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A comparison of two notions of porosity

Abstract. In the paper we compare two notions of porosity: the R-ball porosity ($R > 0$) defined by Preiss and Zajíček, and the porosity which was introduced by Olevskii (here it will be called the O-porosity). We find this comparison interesting since in the literature there are two similar results concerning these two notions. We restrict our discussion to normed linear spaces since the R-ball porosity was originally defined in such spaces.

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1. Introduction. Olevskii [2] proved that every convex nowhere dense subset of a Banach space is O-porous. On the other hand, Zajíček [6] in his survey paper on porosity recalled the definition of R-ball porosity ($R > 0$) and observed (without giving a proof) that a convex nowhere dense subset of a Banach space is R-ball porous for every $R > 0$ (cf. [6, p. 518]). Zajíček also considered the definition of the ball smallness (a set M is ball small if $M = \bigcup_{n\in\mathbb{N}} M_n$ and each M_n is R_n-ball porous for some $R_n > 0$).

In the paper we prove that every R-ball porous subset of any normed linear space is O-porous; moreover, with the help of the Baire Category Theorem, we show that in every nontrivial Banach space there exists an O-porous set which is not ball small (hence not R-ball porous for any $R > 0$).

We also show that, in general, the notion of O-porosity is more restrictive than the most "natural" notions of porosity ((c-)lower or (c-)upper porosity; see [6]).

Also, we would like to point out that Olevskii, as an application of his result, gave a few natural examples of sets which are countable unions of convex nowhere dense subsets of Banach spaces (one of them deals with the Banach-Steinhaus Theorem), so Zajíček's observation implies that these sets are ball small. This seems to be interesting since Zajíček [6, p. 516] stated that there was no result in the literature which asserted that an "interesting" set of singular points is ball small.

This paper is organized as follows. First, in the next section, we give exact definitions of some types of porosity and discuss basic relationships between them. Then, in Section 3, we prove the main result of this note.

2. Some notions of porosity. We start with presenting definitions of (σ-)O-porosity, R-ball porosity, the ball smallness, lower and c-lower porosity ($c > 0$).

Let $(X, \|\cdot\|)$ be a normed linear space and $M \subset X$. Given $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball with center x and radius r.

Definition 2.1 ([2]) M is called *O-porous* if

$$\forall_{\alpha\in(0,1)}\ \exists_{R_0>0}\ \forall_{x\in M}\ \forall_{R\in(0,R_0)}\ \exists_{y\in X}\ \left(\|x-y\| = R \textit{ and } B(y,\alpha R)\cap M = \varnothing\right).$$

Remark 2.2 In fact, Definition 2.1 was suggested to Olevskii by K. Saxe (see [2, Remark 1]).

Definition 2.3 ([6]) Let $R > 0$. We say that M is *R-ball porous* if

$$\forall_{x\in M}\ \forall_{\alpha\in(0,1)}\ \exists_{y\in X}\ \left(\|x-y\| = R \textit{ and } B(y,\alpha R)\cap M = \varnothing\right).$$

Remark 2.4 The definition of R-ball porosity presented in [6] ([6, p. 516]) is slightly different from the above one. Namely, M is R-ball porous if

$$\forall_{x\in M}\ \forall_{\varepsilon\in(0,R)}\ \exists_{y\in X}\ \left(\|x-y\| = R \textit{ and } B(y,R-\varepsilon)\cap M = \varnothing\right).$$

However, it is easy to see that they are equivalent.

Definition 2.5 ([6]) Let $c > 0$. M is called *c-lower porous* if for any $x \in M$, we have that

$$2\liminf_{R\to 0^+} \frac{\gamma(x,R,M)}{R} \geq c,$$

where $\gamma(x,R,M) := \sup\{r \geq 0 : \exists_{y\in X}\ B(y,r) \subset B(x,R)\backslash M\}$.

