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Equivalent forms of $\mathfrak{m}$-paracompactness

0. Introduction. In this paper we give new generalizations of the notions of a space to be fully normal or even. These generalizations we call fully $\mathfrak{m}$-normal and $\mathfrak{m}$-even where $\mathfrak{m}$ is an infinite cardinal. It is then established that these are equivalent to $\mathfrak{m}$-paracompactness under suitable circumstances. The above mentioned definitions are the $\mathrm{FN}_1(\mathfrak{m})$ and $\mathrm{E}_1(\mathfrak{m})$ of paragraph 4. It should be noted that these are different from similar concepts defined in [3]. The main theorem (7.7) gives 16 conditions each of which is equivalent to $\mathfrak{m}$-paracompactness and normality. Some of these are known (see [5]); however, since they fit naturally into our chain of implications (see Fig. 1) we have included them. Paragraph 8 gives an additional condition.

In the course of our investigation we introduce a simple formalism (3.4-3.7) which makes the connections between star coverings and neighborhoods of the diagonal transparent. It is through this formalism which many of our equivalences are established. Although it is not shown here the same technique gives an almost immediate proof of the equivalence between the covering definition of uniformity found in [6] and the neighborhood of the diagonal definition of uniformity found in [1].

For the convenience of the reader we have made the paper essentially self-contained.

1. Preliminaries. Throughout the paper any collection of subsets of a set will be assumed to be indexed even when an explicit indexing is not in evidence. If A is any set, $|A|$ will denote its cardinality. If $\mathscr{A} = \{A_\alpha | \alpha \in \Gamma\}$ one has $|\mathscr{A}| \leqslant |\Gamma|$. For any set A, A^F is defined to be the set of all the finite subsets of A. When A is infinite it is known that $|A| = |A^F|$. It is clear that an indexing may always be arranged so that $|\Gamma| = |\mathscr{A}|$. If $\mathscr{A} = \{A_\alpha | \alpha \in \Gamma\}$ is a collection of subsets of X and $B \subseteq X$ define $c(B, \Gamma, \mathscr{A}) = \{\alpha | A_\alpha \cap B \neq \emptyset\}$. When $B = \{x\}$ where $x \in X$ we write $c(x, \Gamma, \mathscr{A})$ for $c(B, \Gamma, \mathscr{A})$. If no confusion results $c(B, \Gamma)$ will be written for $c(B, \Gamma, \mathscr{A})$. Define $\mathrm{St}(B, \mathscr{A}) = \bigcup \{A_\alpha | \alpha \in c(B, \Gamma, \mathscr{A})\}$ and write $\mathrm{St}(x, \mathscr{A})$ for $\mathrm{St}(\{x\}, \mathscr{A})$.

Let $\mathscr{A}, \mathscr{B}$ be collections of subsets of X; then define $S(\mathscr{A}) = \{\mathrm{St}(x, \mathscr{A}) \mid x \in X\}$, $S^*(\mathscr{A}, \mathscr{B}) = \{\mathrm{St}(A, \mathscr{B}) \mid A \in \mathscr{A}\}$ and $S^*(\mathscr{A}) = S^*(\mathscr{A}, \mathscr{A})$. $\mathscr{A}$ is said to *refine* $\mathscr{B}$ iff each member of $\mathscr{A}$ is contained in some member of $\mathscr{B}$. In this case we write $\mathscr{A} \leqslant \mathscr{B}$. If $S(\mathscr{A}) \leqslant \mathscr{B}$ we write $\mathscr{A} \leqslant^* \mathscr{B}$ and say that $\mathscr{A}$ star *refines* $\mathscr{B}$. If $S^*(\mathscr{A}) \leqslant \mathscr{B}$ we write $\mathscr{A} \leqslant^{**} \mathscr{B}$ and say that $\mathscr{A}$ *strongly star refines* $\mathscr{B}$. If $A \subseteq X$ and $\mathscr{A}$ is a collection of subsets of X, then $\mathscr{A}$ is said to *cover* A iff $A \subseteq \bigcup \mathscr{A}$.

A collection $\mathscr{A}$ of subsets of a topological space is said to be *open* (*closed*) if each member of $\mathscr{A}$ is open (closed). If $\mathscr{A} = \{A_\alpha \mid \alpha \in \Gamma\}$ we define $\mathscr{A}^- = \{A_\alpha^- \mid \alpha \in \Gamma\}$. If $\mathscr{A}$ is a collection of subsets of a topological space it is said to be *closure preserving* iff for any subcollection $\mathscr{B} \subseteq \mathscr{A}$, $\bigcup \mathscr{B}^- = (\bigcup \mathscr{B})^-$.

We use the terms regularity and normality in the usual manner except that no assumption concerning T_0, T_1 or T_2 is made. Normality will often be abbreviated T_4.

The symbol $\wedge$ when used will stand for the word "and". The term neighborhood will always refer to an open set.

2. Coverings and stars. The following easy lemma will be left to the reader.

2.1 LEMMA. *Let $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$ be collections of subsets of X and let $E, F \subseteq X$. Suppose $\mathscr{A} \leqslant \mathscr{B}$ and $\mathscr{C} \leqslant \mathscr{D}$ then the following hold*:

i) $\mathrm{St}(E, \mathscr{A}) \subseteq \mathrm{St}(E, \mathscr{B})$,

ii) $S(\mathscr{A}) \leqslant S(\mathscr{B})$,

iii) *If* $E \subseteq F$, *then* $\mathrm{St}(E, \mathscr{A}) \subseteq \mathrm{St}(F, \mathscr{A})$,

iv) $S^*(\mathscr{C}, \mathscr{A}) \leqslant S^*(\mathscr{D}, \mathscr{B})$.

Let X be a topological space and $\mathscr{A} = \{A_\alpha \mid \alpha \in \Gamma\}$ be a collection of subsets of X. Let $\mathfrak{m}$ be an infinite cardinal then $\mathscr{A}$ is said to be $\mathfrak{m}$-locally finite ($\mathfrak{m}$-discrete) iff there is an open covering $\mathscr{U}$ of X such that $|\mathscr{U}| \leqslant \mathfrak{m}$ and $|c(U, \Gamma)| < \aleph_0$ $\big(|c(U, \Gamma)| \leqslant 1\big)$ for each $U \in \mathscr{U}$. We define local finiteness and discreteness in the same manner dropping the restriction that $|\mathscr{U}| \leqslant \mathfrak{m}$. It is easily seen that an $\mathfrak{m}$-locally finite collection (locally finite collection) is closure preserving. A collection $\mathscr{A} = \{A_\alpha \mid \alpha \in \Gamma\}$ of subsets of X is called *point finite* iff for each $x \in X$ we have $|c(x, \Gamma)| < \aleph_0$.

2.2 THEOREM. *Let $\mathscr{A} = \{A_\alpha \mid \alpha \in \Gamma\}$ be a locally finite* (*discrete*) *collection of subsets of a space X. Then $\mathscr{A}$ is $\mathfrak{m}$-locally finite* ($\mathfrak{m}$-*discrete*) *iff* $|\Gamma| \leqslant \mathfrak{m}$.

Proof. $\rightarrow$: Suppose $\mathscr{A}$ is $\mathfrak{m}$-locally finite (this includes the $\mathfrak{m}$-discrete case). By hypothesis there is an open covering $\mathscr{U}$ of X such that $c(U, \Gamma)$ is finite for for each $U \in \mathscr{U}$ and $|\mathscr{U}| \leqslant \mathfrak{m}$. Since $\mathscr{U}$ is a covering of X, $\bigcup \{c(U, \Gamma) \mid U \in \mathscr{U}\} = \Gamma$. If Γ is finite, then certainly $|\Gamma| \leqslant \mathfrak{m}$. If Γ is infinite, then $|\Gamma| \leqslant \aleph_0 |\mathscr{U}| \leqslant \aleph_0 \mathfrak{m} = \mathfrak{m}$.

