n, k, t) = p(n-k+1, k, t-1) + p(n-1, k, t) + p(n-k, k, t-1)$.

Thus the Theorem is proved. ■

THEOREM 2.2 *Let $k \geq 2$, $n \geq 3$ be integers. Then* $P(n, k) = 2k-2$ *for* $n \leq k$,
$P(k+1, k) = 2k+1$,
$$P(k+2, k) = \begin{cases} 12 & if\ k = 2 \\ 2k+7 & if\ k \geq 3, \end{cases}$$
and for $n \geq k+3$
$P(n, k) = P(n-k+1, k) + P(n-1, k) + P(n-k, k)$.

PROOF From Theorem 2.1 we have that
if $n \leq k$, then $P(n, k) = \sum_{t \geq 0} p(n, k, t) = p(n, k, 0) + p(n, k, 1) = 2n-2$.

If $n = k+1$, then $P(n, k) = \sum_{t=0}^{2} p(k+1, k, t) = 1 + 2(k+1) - 3 + 1 = 2k+1$.

If $n = k+2$, then for $k \geq 3$ we have $P(n, k) = \sum_{t=0}^{2} p(k+2, k, t) = 1 + 2(k+2) - 3 + 5 = 2k+7$. For $k = 2$, $P(k+2, 2) = P(4, 2) = \sum_{t=0}^{3} (4, 2, t) = 12$.

Let $n \geq k+3$. Then $P(n, k) = \sum_{t \geq 0} p(n, k, t) = p(n, k, 0) + p(n, k, 1) + \sum_{t \geq 2} p(n, k, t)$.

Using Theorem 2.1 we obtain that

$$P(n,k) = 1 + 2n - 3 + \sum_{t\geq 2}(p(n-k+1,k,t-1) + p(n-1,k,t) + p(n-k,k,t-1)) =$$

$$2n-2+\sum_{t\geq 1}p(n-k+1,k,t)+\sum_{t\geq 2}p(n-1,k,t)+\sum_{t\geq 1}p(n-k,k,t) =$$

$$2n-2-1+\sum_{t\geq 0}p(n-k+1,k,t)-1-(2n-5)+\sum_{t\geq 0}p(n-1,k,t)-1+\sum_{t\geq 0}p(n-k,k,t) =$$

$$\sum_{t\geq 0}p(n-k+1,k,t)+\sum_{t\geq 0}p(n-1,k,t)+\sum_{t\geq 0}p(n-k,k,t) =$$

$$P(n-k+1,k)+P(n-1,k)+P(n-k,k).$$

Thus the Theorem is proved. ∎

The numbers $P(n,k)$ we will called the *generalized Pell numbers.*
If $k = 2$ and $n \geq 3$, then $P(n,2)$ is the Pell number P_n with the initial conditions $P_3 = 5$ and $P_4 = 12$.
It may be interesting to note that the generalized Pell numbers are defined by k-th order linear recurrence relations. The characteristic equation is $r^k - r^{k-1} - r - 1 = 0$. Clearly for $k = 2$ it has a solution of the form $P_n = \frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$.

The family $\mathcal{X}$ can be regarded as the vertex set of the graph G_n of order $2n-3$ in Figure 1, where vertices from $V(G_n)$ are labeled by integers belonging to corresponding subsets from $\mathcal{X}$.

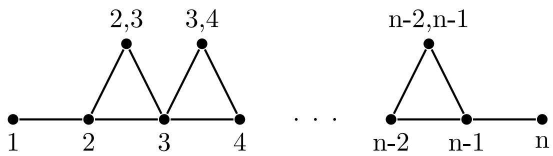

Fig.1. Graph G_n

Thus in the graph terminology, the number $P(n,k)$, for $n \geq 3$, $k \geq 2$ is equal to the total number of subsets $S \subseteq V(G_n)$ such that for each two vertices $x_i, x_j \in S$, $d_{G_n}(x_i, x_j) \geq k$. In the other words for $n \geq 3$, $k \geq 2$, $P(n,k)$ is the total number of k-independent sets of the graph G_n, that means $NI_k(G_n) = P(n,k)$.