Definition 2.6 ([6]) We say that M is *lower porous* if for any $x \in M$, we have that

$$\liminf_{R\to 0^+} \frac{\gamma(x,R,M)}{R} > 0.$$

If we substitute the upper limit for the lower limit in the above definitions, we get definitions of *c-upper porosity* and *upper porosity.*

Remark 2.7 Clearly, the following implications hold:

1 c-lower porosity $\Longrightarrow$ lower porosity $\Longrightarrow$ upper porosity;

2 c-lower porosity $\Longrightarrow$ c-upper porosity $\Longrightarrow$ upper porosity.

Definition 2.8 ([6]) M is called *ball small* if there exist a sequence $(M_n)_{n\in\mathbb{N}}$ of sets and a sequence $(R_n)_{n\in\mathbb{N}}$ of positive reals such that $M = \bigcup_{n\in\mathbb{N}} M_n$ and each M_n is R_n-ball porous.

Definition 2.9 ([2]) M is called *σ-O-porous* if M is a countable union of O-porous sets.

It is obvious that the above definitions can be formulated in any metric space. However, this paper deals only with normed linear spaces and from now on we assume that the letter X denotes a normed linear space (some results presented below may be false for metric spaces).

We omit an obvious proof of the following

Proposition 2.10 *If $x \in X$ and $M \subset X$ is c-lower porous [O-porous, σ-O-porous, R-ball porous, ball small], so is its translation $M + x$.*

Proposition 2.11 *Let $M \subset X$ and $c > 0$. The following conditions are equivalent:*

(i) M is c-lower porous;

(ii) $\forall_{x\in M}\ \forall_{\beta\in(0,\frac{1}{2}c)}\ \exists_{R_0>0}\ \forall_{R\in(0,R_0)}\ \exists_{y\in X}\ B(y,\beta R) \subset B(x,R)\setminus M.$

Proof $(i) \Longrightarrow (ii)$

Take any $x \in M$, $\beta \in (0, \frac{c}{2})$ and $\delta \in (\beta, \frac{c}{2})$. By (i), there exists $R_0 > 0$ such that

$$\inf_{R\in(0,R_0)} \frac{\gamma(x,R,M)}{R} \geq \delta. \tag{1}$$

Now let $R \in (0, R_0)$. By (1), $\gamma(x,R,M) \geq \delta R$. Hence and by the fact that $\delta R > \beta R$, we have that there exists $y \in X$ such that $B(y,\beta R) \subset B(x,R)\setminus M$.

$(ii) \Longrightarrow (i)$

Fix any $x \in M$ and $\beta \in (0, \frac{c}{2})$. Take R_0 as in (ii). It suffices to show that

$$\inf_{R\in(0,R_0)} \frac{\gamma(x,R,M)}{R} \geq \beta.$$

Take any $R \in (0, R_0)$. By (ii), there exists $z \in X$ such that $B(z,\beta R) \subset B(x,R)\setminus M$. Hence $\beta R \in \{r \geq 0 : \exists_{y\in X}\ B(y,r) \subset B(x,R)\setminus M\}$, so $\gamma(x,R,M) \geq \beta R$ and, in particular, $\frac{\gamma(x,R,M)}{R} \geq \beta$. ∎

Proposition 2.12 *If $\varnothing \neq M \subset X$ is c-lower porous, then $c \leq 1$.*

Proof Since X is a normed space, the following clearly holds:

$$\forall_{d>\frac{1}{2}}\ \forall_{R>0}\ \forall_{x\in X}\ \forall_{z\in X}\ \left(x \notin B(z,dR) \Longrightarrow B(z,dR) \subsetneqq B(x,R)\right).$$

Hence we infer $2\gamma(x,R,M) \leq R$ for any $x \in M$ and $R > 0$. ∎

Proposition 2.13 *Every O-porous set $M \subset X$ is 1-lower porous.*

Proof Let M be O-porous. By Proposition 2.11, it suffices to prove

$$\forall_{x\in M}\ \forall_{\beta\in(0,\frac{1}{2})}\ \exists_{R_0>0}\ \forall_{R\in(0,R_0)}\ \exists_{y\in X}\ B(y,\beta R) \subset B(x,R)\backslash M.$$