$\leftarrow$: Now suppose that $\mathscr{A}$ is locally finite (discrete) and $|\Gamma| \leqslant \mathfrak{m}$. By hypothesis there is an open covering $\mathscr{U} = \{U_\lambda | \ \lambda \in \Lambda\}$ such that $|c(U_\lambda, \Gamma)| < \aleph_0$ ($|c(U_\lambda, \Gamma)| \leqslant 1$) for each $\lambda \in \Lambda$. Let $\sigma(\lambda) = c(U_\lambda, \Gamma)$ for each $\lambda \in \Lambda$ and let $\Sigma = \{\sigma(\lambda) | \ \lambda \in \Lambda\}$ then $\Sigma \subseteq \Gamma^F$ and thus $|\Sigma| \leqslant \mathfrak{m}$. For each $\sigma \in \Sigma$ let $V_\sigma = \bigcup \{U_\lambda | \ \sigma(\lambda) = \sigma\}$ then $\mathscr{V} = \{V_\sigma | \ \sigma \in \Sigma\}$ is an open covering of X. By the construction we see that if $\sigma \in \Sigma$, then $c(V_\sigma, \Gamma) = c(U_\lambda, \Gamma)$ when $\sigma = \sigma(\lambda)$. Thus $\mathscr{A}$ is $\mathfrak{m}$-locally finite ($\mathfrak{m}$-discrete).

Let $\mathscr{A} = \{A_\alpha | \ \alpha \in \Gamma\}$, $\mathscr{B} = \{B_\lambda | \ \lambda \in \Lambda\}$ be collections of subsets of X such that $\mathscr{B} \leqslant \mathscr{A}$. For each $\lambda \in \Lambda$ choose $\alpha(\lambda) \in \Gamma$ such that $B_\lambda \subseteq A_{\alpha(\lambda)}$. Let $\alpha \in \Gamma$ and define $B'_\alpha = \bigcup \{B_\lambda | \ \alpha(\lambda) = \alpha\}$. It is clear that $B'_\alpha \subseteq A_\alpha$. Let $\mathscr{B}' = \{B'_\alpha | \ \alpha \in \Gamma\}$ then $\mathscr{B}'$ is called a *$\mathscr{B}$ associated precise refinement of $\mathscr{A}$*. The following lemma then holds.

2.3 LEMMA. *With $\mathscr{A}$, $\mathscr{B}$, $\mathscr{B}'$ as in the previous paragraph*

i) $\bigcup \mathscr{B} = \bigcup \mathscr{B}'$,

ii) *If $\mathscr{B}$ satisfies any one of the following properties so does $\mathscr{B}'$:*

a) *open*

b) *point finite,*

c) *discrete,*

d) *locally finite,*

e) *closure preserving,*

f) *closed and closure preserving.*

2.4 LEMMA. *Let $\mathscr{A} = \{A_\gamma | \ \gamma \in \Gamma\}$, $\mathscr{B} = \{B_\delta | \ \delta \in \Delta\}$, $\mathscr{C} = \{C_\lambda | \ \lambda \in \Lambda\}$ be collections of subsets of X such that*

i) $\bigcup \mathscr{C} \subseteq \bigcup \mathscr{B}$,

ii) $|c(B_\delta, \Gamma)| < \aleph_\alpha$ *for each* $\delta \in \Delta$,

iii) $|c(C_\lambda, \Delta)| < \aleph_\beta$ *for each* $\lambda \in \Lambda$.

Then $|c(C_\lambda, \Gamma)| < \aleph_\alpha \aleph_\beta$ *for each* $\lambda \in \Lambda$.

Proof. Since $\bigcup \mathscr{C} \subseteq \bigcup \mathscr{B}$, for each $\lambda \in \Lambda$ we have $C_\lambda \subseteq \bigcup \{B_\delta | \ \delta \in c(C_\lambda, \Delta)\}$. Thus $\gamma \in c(C_\lambda, \Gamma)$ implies that $\gamma \in c(B_\delta, \Gamma)$ for some $\delta \in c(C_\lambda, \Delta)$. Hence $c(C_\lambda, \Gamma) \subseteq \bigcup \{c(B_\delta, \Gamma) | \ \delta \in c(C_\lambda, \Delta)\}$. Thus $|c(C_\lambda, \Gamma)| < \aleph_\alpha |c(C_\lambda, \Delta)| < \aleph_\alpha \aleph_\beta$.

3. Relations. A relation on a set X is a subset $\hat{R} \subseteq X \times X$. A particular relation is the diagonal $\hat{D}(X) = \{(x, x) | \ x \in X\}$. We define $\hat{R}^{-1}$, the inverse of $\hat{R}$, in the usual manner and call $\hat{R}$ *symmetric* iff $\hat{R} = \hat{R}^{-1}$. If $\hat{R}$, $\hat{S}$ are relations on X we define

$$\hat{R} \circ \hat{S} = \{(x, y) | \ \exists z [(x, z) \in \hat{R} \wedge (z, y) \in \hat{S}]\}.$$

If $A \subseteq X$ define

$$\hat{R}[A] = \{y | \exists x [x \in A \wedge (y, x) \in \hat{R}]\}.$$

If $A = \{x\}$ we write $\hat{R}[x]$ instead of $\hat{R}[\{x\}]$. If X is a topological space a relation $\hat{U}$ on X is called a *neighborhood of the diagonal* iff $\hat{D}(X) \subseteq \hat{U}$ and $\hat{U}$ is open in the product space $X \times X$. The following lemmas are well known.

3.1 Lemma. *Let $\hat{R}, \hat{S}$ be relations on X then*

i) $\hat{R} \circ \hat{S} = \bigcup \{\hat{R}[x] \times \hat{S}^{-1}[x] \mid x \in X\}$,

ii) *For any $A, B \subseteq X$, $\hat{R}[A] \cap B \neq \emptyset$ iff $A \cap \hat{R}^{-1}[B] \neq \emptyset$.*

3.2 Lemma. *If X is a topological space and $\hat{U}, \hat{V}$ are neighborhoods of the diagonal, then*

i) *for any $A \subseteq X$, $\hat{U}[A]$ is open,*

ii) *$\hat{U}^{-1}$ and $\hat{U} \circ \hat{V}$ are neighborhoods of the diagonal.*

3.3 Definition. If $\mathscr{A}$ is a collection of subsets of X we define a *symmetric relation*

$$\hat{K}(\mathscr{A}) = \bigcup \{A \times A \mid A \in \mathscr{A}\}.$$

If $\hat{R}$ is any relation on X we define a *collection of subsets* $\mathscr{L}(\hat{R})$ of X by

$$\mathscr{L}(\hat{R}) = \{\hat{R}[x] \mid x \in X\}.$$

3.4 Theorem. *Let $\mathscr{A}, \mathscr{B}$ be collections of subsets of X and let $\hat{R}, \hat{S}$ be relations on X then*

i) $\hat{K}(\mathscr{A})[x] = \mathrm{St}(x, \mathscr{A})$ *and* $\hat{K}(\mathscr{A})[B] = \mathrm{St}(B, \mathscr{A})$ *for* $B \subseteq X$,

ii) $\mathscr{L}(\hat{K}(\mathscr{A})) = S(\mathscr{A})$,

iii) $\mathscr{A} \ll^* \mathscr{B}$ *iff* $\mathscr{L}(\hat{K}(\mathscr{A})) \ll \mathscr{B}$,

iv) $\hat{K}(\mathscr{L}(\hat{R})) = \hat{R} \circ \hat{R}^{-1}$.

v) $S(\mathscr{L}(\hat{R})) = \mathscr{L}(\hat{R} \circ \hat{R})^{-1}$,

vi) *if* $\hat{R} \subseteq \hat{S}$, *then* $\mathscr{L}(\hat{R}) \ll \mathscr{L}(\hat{S})$,

vii) $S(S(\mathscr{A})) = \mathscr{L}\Big(\hat{K}\big(\mathscr{L}(\hat{K}(\mathscr{A}))\big)\Big) = S^*(S(\mathscr{A}), \mathscr{A})$,

viii) *if* $\mathscr{A} \ll \mathscr{B}$, *then* $\hat{K}(\mathscr{A}) \subseteq \hat{K}(\mathscr{B})$.