Let $X = \{1,2,...,n\}$, $n \geq 3$, and let $\mathcal{F}$ be a family of subsets of X such that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1 = \{\{i\}; i = 1,...,n\}$ and $\mathcal{F}_2 = \{\{i,i+1\}; i = 1,...,n-1\} \cup \{n,1\}$.
Let $\mathcal{I} \subset \mathcal{F}$ such that
(iii). $|\mathcal{I}| = t$, for fixed $t \geq 0$ and
(iv). for each $Y, Y' \in \mathcal{I}$ there exist $i \in Y$ and $j \in Y'$ such that $k \leq |i-j| \leq n-k$.
By $q(n,k,t)$ we denote the number of all subfamilies $\mathcal{I}$ having exactly t elements and further let $Q(n,k) = \sum_{t\geq 0} q(n,k,t)$.

THEOREM 2.3 *Let $k \geq 2$, $t \geq 0$, $n \geq 3$ be integers. Then*
$q(n,k,0) = 1$, $q(n,k,1) = 2n$.
For $t \geq 2$ *and* $n < t(k-1)$, $q(n,k,t) = 0$.
For $t \geq 2$ *and* $t(k-1) \leq n \leq 2k-1$ *we have*
$$q(n,k,t) = \begin{cases} k-1 & for\ n = 2k-2\ and\ k \geq 3\ and\ t = 2 \\ 4k-2 & for\ n = 2k-1\ and\ k \geq 2\ and\ t = 2 \\ 1 & for\ n = 2k-1\ and\ k = 2\ and\ t = 3. \end{cases}$$
For $t \geq 2$ *and* $n \geq 2k$ *we have*
$q(n,k,t) = k \cdot p(n-2k+3,k,t-1) + (k-1) \cdot p(n-2k+4,k,t-1) + p(n-k+2,k,t)$.