Fix $x \in M$ and $\beta \in (0,\frac{1}{2})$. Take $R_0 > 0$ as in the definition of O-porosity, chosen for $\alpha := 2\beta$. Take any $R < 2R_0$. Since $\frac{1}{2}R < R_0$, there exists $y \in X$ such that $\| y - x \| = \frac{1}{2}R$ and $B(y,\alpha\frac{1}{2}R) \cap M = \varnothing$. Since $B(y,\alpha\frac{1}{2}R) = B(y,\beta R)$, it is enough to show that

$$B(y,\beta R) \subset B(x,R). \tag{2}$$

For any $z \in B(y,\beta R)$, we have

$$\| x - z \| \ \leq\ \| x - y \| + \| y - z \| \ \leq \frac{1}{2}R + \beta R < R,$$

so (2) holds. ∎

Now we give an example of 1-lower porous subset of $\mathbb{R}$ which is not O-porous. This example and Proposition 2.13 show that, in general, the notion of O-porosity is more restrictive than the notion of 1-lower porosity.

Example 2.14 Consider the set $M := \bigcup_{n\in\mathbb{N}} \left\{n + \frac{k}{n} : k = 0, ..., n-1\right\}$. It is easy to see that M is 1-lower porous and is not O-porous.

By Proposition 2.13, every σ-O-porous set is σ-1-lower porous (a countable union of 1-lower porous sets). However, we do not know if there exists a set which is σ-1-lower porous and is not σ-O-porous. We leave it as an open question.

3. Main results. We first show that any R-ball porous subset of a normed linear space is O-porous. We need the following result which is also given (without a proof) in [1].

Proposition 3.1 *Let X be a normed linear space. If $M \subset X$ is R-ball porous, then M is r-ball porous for all $r \in (0,R]$.*

PROOF Fix $r \in (0, R]$, $x \in M$ and $\alpha \in (0,1)$. Define $\beta := 1 - \frac{r}{R}(1-\alpha)$. By hypothesis, there exists $y \in X$ such that $\|x - y\| = R$ and $B(y, \beta R) \cap M = \varnothing$. Let $y_1 \in [x, y]$ be such that $\| y_1 - x \| = r$. It suffices to prove that

$$B(y_1, \alpha r) \subset B(y, \beta R). \tag{3}$$

For any $z \in B(y_1, \alpha r)$, we have

$$\| y - z \| \leq \| y - y_1 \| + \| y_1 - z \| < R - r + \alpha r = \beta R$$

which yields (3). ■

COROLLARY 3.2 *Any R-ball porous subset of a normed linear space is O-porous.*

Now we will show that in any nontrivial Banach space there exists an O-porous set which is not ball small. In particular, the class of σ-O-porous sets in Banach spaces is essentially wider (with respect to the inclusion) than the class of ball small sets. Hence we infer Olevskii's result does not imply that a countable union of nowhere dense convex subsets of any Banach space is ball small.

We start with some auxiliary results.

PROPOSITION 3.3 *Let $R > 0$ and $M \subset \mathbb{R}$.*

1 *If M is R-ball porous, then for any $x \in \mathbb{R}$, the set $[x - R, x + R] \cap M$ contains at most 2 elements.*

2 *If for any $x \in \mathbb{R}$, the set $[x - R, x + R] \cap M$ contains at most 2 elements, then M is $\frac{1}{2}R$-ball porous.*

PROOF *Ad* 1. Assume that for some $x \in \mathbb{R}$, the set $[x - R, x + R] \cap M$ contains more than two elements. Let a, b, $c \in \mathbb{R}$ be such that $a < b < c$ and $\{a, b, c\} \subset [x - R, x + R] \cap M$. Therefore $c - b < 2R$ and $b - a < 2R$. Take $\alpha \in (0,1)$ such that

$$\alpha > \max\left\{ \frac{| c - b - R |}{R}, \frac{| b - a - R |}{R} \right\}.$$

It is easy to see that $a \in B(b - R, \alpha R)$ and $c \in B(b + R, \alpha R)$. Hence M is not R-ball porous.