Proof. i) $y \in \hat{K}(\mathscr{A})[x] \leftrightarrow (y, x) \in \hat{K}(\mathscr{A}) \leftrightarrow (y, x) \in A \times A$ for some $A \in \mathscr{A}$ $\leftrightarrow y, x \in A$ for some $A \in \mathscr{A} \leftrightarrow y \in \mathrm{St}(x, \mathscr{A})$,

$$\hat{K}(\mathscr{A})[B] = \bigcup \{\hat{K}(\mathscr{A})[x] \mid x \in B\} = \bigcup \{\mathrm{St}(x, \mathscr{A}) \mid x \in B\} = \mathrm{St}(B, \mathscr{A}).$$

ii) Follows immediately from i) and the definition of $S(\mathscr{A})$.

iii) Follows immediately from ii) and the definition of $\ll^*$.

iv) $\hat{K}(\mathscr{L}(\hat{R})) = \bigcup \{\hat{R}[x] \times \hat{R}[x] \mid x \in X\} = \hat{R} \circ \hat{R}^{-1}$ by 3.1.

v) By ii) $S(\mathscr{L}(\hat{R})) = \mathscr{L}\big(\hat{K}(\mathscr{L}(\hat{R}))\big) = \mathscr{L}(\hat{R} \circ \hat{R}^{-1})$ by iv).

vi) If $\hat{R} \subseteq \hat{S}$ then $\hat{R}[x] \subseteq \hat{S}[x]$ for each $x \in X$.

vii) $S(S(\mathscr{A})) = S\big(\mathscr{L}(\hat{K}(\mathscr{A}))\big) = \mathscr{L}\Big(\hat{K}\big(\mathscr{L}(\hat{K}(\mathscr{A}))\big)\Big)$. By iv) the latter is $\mathscr{L}\big(\hat{K}(\mathscr{A})\circ\hat{K}(\mathscr{A})\big)$. Let $x \in X$, $\hat{K}(\mathscr{A})\circ\hat{K}(\mathscr{A})[x] = \hat{K}(\mathscr{A})\big[\hat{K}(\mathscr{A})[x]\big] = \hat{K}(\mathscr{A})[\mathrm{St}(x, \mathscr{A})] = \mathrm{St}\big(\mathrm{St}(x, \mathscr{A}), \mathscr{A}\big)$ by i). Thus $\mathscr{L}\big(\hat{K}(\mathscr{A})\circ\hat{K}(\mathscr{A})\big) = S^*\big(S(\mathscr{A}), \mathscr{A}\big)$.

viii) For each $A \in \mathscr{A}$ there is a $B \in \mathscr{B}$ such that $A \subseteq B$ and hence $A \times A \subseteq B \times B$.

The following corollary is well known; however, we include it as an application of 3.4 as well as for its use later on.

3.5 Corollary. *Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be collections of subsets of X. If $\mathscr{A} \leqslant^* \mathscr{B}$ and $\mathscr{B} \leqslant^* \mathscr{C}$, then $\mathscr{A} \leqslant^{**} \mathscr{C}$.*

Proof. By 3.4 iii) $\mathscr{L}\big(\hat{K}(\mathscr{A})\big) \leqslant \mathscr{B}$ and $\mathscr{L}\big(\hat{K}(\mathscr{B})\big) \leqslant \mathscr{C}$. Thus by 3.4 viii) and vi) $\mathscr{L}\Big(\hat{K}\big(\mathscr{L}(\hat{K}(\mathscr{A}))\big)\Big) \leqslant \mathscr{L}\big(\hat{K}(\mathscr{B})\big)$. Hence by 3.4 vii) $S^*\big(S(\mathscr{A}), \mathscr{A}\big) \leqslant \mathscr{C}$. Since $\mathscr{A} \leqslant S(\mathscr{A})$, by 2.1 iv) $S^*(\mathscr{A}, \mathscr{A}) \leqslant \mathscr{C}$. Thus $S^*(\mathscr{A}) \leqslant \mathscr{C}$ and hence by definition $\mathscr{A} \leqslant^{**} \mathscr{C}$.

3.6 Corollary. *If $\mathscr{A}, \mathscr{B}$ are collections of subsets of X and $\mathscr{A} \leqslant^* \mathscr{B}$, then $\hat{K}(\mathscr{A})\circ\hat{K}(\mathscr{A}) \subseteq \hat{K}(\mathscr{B})$.*

Proof. $\hat{K}(\mathscr{A})\circ\hat{K}(\mathscr{A}) = \hat{K}\big(\mathscr{L}(\hat{K}(\mathscr{A}))\big) = \hat{K}\big(S(\mathscr{A})\big)$ by Theorem 3.4 iv) and ii). But $S(\mathscr{A}) \leqslant \mathscr{B}$ hence $\hat{K}\big(S(\mathscr{A})\big) \subseteq \hat{K}(\mathscr{B})$. Thus $\hat{K}(\mathscr{A})\circ\hat{K}(\mathscr{A}) \subseteq \hat{K}(\mathscr{B})$.

3.7 Corollary. *If $\hat{U}, \hat{V}, \hat{W}$ are relations on X where $\hat{U}, \hat{V}$ are symmetric and $\hat{U}\circ\hat{U} \subseteq \hat{V}$, $\hat{V}\circ\hat{V} \subseteq \hat{W}$, then $\mathscr{L}(\hat{U}) \leqslant^{**} \mathscr{L}(\hat{W})$.*

Proof. Since $\hat{U}\circ\hat{U} = \hat{K}\big(\mathscr{L}(\hat{U})\big)$ and $\hat{V}\circ\hat{V} = \hat{K}\big(\mathscr{L}(\hat{V})\big)$ we have $\hat{K}\big(\mathscr{L}(\hat{U})\big) \subseteq \hat{V}$ and $\hat{K}\big(\mathscr{L}(\hat{V})\big) \subseteq \hat{W}$. But

$$\mathscr{L}\Big(\hat{K}\Big(\mathscr{L}\big(\hat{K}(\mathscr{L}(\hat{U}))\big)\Big)\Big) \leqslant \mathscr{L}\big(\hat{K}(\mathscr{L}(\hat{V}))\big) \leqslant \mathscr{L}(\hat{W}).$$

Thus $S^*\big(S(\mathscr{L}(\hat{U})), \mathscr{L}(\hat{U})\big) \leqslant \mathscr{L}(\hat{W})$. Hence $S^*\big(\mathscr{L}(\hat{U})\big) \leqslant \mathscr{L}(\hat{W})$ and thus

$$\mathscr{L}(\hat{U}) \leqslant^{**} \mathscr{L}(\hat{W}).$$

4. The basic definitions. Let X be a topological space and $\mathfrak{m}$ an infinite cardinal. We now define a collection of properties. In all the definitions of this section $\mathscr{U}$ is any open covering of X with $|\mathscr{U}| \leqslant \mathfrak{m}$.