PROOF If $t = 0,1$, then the results are obvious. Assume that $t \geq 2$. Let $\mathcal{F} \supset \mathcal{I}_0 = \{\{1,2\}, \{\{ik-(i-1), ik-(i-2)\}; i = 1,...,t-1\}\}$. Since $\mathcal{I}_0 \subset \mathcal{F}$, so we deduce that $n+1-((t-1)k-(t-3)) \geq k-2$, hence to construct a family $\mathcal{I}_0$ we need $n \geq t(k-1)$. Otherwise if $n < t(k-1)$ it is easy to observe that $q(n,k,t) = 0$. Moreover for $n = 2k-2$ and $n = 2k-1$ we can find that if $k \geq 3$ then $q(2k-2,k,2) = k-1$ and for $k \geq 2$ we have $q(2k-1,k,2) = 4k-2$ and $q(2k-1,2,3) = 1$.
Let $t \geq 2$ and $n \geq 2k$. Let $\mathcal{I} \subset \mathcal{F}$ be a subfamily satisfying conditions (iii) and (iv) . Clearly $|\mathcal{I}| = t$ and for every $Y, Y' \in \mathcal{I}$ there are $i \in Y$ and $j \in Y'$ such that $k \leq |i-j| \leq n-k$. Let $\mathcal{F} \supset \mathcal{I}^* = \{\{n-s\}; s = 0,1,...,k-1\} \cup \{\{n-r, n-r+1\}; r = 1,...,k-1\}$. We distinguish the following cases:
(1). $\mathcal{I}^* \cap \mathcal{I} = \emptyset$.
Then $\mathcal{I} \subset \mathcal{F} \setminus \mathcal{I}^* = \{\{m\}; m = 1,...,n-k\} \cup \{\{z,z+1\}; z = 1,...,n-k\} \cup \{n,1\}$. Since in the family $\mathcal{F} \setminus \mathcal{I}^*$ the integer n belongs only to one subset $\{n,1\} \in \mathcal{F} \setminus \mathcal{I}^*$ and the integer $n-k+1$ belongs only to one subset $\{n-k, n-k+1\}$ so the number of considered subfamilies in $\mathcal{F} \setminus \mathcal{I}^*$ is the same as in the family $\mathcal{F}^* = \{\{m\}; m = n,1,...,n-k+1\} \cup \{\{z,z+1\}; z = 1,...,n-k-1\}$. Clearly $|\mathcal{I}| = t$ and the definition of $\mathcal{F}^*$ guarantees $|i-j| \leq n-k$ hence to find the number of subfamilies $\mathcal{I}$ we take into considerations only condition $|i-j| \geq k$. This implies that we have exactly $p(n-k+2,k,t)$ subfamilies $\mathcal{I}$ such that $\mathcal{I}^* \cap \mathcal{I} = \emptyset$.
(2). $\mathcal{I}^* \cap \mathcal{I} \neq \emptyset$.
We distinguish two possibilities:
(2.1). $\mathcal{I}^* \cap \mathcal{I} \subset \{\{n-s\}; s = 0,...,k-1\}$.
Without loos of the generalizations assume that $\mathcal{I}^* \cap \mathcal{I} = \{n\}$. Then the definition of the family $\mathcal{I}$ gives that $\{n-i\} \cup \{i\} \notin \mathcal{I}$, for $i = 1,...,k-1$ and $\{1,n\} \cup \{n-j, n-j-1\} \notin \mathcal{I}$, $j = 0,...,k-2$ and $\{l,l+1\} \notin \mathcal{I}$ for $l = 1,...,k-2$. Then $\mathcal{I} = \mathcal{I}' \cup \{n\}$, where $|\mathcal{I}'| = t-1$, $\mathcal{I}' \subset \{\{r\}; r = k,...,n-k\} \cup \{\{s,s+1\}; s = k-1,...,n-k\}$ and $\mathcal{I}'$ satisfies the condition (iv). Since in the family $\{\{r\}; r = k,...,n-k\} \cup \{\{s,s+1\}; s = k-1,...,n-k\}$ the integer $k-1$ belongs only to the one subset $\{k-1,k\}$ and the integer $n-k+1$ belongs only to the subset $\{n-k, n-k+1\}$, hence we can find the number of the subfamilies $\mathcal{I}'$ of the family $\{\{r\}; r = k-1,...,n-k+1\} \cup \{\{s,s+1\}; s = k,...,n-k-1\}$. Evidently for every subsets $Y, Y' \in \mathcal{I}'$ and for every $i \in Y$ and $j \in Y'$ we have $|i-j| \leq n-k$. This implies that there are exactly $p(n-2k+3,k,t-1)$ subfamilies containing the subset $\{n\}$. Since we can choose exactly k subsets belonging to the $\{\{n-s\}; s = 0,...,k-1\}$, so we deduce that there are exactly $k \cdot p(n-2k+3,k,t-1)$ subfamilies $\mathcal{I}$ such that $\mathcal{I}^* \cap \mathcal{I} \subset \{\{n-s\}; s = 0,...,k-1\}$.

(2.2). $\mathcal{I}^* \cap \mathcal{I} \subset \{\{n-r, n-r+1\}; r = 1, ..., k-1\}$.
Without loos of the generalizations assume that $\mathcal{I}^* \cap \mathcal{I} = \{n-1, n\}$. Clearly $\{n-i\} \notin \mathcal{I}$, for $i = 0, ..., k-1$ and $\{l\} \notin \mathcal{I}$, for $l = 1, ..., k-2$. Moreover $\{n-j, n-j+1\} \notin \mathcal{I}$, for $j = 2, ..., k-1\}$ and $\{n, 1\} \cup \{l, l+1\} \notin \mathcal{I}$, for $l = 1, ..., k-3$ and $k \geq 3$. Proving analogously as in subcase (2.1) we obtain that there are exactly $(k-1) \cdot p(n-2k+4, k, t-1)$ subfamilies $\mathcal{I}$ such that $\mathcal{I}^* \cap \mathcal{I} \subset \{\{n-r, n-r+1\}; r = 1, ..., k-1\}$.