Ad 2. Fix any $x \in M$ and $\alpha \in (0,1)$. Since $[x - R, x + R] \cap M$ contains at most two elements, we see that $(x, x + R] \cap M = \varnothing$ or $[x - R, x) \cap M = \varnothing$. Therefore $B(x - \frac{1}{2}R, \alpha\frac{1}{2}R) \cap M = \varnothing$ or $B(x + \frac{1}{2}R, \alpha\frac{1}{2}R) \cap M = \varnothing$. Hence M is $\frac{1}{2}R$-ball porous. ■

COROLLARY 3.4 *Any R-ball porous subset of $\mathbb{R}$ is countable.*

Hence, as an immediate consequence, we get the following characterization of ball small subsets of $\mathbb{R}$. Note that the folowing result was also given (without a proof) in [3].

COROLLARY 3.5 *Let $M \subset \mathbb{R}$. M is ball small iff M is countable.*

Now we will give an example of an O-porous subset of $\mathbb{R}$ which is not ball small. In the following $\{0,1\}^{<\mathbb{N}}$ is the set of all finite sequences of ones and zeros, $\mid s \mid$ means the length of s, i.e., $\mid s \mid = \mid (s(0),...,s(n-1)) \mid = n$, and diam$A$ denotes the diameter of A.

EXAMPLE 3.6 Fix a sequence $\gamma = (\gamma_n)_{n\in\mathbb{N}\cup\{0\}}$ such that $\gamma_n > \frac{1}{2}$ and $\gamma_n \nearrow 1$. Consider the symmetric perfect set $C(\gamma)$ (cf. [5, p. 318]). The set $C(\gamma)$ is defined similarly to the Cantor ternary set: by induction, we define the sequence of sets $\{C_s\}_{s\in\{0,1\}^{<\mathbb{N}}}$ satisfying the following conditions :

(c1) $C_\varnothing := [0,1]$;

(c2) if $s \in \{0,1\}^{<\mathbb{N}}$ and H_s is the concentric open subinterval of C_s of length $\gamma_{|s|}l_{|s|}$, where $l_{|s|}$ =diamC_s, then $C_{s\hat{}0}$ is the left and $C_{s\hat{}1}$ is the right subinterval of $C_s \backslash H_s$.

Finally, set $C(\gamma) := \bigcap_{n\in\mathbb{N}} \bigcup_{|s|=n} C_s$. Since $C(\gamma)$ is uncountable, it is not ball small in view of Corollary 3.5. We will show that $C(\gamma)$ is O-porous. It is easy to see that for any $n \in \mathbb{N}$, $l_n = \frac{1-\gamma_{n-1}}{2} l_{n-1}$. Fix $\alpha \in (0,1)$ and let $n_0 \in \mathbb{N}$ be such that

$$\gamma_{n_0} > \frac{\alpha+1}{2}. \tag{4}$$

Set $R_0 := \frac{1}{2} l_{n_0+1}$ and let $R \in (0, R_0)$. Since $l_n \searrow 0$, there is $n \in \mathbb{N}$ such that

$$\frac{1}{2} l_{n+1} < R \le \frac{1}{2} l_n. \tag{5}$$

Since $R < R_0$, we have that $n \ge n_0 + 1$. Now take any $x \in C(\gamma)$. In particular, $x \in C_s$ for some s with $\mid s \mid = n$. Consider four cases:

Case 1: $x \in C_{s\hat{}0\hat{}1}$. Set $y := x + R$. We will prove that $B(y, \alpha R) \cap C(\gamma) = \varnothing$. By (4) and (5), we have

$$\text{dist}(C_{s\hat{}0\hat{}1}, C_{s\hat{}1\hat{}0}) = \gamma_n l_n > \frac{\alpha+1}{2} l_n \ge (\alpha+1)R,$$

so it suffices to show that for any $z \in C_{s\hat{}0\hat{}1}$, $\mid z - y \mid \ge \alpha R$.
Since diam$C_{s\hat{}0\hat{}1} = l_{n+2} < R$, for any $z \in C_{s\hat{}0\hat{}1}$, by (4) and (5), we have

$$\mid z - y \mid \ge R - \text{diam} C_{s\hat{}0\hat{}1} = R - l_{n+2}$$

$$= R - \frac{1-\gamma_{n+1}}{2} l_{n+1} > R - (1-\gamma_{n+1})R = \gamma_{n+1} R > \alpha R.$$