R$(\mathfrak{m})$: There is an open covering $\mathscr{V}$ of X such that $|\mathscr{V}| \leqslant \mathfrak{m}$ and $\mathscr{V}^- \leqslant \mathscr{U}$.

In the next three definitions we take $j = 1, 2, 3$, Q_1 to stand for the word "open", Q_2 to stand for the word "arbitrary" and Q_3 to stand for the word "closed". The notation N^+ is introduced to stand for the set of positive integers.

P$_j(\mathfrak{m})$: There is a covering $\mathscr{V} \leqslant \mathscr{U}$ of X such that $\mathscr{V}$ is Q$_j$ and $\mathscr{V}$ is locally finite.

$\sigma LP_j(\mathfrak{m})$: There is a covering $\mathscr{V} \leqslant \mathscr{U}$ of X such that $\mathscr{V}$ is Q_j and $\mathscr{V} = \bigcup\{\mathscr{V}_n | n \in N^+\}$ where each $\mathscr{V}_n$ is locally finite.

$\sigma DP_j(\mathfrak{m})$: There is a covering $\mathscr{V}$ of X such that $\mathscr{V}$ is Q_j, $\mathscr{V} = \bigcup\{\mathscr{V}_n | n \in N^+\}$ where each $\mathscr{V}_n$ is discrete, and $\mathscr{V} \leqslant \mathscr{U}$.

$FN_1(\mathfrak{m})$: There is an open covering $\mathscr{V}$ of X with $|\mathscr{V}| \leqslant \mathfrak{m}$ and $\mathscr{V} \leqslant^* \mathscr{U}$.

$FN_2(\mathfrak{m})$: There is an open covering $\mathscr{V}$ of X with $|\mathscr{V}| \leqslant \mathfrak{m}$ and $\mathscr{V} \leqslant^{**} \mathscr{U}$.

$FN_3(\mathfrak{m})$: There is a sequence $(\mathscr{V}_n | \ n \in N^+)$ of open coverings of X such that $\mathscr{V}_1 \leqslant^* \mathscr{U}$ and $\mathscr{V}_{n+1} \leqslant^* \mathscr{V}_n$ for all $n \in N^+$.

$FN_4(\mathfrak{m})$: There is a sequence $(\mathscr{V}_n | \ n \in N^+)$ of open coverings of X such that $\mathscr{V}_1 \leqslant^{**} \mathscr{U}$ and $\mathscr{V}_{n+1} \leqslant^{**} \mathscr{V}_n$ for all $n \in N^+$.

A neighborhood $\hat{U}$ of the diagonal of X is called an $\mathfrak{m}$-*neighborhood of the diagonal* iff there is an open covering $\mathscr{W}$ of X such that $|\mathscr{W}| \leqslant \mathfrak{m}$ and $\hat{U} = \hat{K}(\mathscr{W})$.

With $X, \mathfrak{m}, \mathscr{U}$ as above we define

$E_1(\mathfrak{m})$: There is an $\mathfrak{m}$-neighborhood $\hat{U}$ of the diagonal such that $\mathscr{L}(\hat{U}) \leqslant \mathscr{U}$.

$E_2(\mathfrak{m})$: There is a sequence $(\hat{U}_n | \ n \in N^+)$ of symmetric neighborhoods of the diagonal such that $\mathscr{L}(\hat{U}_1) \leqslant \mathscr{U}$ and $\hat{U}_{n+1} \circ \hat{U}_{n+1} \subseteq \hat{U}_n$ for each $n \in N^+$.

$CN_1(\mathfrak{m})$: If $\mathscr{A} = \{A_\alpha | \ \alpha \in \Gamma\}$ is an $\mathfrak{m}$-discrete collection of subsets of X there is an $\mathfrak{m}$-discrete open collection $\mathscr{V} = \{V_\alpha | \ \alpha \in \Gamma\}$ such that $A_\alpha \subseteq V_\alpha$ for each $\alpha \in \Gamma$.

$CN_2(\mathfrak{m})$: If $\mathscr{A} = \{A_\alpha | \ \alpha \in \Gamma\}$ is an $\mathfrak{m}$-locally finite collection of subsets of X there is an $\mathfrak{m}$-locally finite collection $\mathscr{V} = \{V_\alpha | \ \alpha \in \Gamma\}$ of open sets such that $A_\alpha \subseteq V_\alpha$ for each $\alpha \in \Gamma$.

If a space satisfies $P_1(\mathfrak{m})$ we call it $\mathfrak{m}$-*paracompact*. (Note that no separation assumptions are made.) When $FN_1(\mathfrak{m})$ is satisfied the space is called *fully* $\mathfrak{m}$-*normal*. (The term "$\mathfrak{m}$-fully normal" has already been used in [3] for a somewhat different concept.) When $E_1(\mathfrak{m})$ is satisfied the space is called $\mathfrak{m}$-*even*.

The covering $\mathscr{V}$ which appears in $\sigma LP_j(\mathfrak{m})$ (resp. $\sigma DP_j(\mathfrak{m})$) is usually called *σ-locally finite* (resp. *σ-discrete*)

We may make all of the above definitions absolute by dropping the "$\mathfrak{m}$" from their statements. In this case we designate them by R, $P_1, \ldots$ $\ldots, FN_1, \ldots$ etc. P_1 is usually known as paracompactness, (again no separation assumptions made) FN_1 is called full normality.

A space which satisfies $R(\mathfrak{m})$ is called $\mathfrak{m}$-*regular*. It should be observed that R is not precisely equivalent to regularity. We investigate this in somewhat more detail in the next paragraph.

When a space satisfies $\mathrm{CN}_1(\mathfrak{m})$ we call it $\mathfrak{m}$-*collectionwise normal.* A CN_1 space is usually called *collectionwise normal.*

5. $\mathrm{R}(\mathfrak{m})$. The following is classical:

5.1 LEMMA. *For any space X, if X is regular then X satisfies* R.

5.2 LEMMA [1]. *If X is a T_1 space and satisfies* $\mathrm{R}(\mathfrak{m})$ *for some* $\mathfrak{m}$, *then X is regular.*

Proof: Let $x \in X$ and let U be a neighborhood of x. Then $\mathscr{U} = \{U, X \sim \{x\}\}$ is an open covering of X such that $|\mathscr{U}| \leqslant \mathfrak{m}$. Thus there is an open covering $\mathscr{V}$ of X such that $\mathscr{V}^- \ll \mathscr{U}$. There is a $V \in \mathscr{V}$ such that $x \in V$ hence $V^- \subseteq U$.