Finally from the above cases we obtain that
$q(n, k, t) = k \cdot p(n-2k+3, k, t-1) + (k-1) \cdot p(n-2k+4, k, t-1) + p(n-k+2, k, t)$.

Thus the Theorem is proved. ∎

Theorem 2.4 *Let $k \geq 2$, $n \geq 3$ be integers. Then*
$Q(n, k) = 2n + 1$ *for* $n \leq 2k - 3$
$Q(2k-2, k) = 3k - 4,$
$Q(2k-1, k) = \begin{cases} 14 & if\ k = 2 \\ 8k-3 & if\ k \geq 3, \end{cases}$ *and for* $n \geq 2k$
$Q(n, k) = k \cdot P(n-2k+3, k) + (k-1) \cdot P(n-2k+4, k) + P(n-k+2, k)$.

Proof The initial conditions follow by Theorem 2.3. Assume that $n \geq 2k$. Then by the definition of $Q(n, k)$ and by Theorem 2.3 we have that

$$Q(n, k) = \sum_{t \geq 0} q(n, k, t) = q(n, k, 0) + q(n, k, 1) + \sum_{t \geq 2} q(n, k, t) =$$

$$1 + 2n + \sum_{t \geq 2} (k \cdot p(n-2k+3, k, t-1) + (k-1) \cdot p(n-2k+4, k, t-1) + p(n-k+2, k, t) =$$

$$1 + 2n + k \sum_{t \geq 1} p(n-2k+3, k, t) + (k-1) \sum_{t \geq 1} p(n-2k+4, k, t) + \sum_{t \geq 2} p(n-k+2, k, t) =$$

$$1 + 2n - k + k \sum_{t \geq 0} p(n-2k+3, k, t) - (k-1) + (k-1) \sum_{t \geq 0} p(n-2k+4, k, t) -$$

$$-1 - 2(n-k+2) + 3 + \sum_{t \geq 0} p(n-k+2, k, t) =$$

$$k \cdot P(n-2k+3, k) + (k-1) \cdot P(n-2k+4, k) + P(n-k+2, k),$$

which ends the proof. ∎

The numbers $Q(n, k)$ we will call the *generalized Pell-Lucas numbers.*
If $k = 2$, then for $n = 3$, $Q(3, 2) = Q_3$ and for $n \geq 4$, $Q(n, 2)$ is the Pell-Lucas number $Q_n = 2P_{n-1} + 2P_n$.

The family $\mathcal{F}$ can be regarded as the vertex set of the graph R_n of order $2n$ in Figure 2.

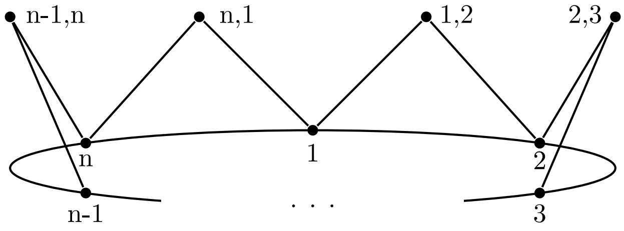

Fig. 2. Graph R_n

Consequently in the graph terminology the generalized Pell-Lucas number $Q(n,k)$, for $n \geq 3$ and $k \geq 2$ is equal to the total number of subset $S \subset V(R_n)$ such that for each two vertices $x_i, x_j \in S$, $d_{R_n}(x_i, x_j) \geq k$. In the other words the number of all k-independent sets of the graph R_n is equal to the generalized Pell-Lucas number, that means $NI_k(R_n) = Q(n,k)$.

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