Case 2: $x \in C_{s\hat{}1\hat{}0}$. Set $y := x - R$. Using a similar argument as in Case 1, we may infer $B(y, \alpha R) \cap C(\gamma) = \varnothing$.

Case 3: $x \in C_{s\hat{\ }0\hat{\ }0}$. Set $y := x - R$. By (4) and (5), we have

$$2R \leq l_n = \frac{1-\gamma_{n-1}}{2} l_{n-1} < \gamma_{n-1} l_{n-1} = \min\{\mathrm{dist}(C_q, C_p) : p \neq q \text{ and } \mid p \mid = \mid q \mid = n\},$$

and we can show that $B(y, \alpha R) \cap C(\gamma) = \varnothing$ in a similar way as in Case 1.

Case 4: $x \in C_{s\hat{\ }1\hat{\ }1}$. *Set* $y := x + R$. Repeating an argument from Case 3, we infer $B(y, \alpha R) \cap C(\gamma) = \varnothing$.

As a conclusion we see that $C(\gamma)$ is O-porous.

Lemma 3.7 *Any nontrivial real Banach space* $(Y, \|\cdot\|_0)$ *is isometrically isomorphic to some space* $(\mathbb{R} \times X, \|\cdot\|)$, *where* X *is a closed linear subspace of* Y *and the norm* $\|\cdot\|$ *satisfies the following conditions:*

(a) $\|\cdot\|$ *is equivalent to the norm max* $\|\cdot\|_{\max}$, *where*
$\|(t,x)\|_{\max} := \max\{\mid t \mid, \|x\|_0\}$ *for* $(t,x) \in \mathbb{R} \times X$;

(b) $\forall_{t\in\mathbb{R}}\ \|(t,0)\| = \mid t \mid$ *and* $\forall_{x \in X}\ \|(0,x)\| = \|x\|_0$.

Proof Take any $x_0 \in Y$ with $\|x_0\|_0 = 1$ and consider one dimensional subspace $Y_1 := \{tx_0 : t \in \mathbb{R}\} \subset Y$. Since $\dim Y_1 < \infty$, there exists a closed subspace $Y_2 \subset Y$ such that $Y = Y_1 \oplus Y_2$ (see, e.g., [4]). Set $X = Y_2$. Note that for every $y \in Y$ there are $t \in \mathbb{R}$ and $x \in X$ such that $y = tx_0 + x$. For $(t,x) \in \mathbb{R} \times X$ define $\|(t,x)\| := \|tx_0 + x\|_0$. Observe that $\|\cdot\|$ is a norm on $\mathbb{R} \times X$, equivalent to the norm $\|\cdot\|_{\max}$. Then the function $(t,x) \mapsto tx_0 + x$ is an isometrical isomorphism between $(\mathbb{R} \times X, \|\cdot\|)$ and $(Y, \|\cdot\|_0)$. ■

Remark 3.8 Let $(X, \|\cdot\|)$ be a complex space. Then we can consider it as a real space (it is obvious how to formalize this statement). It is clear that for any $M \subset X$, the following statements are equivalent:

1. M is O-porous [ball small] in $(X, \|\cdot\|)$;
2. M is O-porous [ball small] in $(X, \|\cdot\|)$ considered as a real space.

From now on the symbol $(\mathbb{R} \times X, \|\cdot\|)$ denotes a real normed linear space $\mathbb{R} \times X$, where X is a real normed linear space and the norm $\|\cdot\|$ satisfies conditions (a) and (b) from Lemma 3.7.

REMARK 3.9 By the above Remark and Lemma 3.7, in order to prove that any nontrivial Banach space admits an O-porous set which is not ball small, it suffices to prove it for every space $(\mathbb{R} \times X, \|\cdot\|)$, where X is a real Banach space.