5.3 COROLLARY. *If X is a T_1 space, then X satisfies* R *iff X is regular.*

5.4 THEOREM. *Let X be an* $\mathrm{R}(\mathfrak{m})$ *space and let Y be a subspace of X such that $Y = \bigcup \{C_a|\ a \in \Gamma\}$ where $|\Gamma| \leqslant \mathfrak{m}$ and C_a is closed in X for each a. Then Y is an* $\mathrm{R}(\mathfrak{m})$ *space.*

Proof. Let $\mathscr{A} = \{A_\lambda|\ \lambda \in \Lambda\}$ be a relative open covering of Y such that $|\Lambda| \leqslant \mathfrak{m}$. For each λ we can find an open set U_λ of X such that $A_\lambda = Y \cap U_\lambda$. Let $\mathscr{U} = \{U_\lambda|\ \lambda \in \Lambda\}$. For each a, $\mathscr{U}_a = \mathscr{U} \cup \{X \sim C_a\}$ is an open covering of X with $|\mathscr{U}_a| \leqslant \mathfrak{m}$. Hence we can find an open covering $\mathscr{V}_a = \{V^a_\lambda | \lambda \in \Lambda_a\}$ of X such that $|\Lambda_a| \leqslant \mathfrak{m}$ and $\mathscr{V}_a^- \ll \mathscr{U}_a$. We may assume $\Lambda_a \cap \Lambda_\beta = \emptyset$ for $a \neq \beta$. Let $\Lambda'_a = \{\lambda|\ V^a_\lambda \in \mathscr{V}_a, V^a_\lambda \cap C_a \neq \emptyset\}$ and $\mathscr{W}_a = \{V^a_\lambda|\ \lambda \in \Lambda'_a\}$. Clearly $C_a \subseteq \bigcup \mathscr{W}_a$ and $\mathscr{W}_a^- \ll \mathscr{U}$. If $\mathscr{W} = \bigcup \{\mathscr{W}_a|\ a \in \Gamma\}$, then $Y \subseteq \bigcup \mathscr{W}, \mathscr{W} \ll \mathscr{U}$ and $|\mathscr{W}| \leqslant \mathfrak{m}^2 = \mathfrak{m}$. Let $\mathscr{B} = \{V^a_\lambda \cap Y | V^a_\lambda \in \mathscr{W}_a, a \in \Gamma\}$ then clearly $Y = \bigcup \mathscr{B}$ and $\mathrm{Cl}_Y(\mathscr{B}) \ll \mathscr{A}$. (Here Cl_Y means the closure relative to Y.)

5.5 COROLLARY. *Any F_σ subset of an* $\mathrm{R}(\mathfrak{m})$ *space is* $\mathrm{R}(\mathfrak{m})$.

5.6 LEMMA. *If X is* $\mathrm{R}(\mathfrak{m})$ *and* $\mathrm{P}_1(\mathfrak{m})$, *then X is T_4.*

Proof. Let C be a closed subset of X and U an open subset such that $C \subseteq U$. Let $\mathscr{U} = \{U, X \sim C\}$ then $\mathscr{U}$ is an open covering of X such that $|\mathscr{U}| \leqslant \mathfrak{m}$. Since X is $R(\mathfrak{m})$ there is an open covering $\mathscr{V}$ of X such that $|\mathscr{V}| \leqslant \mathfrak{m}$ and $\mathscr{V}^- \ll \mathscr{U}$. Since X is $\mathrm{P}_1(\mathfrak{m})$ there is a locally finite open covering $\mathscr{W}$ of X such that $\mathscr{W} \ll \mathscr{V}$. Let $\mathscr{S} = \{W \mid W \in \mathscr{W}, W^- \cap C \neq \emptyset\}$. Clearly $C \subseteq \bigcup \mathscr{S}$. If $W \in \mathscr{S}$ there is a $V \in \mathscr{V}$ such that $W \subseteq V$ and hence $C \cap V^- \neq \emptyset$. But $\mathscr{V}^- \ll \mathscr{U}$ thus $V^- \subseteq U$. Hence $\bigcup \mathscr{S}^- \subseteq U$. But $\mathscr{S}$ is closure preserving, thus $\bigcup \mathscr{S}^- = (\bigcup \mathscr{S})^-$. Since $\bigcup \mathscr{S}$ is open we are done.

We recall the following well-known theorem [2].

5.7 THEOREM. *Let $\mathscr{U} = \{U_a|\ a \in \Gamma\}$ be a point finite open covering of a* T_4 *space X. Then there is an open covering $\mathscr{V} = \{V_a|\ a \in \Gamma\}$ such that $V_a^- \subseteq U_a$ for each $a \in \Gamma$.*

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[1] This lemma was brought to the author's attention by W. M. Fleischman.

From this the following corollary follows immediately.

5.8 COROLLARY. *If X is $\mathrm{P}_1(\mathfrak{m})$ and T_4, then X is $\mathrm{R}(\mathfrak{m})$.*

Hence we may state the following theorem.

5.9 THEOREM. *If X is $\mathrm{P}_1(\mathfrak{m})$, then X is T_4 iff X is $\mathrm{R}(\mathfrak{m})$.*

5.10 LEMMA. *Let X be a topological space and let $\mathscr{U} = \{U_a \mid a \in \Gamma\}$ be an open covering of X. If there exist open coverings $\mathscr{V}_1, \mathscr{V}_2$ of X such that $S^*(\mathscr{V}_1, \mathscr{V}_2) \leqslant \mathscr{U}$, there is an open covering $\mathscr{W} = \{W_a \mid a \in \Gamma\}$ such that $W_a^- \subseteq U_a$ for each $a \in \Gamma$.*

Proof. Suppose $\mathscr{V}_1 = \{V_\lambda \mid \lambda \in \Lambda\}$. For each $\lambda \in \Lambda$ choose $a(\lambda) \in \Gamma$ such that $\mathrm{St}(V_\lambda, \mathscr{V}_2) \subseteq U_{a(\lambda)}$. If $a \in \Gamma$ define $W_a = \bigcup \{V_\lambda \mid a(\lambda) = a\}$. Clearly $W_a \subseteq U_a$ for each a and $\mathscr{W} = \{W_a \mid a \in \Gamma\}$ is an open covering of X. If $x \in W_a^-$ there is a V' in $\mathscr{V}_2$ such that $x \in V'$. Hence $V' \cap W_a \neq \emptyset$. Thus $V' \subseteq \mathrm{St}(W_a, \mathscr{V}_2)$. But $\mathrm{St}(W_a, \mathscr{V}_2) = \bigcup \{\mathrm{St}(V_\lambda, \mathscr{V}_2) \mid a(\lambda) = a\} \subseteq U_a$. Hence $x \in U_a$ and thus $W_a^- \subseteq U_a$.

5.11 COROLLARY. *If X is $\mathrm{FN}_4(\mathfrak{m})$, then X is $\mathrm{R}(\mathfrak{m})$ and T_4.*

6. Some simple relationships. The following theorem is trivial.

6.1 THEOREM. *For any space X the following implications hold:*

i) $\mathrm{P}_1(\mathfrak{m}) \to \mathrm{P}_2(\mathfrak{m})$, $\sigma\mathrm{LP}_1(\mathfrak{m}) \to \sigma\mathrm{LP}_2(\mathfrak{m})$, $\sigma\mathrm{DP}_1(\mathfrak{m}) \to \sigma\mathrm{DP}_2(\mathfrak{m})$,

ii) $\mathrm{P}_3(\mathfrak{m}) \to \mathrm{P}_2(\mathfrak{m})$, $\sigma\mathrm{LP}_3(\mathfrak{m}) \to \sigma\mathrm{LP}_2(\mathfrak{m})$, $\sigma\mathrm{DP}_3(\mathfrak{m}) \to \sigma\mathrm{DP}_2(\mathfrak{m})$,

iii) $\mathrm{P}_j(\mathfrak{m}) \to \sigma\mathrm{LP}_j(\mathfrak{m})$, $j = 1, 2, 3$,

iv) $\sigma\mathrm{DP}_j(\mathfrak{m}) \to \sigma\mathrm{LP}_j(\mathfrak{m})$, $j = 1, 2, 3$.

6.2 THEOREM. *In any space X the following hold:*

i) $\mathrm{FN}_1(\mathfrak{m}) \leftrightarrow \mathrm{FN}_2(\mathfrak{m})$,

ii) $\mathrm{FN}_3(\mathfrak{m}) \leftrightarrow \mathrm{FN}_4(\mathfrak{m})$,

iii) $\mathrm{FN}_3(\mathfrak{m}) \leftrightarrow \mathrm{E}_2(\mathfrak{m})$,

iv) $\mathrm{FN}_1(\mathfrak{m}) \leftrightarrow \mathrm{E}_1(\mathfrak{m})$,

v) $\mathrm{FN}_1(\mathfrak{m}) \to \mathrm{FN}_3(\mathfrak{m})$,

vi) $\mathrm{E}_1(\mathfrak{m}) \to E_2(\mathfrak{m})$.

Proof. i) $\mathrm{FN}_2(\mathfrak{m}) \to \mathrm{FN}_1(\mathfrak{m})$ is trivial. To see $\mathrm{FN}_1(\mathfrak{m}) \to \mathrm{FN}_2(\mathfrak{m})$ let $\mathscr{U}$ be an open covering of X with $|\mathscr{U}| \leqslant \mathfrak{m}$. By hypothesis there is an open convering $\mathscr{W}$ of X such that $\mathscr{W} \leqslant^* \mathscr{U}$ and $|\mathscr{W}| \leqslant \mathfrak{m}$. Again by hypothesis there is an open covering $\mathscr{V}$ of X such that $\mathscr{V} \leqslant^* \mathscr{W}$ and $|\mathscr{V}| \leqslant \mathfrak{m}$. By 3.5 $\mathscr{V} \leqslant^{**} \mathscr{U}$.

ii) $\mathrm{FN}_4(\mathfrak{m}) \to \mathrm{FN}_3(\mathfrak{m})$ is trivial. To see $\mathrm{FN}_3(\mathfrak{m}) \to \mathrm{FN}_4(\mathfrak{m})$ let $\mathscr{U}$ be an open covering of X with $|\mathscr{U}| \leqslant \mathfrak{m}$. By hypothesis there is a sequence $(\mathscr{W}_n \mid n \in N^+)$ of open coverings of X such that $\mathscr{W}_1 \leqslant^* \mathscr{U}$ and $\mathscr{W}_{n+1} \leqslant^* \mathscr{W}_n$ for all $n \in N^+$. Let $\mathscr{V}_n = \mathscr{W}_{2n}$ for $n \in N^+$. Then $\mathscr{V}_1 \leqslant^* \mathscr{W}_1 \leqslant^* \mathscr{U}$ and hence by 3.5 $\mathscr{V}_1 \leqslant^{**} \mathscr{U}$. Further $\mathscr{V}_{n+1} \leqslant^* \mathscr{W}_{2n+1} \leqslant^* \mathscr{V}_n$ and hence $\mathscr{V}_{n+1} \leqslant^{**} \mathscr{V}_n$ for all $n \in N^+$.

iii) $\mathrm{FN}_3(\mathfrak{m}) \to \mathrm{E}_2(\mathfrak{m})$: If $\mathscr{U}$ is an open covering of X with $|\mathscr{U}| \leqslant \mathfrak{m}$ we can find a sequence $(\mathscr{V}_n|\ n \in N^+)$ such that $\mathscr{V}_1 \ll^* \mathscr{U}$ and $\mathscr{V}_{n+1} \ll^* \mathscr{V}_n$ for all $n \in N^+$. Let $\hat{U}_n = \hat{K}(\mathscr{V}_n)$. By 3.4 iii) $\mathscr{L}(\hat{U}_1) \ll \mathscr{U}$. By 3.6 $\hat{U}_{n+1} \circ \hat{U}_{n+1} \subseteq \hat{U}_n$ for all $n \in N^+$.

$\mathrm{E}_2(\mathfrak{m}) \to \mathrm{FN}_3(\mathfrak{m})$: Let $\mathscr{U}$ be an open covering of X such that $|\mathscr{U}| \leqslant \mathfrak{m}$. Let $(\hat{U}_n|\ n \in N^+)$ be the sequence of symmetric neighborhoods of the diagonal provided by $\mathrm{E}_2(\mathfrak{m})$. Let $\mathscr{V}_n = \mathscr{L}(\hat{U}_{2n+1})$ for $n \in N^+$. By 3.7, $\mathscr{V}_{n+1} \ll^{**} \mathscr{V}_n$ for all $n \in N^+$. Again by 3.7 $\mathscr{L}(\hat{U}_3) \ll^{**} \mathscr{L}(\hat{U}_1)$. Since $\mathscr{L}(\hat{U}_1) \ll \mathscr{U}$ we see that $\mathscr{V}_1 \ll^{**} \mathscr{U}$. Thus $\mathrm{E}_2(\mathfrak{m}) \to \mathrm{FN}_4(\mathfrak{m})$. By ii) $\mathrm{FN}_4(\mathfrak{m}) \to \mathrm{FN}_3(\mathfrak{m})$ and hence the result.

iv) $\mathrm{FN}_1(\mathfrak{m}) \to \mathrm{E}_1(\mathfrak{m})$: Let $\mathscr{U}$ be an open covering of X with $|\mathscr{U}| \leqslant \mathfrak{m}$ and let $\mathscr{V}$ be the open covering provided by $\mathrm{FN}_1(\mathfrak{m})$. Let $\hat{U} = \hat{K}(\mathscr{V})$ then $\hat{U}$ is an $\mathfrak{m}$ — neighborhood of the diagonal. By 3.4 iii) $\mathscr{L}(\hat{U}) \ll \mathscr{U}$.

$\mathrm{E}_1(\mathfrak{m}) \to \mathrm{FN}_1(\mathfrak{m})$: Let $\mathscr{U}$ be an open covering of X with $|\mathscr{U}| \leqslant \mathfrak{m}$ and let $\hat{U}$ be the $\mathfrak{m}$ — neighborhood of the diagonal provided by $\mathrm{E}_1(\mathfrak{m})$. Then $\hat{U} = \hat{K}(\mathscr{V})$ where $\mathscr{V}$ is some open covering of X with $|\mathscr{V}| \leqslant \mathfrak{m}$. By 3.4 iii) $S(\mathscr{V}) \ll \mathscr{U}$ and hence $\mathscr{V} \ll^* \mathscr{U}$.

v) $\mathrm{FN}_1(\mathfrak{m}) \to \mathrm{FN}_3(\mathfrak{m})$. This follows immediately by iteration.

vi) $\mathrm{E}_1(\mathfrak{m}) \to \mathrm{E}_2(\mathfrak{m})$. Follows from the above implications.

6.3 THEOREM. *For any topological space X, if X is* $\mathrm{R}(\mathfrak{m})$, *then the following hold*:

i) $\mathrm{P}_2(\mathfrak{m}) \to \mathrm{P}_3(\mathfrak{m})$,

ii) $\sigma\mathrm{LP}_2(\mathfrak{m}) \to \sigma\mathrm{LP}_3(\mathfrak{m})$,

iii) $\sigma\mathrm{DP}_2(\mathfrak{m}) \to \sigma\mathrm{DP}_3(\mathfrak{m})$.