LEMMA 3.10 *Let $(X, \|\cdot\|_0)$ be a real normed linear space. For any $R > 0$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $p \in X$ and any t_1, t_0, $t_2 \in \mathbb{R}$, if $t_1 < t_0 < t_2$, $| t_1 - t_0 | < \delta$ and $| t_2 - t_0 | < \delta$, then the set $N := \{t_1, t_0, t_2\} \times B(p, \varepsilon)$ is not R-ball porous in $(\mathbb{R} \times X, \|\cdot\|)$.*

PROOF There exists $\gamma \in (0, 1]$ such that for any $(s, x) \in \mathbb{R} \times X$,

$$\gamma \max\{| s |, \|x\|_0\} \leq \|(s, x)\|. \tag{6}$$

Fix $R > 0$ and $\varepsilon > 0$. Let

$$\overline{\varepsilon} := \min\left\{\frac{R}{4}, \frac{\varepsilon}{4}\right\}. \tag{7}$$

We are ready to define δ

$$\delta := \overline{\varepsilon} \min\left\{\frac{\varepsilon\gamma}{R}, 1\right\}. \tag{8}$$

In particular,

$$\delta \leq \overline{\varepsilon} \leq \frac{R}{4} < R.$$

Take any $p \in X$ and any t_1, t_0, $t_2 \in \mathbb{R}$ such that $t_1 < t_0 < t_2$, $| t_1 - t_0 | < \delta$ and $| t_2 - t_0 | < \delta$. By Proposition 2.10, we may assume that $(t_0, p) = (0, 0)$. It suffices to show that there is $\alpha \in (0, 1)$ such that for any $(s, y) \in \mathbb{R} \times X$,

$$\|(s, y)\| = R \Longrightarrow B((s, y), \alpha R) \cap N \neq \varnothing.$$

Let $\alpha \in (0, 1)$ be such that

$$\alpha > \max\left\{\frac{R - \overline{\varepsilon}}{R}, \frac{R - \gamma \mid t_1 \mid}{R}, \frac{R - \gamma \mid t_2 \mid}{R}\right\}. \tag{9}$$

Fix any (s, y) such that $\|(s, y)\| = R$. Now we will consider three cases.

Case 1: $y = 0$. Since $| s | = R > \delta$, we see that $s > t_2 > 0$ or $s < t_1 < 0$. Assume that $s > t_2 > 0$. Then $(t_2, 0) \in N$ and, by (9), we have

$$\|(s, 0) - (t_2, 0)\| = s - t_2 = R - t_2 \leq R - \gamma t_2 < \alpha R.$$

If $s < t_1 < 0$, we proceed similarly.

Case 2: $| s | < \overline{\varepsilon}$ and $y \neq 0$. Since

$$R = \|(s, y)\| = \|(s, 0) + (0, y)\| \leq \|(s, 0)\| + \|(0, y)\| < \overline{\varepsilon} + \|(0, y)\|$$

and

$$\|(0,y)\| = \|(s,y)-(s,0)\| \leq \|(s,y)\| + \|(s,0)\| < R + \overline{\varepsilon},$$

we get, by (7), that

$$0 < 3\overline{\varepsilon} \leq R - \overline{\varepsilon} < \|(0,y)\| < R + \overline{\varepsilon}. \tag{10}$$

Now take $z \in [0,y]$ such that $\|z\|_0 = 3\overline{\varepsilon}$. By (7), (9) and (10), we have $(0,z) \in N$ and also

$$\|(s,y)-(0,z)\| = \|(s,0)+(0,y)-(0,z)\| \leq \|(s,0)\| + \|(0,y)-(0,z)\|$$

$$= \|(s,0)\| + \|(0,y)\| - \|(0,z)\| < \overline{\varepsilon} + (R+\overline{\varepsilon}) - \|z\|_0 = R + 2\overline{\varepsilon} - 3\overline{\varepsilon} < \alpha R.$$