Proof. The arguments establishing all three implications are similar. As an example we give the argument for ii). Let $\mathscr{U}$ be an open covering of X such that $|\mathscr{U}| \leqslant \mathfrak{m}$. By the $\mathrm{R}(\mathfrak{m})$ property there is an open covering $\mathscr{V}$ of X such that $|\mathscr{V}| \leqslant \mathfrak{m}$ and $\mathscr{V}^- \ll \mathscr{U}$. If X is $\sigma\mathrm{LP}_2(\mathfrak{m})$ there is a covering $\mathscr{W}$ of X such that $\mathscr{W} \ll \mathscr{V}$ and we may write $\mathscr{W} = \bigcup \{\mathscr{W}_n|\ n \in N^+\}$ where $\mathscr{W}_n$ is locally finite for each $n \in N^+$. $\mathscr{W}^- \ll \mathscr{V}^-$ and hence $\mathscr{W}^- \ll \mathscr{U}$. Since $\mathscr{W}_n^-$ is locally finite for each $n \in N^+$ it follows that $\mathscr{W}^-$ is the desired σ-locally finite closed covering required for $\sigma\mathrm{LP}_3(\mathfrak{m})$.

6.4 THEOREM. *In any topological space X the following hold.*

i) $\mathrm{P}_2(\mathfrak{m}) \wedge \mathrm{CN}_2(\mathfrak{m}) \to \mathrm{P}_1(\mathfrak{m})$,

ii) $\sigma\mathrm{LP}_2(\mathfrak{m}) \wedge \mathrm{CN}_2(\mathfrak{m}) \to \sigma\mathrm{LP}_1(\mathfrak{m})$,

iii) $\sigma\mathrm{DP}_2(\mathfrak{m}) \wedge \mathrm{CN}_1(\mathfrak{m}) \to \sigma\mathrm{DP}_1(\mathfrak{m})$.

Proof. The proofs of all three are similar and we present only one.

iii) Let $\mathscr{U} = \{U_\alpha |\ \alpha \in \Gamma\}$ be an open covering of X with $|\Gamma| \leqslant \mathfrak{m}$. If X is $\sigma\mathrm{DP}_2(\mathfrak{m})$ there is a covering $\mathscr{W}$ of X such that $\mathscr{W} \leqslant \mathscr{U}$ and $\mathscr{W} = \bigcup\{\mathscr{W}_n |\ n \in N^+\}$ where each $\mathscr{W}_n$ is discrete. Since $\mathscr{W}_n \leqslant \mathscr{U}$, by 2.3 we may replace $\mathscr{W}_n$ by an associated precise refinement $\mathscr{W}'_n = \{W_{n\alpha} |\ \alpha \in \Gamma\}$ which is discrete. Recall that $W_{n\alpha} \subseteq U_\alpha$ for each $\alpha \in \Gamma$ and $\bigcup \mathscr{W}_n = \bigcup \mathscr{W}'_n$. Now $\mathscr{W}'_n$ is $\mathfrak{m}$-discrete by 2.2. The $\mathrm{CN}_1(\mathfrak{m})$ property implies that there is for each $n \in N^+$ an open collection $\mathscr{S}_n = \{S_{n\alpha} |\ \alpha \in \Gamma\}$ which is discrete and such that $W_{n\alpha} \subseteq S_{n\alpha}$ for each $\alpha \in \Gamma$. Let $V_{n\alpha} = S_{n\alpha} \cap U_\alpha$ for each $\alpha \in \Gamma$; if $\mathscr{V}_n = \{V_{n\alpha} |\ \alpha \in \Gamma\}$, then $\mathscr{V}_n$ is discrete, open, $\mathscr{V}_n \leqslant \mathscr{U}$ and $\bigcup \mathscr{W}'_n \subseteq \bigcup \mathscr{V}_n$. Thus if $\mathscr{V} = \bigcup\{\mathscr{V}_n |\ n \in N^+\}$, $\mathscr{V}$ is the desired σ-discrete open refinement of $\mathscr{U}$ which covers X.

6.5 Lemma. *Let $\mathscr{A} = \{A_\alpha |\ \alpha \in \Gamma\}$ be a collection of subsets of X and suppose $\mathscr{U} = \{U_\lambda |\ \lambda \in \Lambda\}$ is an open covering of X. If $\hat{U}_1$, $\hat{U}_2$ are symmetric neighborhoods of the diagonal such that $\hat{U}_2 \circ \hat{U}_2 \subseteq \hat{U}_1$ and $\mathscr{L}(\hat{U}_1) \leqslant \mathscr{U}$ and if $\mathscr{W} = \{W_\alpha |\ \alpha \in \Gamma\}$ where $W_\alpha = \hat{U}_2[A_\alpha]$, then for each $x \in X$ there is a $\lambda(x) \in \Lambda$ such that*

$$|c(\hat{U}_2[x], \Gamma, \mathscr{W})| \leqslant |c(U_{\lambda(x)}, \Gamma, \mathscr{A})|.$$

Proof. Since $\mathscr{L}(\hat{U}_1) \leqslant \mathscr{U}$, for each $x \in X$ there is a $\lambda(x) \in \Lambda$ such that $\hat{U}_1[x] \subseteq U_{\lambda(x)}$ and thus $\hat{U}_2 \circ \hat{U}_2[x] \subseteq U_{\lambda(x)}$. Hence

$$|c(\hat{U}_2 \circ \hat{U}_2[x], \Gamma, \mathscr{A})| \leqslant |c(U_{\lambda(x)}, \Gamma, \mathscr{A})|.$$

By 3.1 ii),

$$\hat{U}_2 \circ \hat{U}_2[x] \cap A_\alpha \neq \emptyset \quad \text{iff} \quad \hat{U}_2[x] \cap \hat{U}_2[A_\alpha] \neq \emptyset$$

and thus

$$|c(\hat{U}_2[x], \Gamma, \mathscr{W})| = |c(\hat{U}_2 \circ \hat{U}_2[x], \Gamma, \mathscr{A})| \leqslant |c(U_{\lambda(x)}, \Gamma, \mathscr{A})|.$$

6.6 Theorem. *If X is $\mathrm{E}_2(\mathfrak{m})$, then X is both $\mathrm{CN}_1(\mathfrak{m})$ and $\mathrm{CN}_2(\mathfrak{m})$.*

Proof. Let $\mathscr{A} = \{A_\alpha |\ \alpha \in \Gamma\}$ be a collection of subsets which is $\mathfrak{m}$-discrete (resp. $\mathfrak{m}$-locally finite). By 2.2, $|\Gamma| \leqslant \mathfrak{m}$. From the definition of $\mathfrak{m}$-discreteness (resp. $\mathfrak{m}$-local finiteness) there is an open covering $\mathscr{U} = \{U_\lambda |\ \lambda \in \Lambda\}$ such that $|c(U_\lambda, \Gamma, \mathscr{A})| \leqslant 1$ (resp. $c(U_\lambda, \Gamma, \mathscr{A})$ is finite), for each $\lambda \in \Lambda$ and moreover, $|\Lambda| \leqslant \mathfrak{m}$. Since X is $\mathrm{E}_2(\mathfrak{m})$, we can find symmetric neighborhoods $\hat{U}_1$, $\hat{U}_2$ of the diagonal such that $\hat{U}_2 \circ \hat{U}_2 \subseteq \hat{U}_1$ and $\mathscr{L}(\hat{U}_1) \leqslant \mathscr{U}$. Hence by 6.5 for each $x \in X$ there is a $\lambda(x) \in \Lambda$ such that $|c(\hat{U}_2[x], \Gamma, \mathscr{W})| \leqslant |c(U_{\lambda(x)}, \Gamma, \mathscr{A})|$ (here $\mathscr{W}$ is defined as in 6.5). Thus $|c(\hat{U}_2[x], \Gamma, \mathscr{W})| \leqslant 1$ (resp. $c(\hat{U}_2[x], \Gamma, \mathscr{W})$ is finite) for each $x \in X$. Since $|\Gamma| \leqslant \mathfrak{m}$, by 2.2 $\mathscr{W}$ is $\mathfrak{m}$-discrete (resp. $\mathfrak{m}$-locally finite). Since $A_\alpha \subseteq W_\alpha$ and W_α is open for each $\alpha \in \Gamma$, X is $\mathrm{CN}_1(\mathfrak{m})$ (resp. $\mathrm{CN}_2(\mathfrak{m})$).