Case 3: $| \, s \, | \geq \overline{\varepsilon}$ and $y \neq 0$. We may assume, without loss of generality, that $s \geq \overline{\varepsilon}$. Set

$$\lambda := \frac{t_2}{s}.$$

By (8), $\delta/\overline{\varepsilon} = \min\left\{\frac{\varepsilon\gamma}{R}, 1\right\}$ and by (6), $\max\{| \, s \, |, \|y\|_0\} \leq \frac{1}{\gamma}R$. Hence

$$\frac{\gamma t_2}{R} \leq \lambda < \frac{\delta}{s} \leq \min\left\{\frac{\varepsilon\gamma}{R}, 1\right\} \leq \min\left\{\frac{\varepsilon}{\|y\|_0}, 1\right\}.$$

In particular, $\|\lambda y\|_0 < \varepsilon$ and $\lambda \in (0,1)$, so $(\lambda s, \lambda y)(=(t_2, \lambda y))$ is an element of N, and, using (9) also, we infer

$$\|(s,y)-(\lambda s, \lambda y)\| = | \, 1-\lambda \, | \, R = (1-\lambda)R \leq \left(1 - \frac{\gamma t_2}{R}\right) R < \alpha R.$$

With the case $s \leq -\overline{\varepsilon}$ we proceed analogously.
Thus we have shown that in each case $B((s,y), \alpha R) \cap N \neq \varnothing$. ■

Lemma 3.11 *Let X be a real normed linear space and $M \subset \mathbb{R}$ be O-porous. Then $M \times X$ is O-porous in the space $(\mathbb{R} \times X, \| \cdot \|)$.*

Proof For any $r > 0$, we put

$$A_r := \sup\left\{| \, t \, | : (t,x) \in \mathbb{R} \times X, \ \|(t,x)\| = r\right\}. \tag{11}$$

Clearly, in the above definition it is enough to consider $t \geq 0$ or $t \leq 0$. Since $\| \cdot \|$ is equivalent to the norm max, we infer $0 < A_r < \infty$ for any $r > 0$. It is also easy to prove that for any $r > 0$, we have:

$$rA_1 = A_r; \tag{12}$$

By (12), we see that for any $(t,x) \in \mathbb{R} \times X$,

$$| \, t \, | \leq A_1 \|(t,x)\|. \tag{13}$$

Now we are ready to prove the Lemma. Fix any $\alpha \in (0,1)$ and $\beta \in (\alpha, 1)$. Since M is O-porous, there exists $R_0 > 0$ such that

$$\forall_{t\in M}\ \forall_{R\in(0,R_0)}\ \exists_{a\in\mathbb{R}}\ (\mid a-t\mid = R \text{ and } B(a,\beta R)\cap M = \varnothing)\,. \tag{14}$$

Define $R_1 := R_0/A_1$, and take any $R \in (0, R_1)$ and $(t,x) \in M\times X$. Since $RA_1 < R_0$, by (14), there exists $a \in \mathbb{R}$ such that $\mid a-t \mid = RA_1$ and

$$B(a, \beta RA_1)\cap M = \varnothing. \tag{15}$$

Assume that $a = t + RA_1$ (we proceed analogously with the case $a = t - RA_1$) and put $\varepsilon := (\beta-\alpha)RA_1$. We will show that there exists $(s,y) \in \mathbb{R}\times X$ such that

$$\|(t,x)-(s,y)\| = R \text{ and } \mid a-s \mid < \varepsilon. \tag{16}$$

By (12) and the statement after (11), there exists $(q,z) \in \mathbb{R}\times X$ such that $q \geq 0$, $\|(q,z)\| = R$ and $RA_1 - \varepsilon < q \leq RA_1$. Now define $s := t+q$ and $y := x+z$. Since $(s,y)-(t,x) = (q,z)$ and $\mid a-s \mid = \mid RA_1 - q \mid$, we get (16). It suffices to observe that

$$B((s,y),\alpha R)\cap(M\times X) = \varnothing.$$

For any $(p,u) \in B((s,y),\alpha R)$, by (13) and (16), we have

$$\mid p-a \mid \leq \mid p-s \mid + \mid s-a \mid < \alpha RA_1 + \varepsilon = \beta RA_1,$$

so, by (15), $p \notin M$. Hence $(p,u) \notin M\times X$. ∎

Proposition 3.12 *Let X be a normed linear space and $R > 0$. If $M \subset X$ is R-ball porous, so is its closure $\overline{M}$.*

Proof Let $M \subset X$ be R-ball porous. Fix $\alpha \in (0,1)$ and $x \in \overline{M}$. Take any $\beta \in (\alpha,1)$ and choose $x_1 \in M$ such that $\| x - x_1 \| < (\beta-\alpha)R$. By hypothesis, we infer there exists $y_1 \in X$ such that $\| y_1 - x_1 \| = R$ and $B(y_1, \beta R)\cap M = \varnothing$. Put $y := y_1 + (x - x_1)$. Then we have $\| y - x \| = R$ and for any $z \in B(y,\alpha R)$, we see that

$$\| y_1 - z \| \leq \| y_1 - y \| + \| y - z \| < (\beta-\alpha)R + \alpha R = \beta R,$$

so $z \in B(y_1,\beta R)$. Hence $B(y,\alpha R) \subset X\backslash M$. Finally, we have

$$B(y,\alpha R) = \mathrm{Int}B(y,\alpha R) \subset \mathrm{Int}(X\backslash M) = X\backslash\overline{M}$$

which shows that $\overline{M}$ is R-ball porous. ∎

Now we are ready to give a more general construction. A very important tool in the proof of the following theorem is the Baire Category Theorem (note that the idea of using the Baire Category Theorem to show that some sets are not σ-porous in some senses, is commonly known – see, e. g. [6, p. 513]).

Theorem 3.13 *In any nontrivial Banach space there exists an O-porous set which is not ball small.*

Proof We will use Remark 3.8. Let X be any real Banach space, and let $C(\gamma) \subset \mathbb{R}$ be the set defined in Example 3.6. By Lemma 3.11, $C(\gamma) \times X$ is O-porous in $(\mathbb{R} \times X, \|\cdot\|)$. We will show that $C(\gamma) \times X$ is not ball small. Assume that $C(\gamma) \times X = \bigcup_{n\in\mathbb{N}} M_n$. In view of Proposition 3.12, it is enough to show that there exists $n_0 \in \mathbb{N}$ such that $\overline{M_{n_0}}$ is not R-ball porous for any $R > 0$. Since $C(\gamma) \times X$ is a closed subset of a Banach space $\mathbb{R} \times X$, it is complete. By the Baire Category Theorem, there exists $n_0 \in \mathbb{N}$ such that M_{n_0} is not nowhere dense in $C(\gamma) \times X$. Hence there exist a set $K \neq \varnothing$ open in $C(\gamma)$, $p \in X$ and $\varepsilon > 0$ such that

$$K \times B(p,\varepsilon) \subset \overline{M_{n_0}}^{C(\gamma)\times X} = \overline{M_{n_0}}. \tag{17}$$

Now we show that for any $R > 0$, the set $\overline{M_{n_0}}$ is not R-ball porous. Fix $R > 0$. It is well known that the family $\{C_s \cap C(\gamma) : s \in \{0,1\}^{<\mathbb{N}}\}$ forms a topological base of $C(\gamma)$. Hence there exists $s \in \{0,1\}^{<\mathbb{N}}$ such that $C_s \cap C(\gamma) \subset K$ and $\operatorname{diam} C_s < \delta$, where $\delta > 0$ is chosen for R and ε as in Lemma 3.10. Since $C_s \cap C(\gamma)$ is infinite, the set $(C_s \cap C(\gamma)) \times B(p,\varepsilon)$ contains some set of the form $\{t_1, t_0, t_2\} \times B(p,\varepsilon)$ which, according to Lemma 3.10, is not R-ball porous. Hence and by (17), we infer $\overline{M_{n_0}}$ is not R-ball porous. ■

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