6.7 Theorem. *If X is $\sigma\mathrm{LP}_1(\mathfrak{m})$, then X is $\mathrm{P}_2(\mathfrak{m})$.* (Compare [1]).

Proof. Let $\mathscr{U}$ be an open covering of X such that $|\mathscr{U}| \leqslant \mathfrak{m}$. By hypothesis there is an open covering $\mathscr{W}$ of X such that $\mathscr{W} \leqslant \mathscr{U}$ and $\mathscr{W} = \bigcup\{\mathscr{W}_n |\ n \in N^+\}$ where $\mathscr{W}_n$ is locally finite for each $n \in N^+$. Now, for each

$n \in N^+$ let $Y_n = \bigcup\{\bigcup \mathscr{W}_j|\ 1 \leqslant j \leqslant n\}$. If $\mathscr{W}_n = \{W_{n\alpha}|\ \alpha \in \Gamma_n\}$, define for each $\alpha \in \Gamma_n$, $V_{n\alpha} = W_{n\alpha} \sim Y_{n-1}$ if $n > 1$ and $V_{1\alpha} = W_{1\alpha}$. Let $\mathscr{V} = \{V_{n\alpha}|\ n \in N^+, \alpha \in \Gamma\}$ where $\Gamma = \bigcup\{\Gamma_n|\ n \in N^+\}$. It is clear that $\mathscr{V} \ll \mathscr{U}$. Now let $x \in X$, since $\mathscr{W}$ is a covering of X there is an n such that $x \in Y_n$. For each j, $1 \leqslant j \leqslant n$, there is a neighborhood K_j of x such that $c(K_j, \Gamma_j, \mathscr{W}_j)$ is finite. Let $K(x) = K_1 \cap \ldots \cap K_n \cap Y_n$, then $K(x)$ is a neighborhood of x. Since $Y_n \cap V_{p\alpha} = \emptyset$ for $p > n$ we see that

$$\big|c\big(K(x), \Gamma, \mathscr{V}\big)\big| \leqslant |c(K_1, \Gamma_1, \mathscr{W}_1)| + \ldots + |c(K_n, \Gamma_n, \mathscr{W}_n)|.$$

Thus $\mathscr{V}$ is locally finite.

We end this section with an easy lemma which we leave to the reader

6.8 Lemma. *If X is $\mathrm{P}_3(\mathfrak{m})$, then X is T_4.*

7. The main theorems.

7.1 Theorem. *If X is $\mathrm{P}_3(\mathfrak{m})$, then X is $\mathrm{P}_1(\mathfrak{m})$.*

Proof. Let $\mathscr{U} = \{U_\alpha|\ \alpha \in \Gamma\}$ be an open covering of X with $|\Gamma| \leqslant \mathfrak{m}$. By the $\mathrm{P}_3(\mathfrak{m})$ property, 6.8 and 2.3 there is a locally finite closed covering $\mathscr{K} = \{K_\alpha|\ \alpha \in \Gamma\}$ of X such that $K_\alpha \subseteq U_\alpha$ for each $\alpha \in \Gamma$. Thus there is an open covering $\mathscr{W} = \{W_\lambda|\ \lambda \in \Lambda\}$ of X such that $c(W_\lambda, \Gamma, \mathscr{K})$ is finite for each $\lambda \in \Lambda$. Let $\Sigma = \{c(W_\lambda, \Gamma, \mathscr{K})|\ \lambda \in \Lambda\}$ then $\Sigma \subseteq \Gamma^F$ and thus $|\Sigma| \leqslant |\Gamma^F| \leqslant \mathfrak{m}$. For each $\sigma \in \Sigma$ let $W'_\sigma = \bigcup\{W_\lambda|\ c(W_\lambda, \Gamma, \mathscr{K}) = \sigma\}$ then $\mathscr{W}' = \{W'_\sigma|\ \sigma \in \Sigma\}$ is an open covering of X. Further we see that $c(W'_\sigma, \Gamma, \mathscr{K}) = \sigma$ for each $\sigma \in \Sigma$. Since $|\Sigma| \leqslant \mathfrak{m}$, by the $\mathrm{P}_3(\mathfrak{m})$ property and 2.3 there is a locally finite closed covering $\mathscr{C} = \{C_\sigma|\ \sigma \in \Sigma\}$ such that $C_\sigma \subseteq W'_\sigma$ for each $\sigma \in \Sigma$. For each $\alpha \in \Gamma$ let $K'_\alpha = X \sim \bigcup\{C_\sigma|\ \sigma \in \Sigma \sim c(K_\alpha, \Sigma, \mathscr{C})\}$. Since $\mathscr{C}$ is closure preserving and covers X, K'_α is open and $K_\alpha \subseteq K'_\alpha$ for each $\alpha \in \Gamma$. Now observe that for each $\sigma \in \Sigma$ and $\alpha \in \Gamma$, $C_\sigma \cap K_\alpha = \emptyset$ iff $C_\sigma \cap K'_\alpha = \emptyset$. For, if $C_\sigma \cap K_\alpha = \emptyset$, $\sigma \in \Sigma \sim c(K_\alpha, \Sigma, \mathscr{C})$ and hence $C_\sigma \cap K'_\alpha = \emptyset$. Thus if $\mathscr{K}' = \{K'_\alpha|\ \alpha \in \Gamma\}$, for each $\sigma \in \Sigma$, $c(C_\sigma, \Gamma, \mathscr{K}') = c(C_\sigma, \Gamma, \mathscr{K}) \subseteq c(W'_\sigma, \Gamma, \mathscr{K}) = \sigma$. Hence $c(C_\sigma, \Gamma, \mathscr{K}')$ is finite for each $\sigma \in \Sigma$. Now for each $\alpha \in \Gamma$ let $V_\alpha = K'_\alpha \cap U_\alpha$ and $\mathscr{V} = \{V_\alpha|\ \alpha \in \Gamma\}$. Then $\mathscr{V}$ is an open covering of X and $\mathscr{V} \ll \mathscr{U}$. For each $\sigma \in \Sigma$ we see that $c(C_\sigma, \Gamma, \mathscr{V}) \subseteq c(C_\sigma, \Gamma, \mathscr{K}')$ and thus $c(C_\sigma, \Gamma, \mathscr{V})$ is finite. Thus we have $|c(C_\sigma, \Gamma, \mathscr{V})| < \aleph_0$ for each $\sigma \in \Sigma$. Since $\mathscr{C}$ is locally finite there is an open covering $\mathscr{S} = \{S_\delta|\ \delta \in \Delta\}$ such that $|c(S_\delta, \Sigma, \mathscr{C})| < \aleph_0$ for each $\delta \in \Delta$. By 2.4 $|c(S_\delta, \Gamma, \mathscr{V})| < \aleph_0$ for each $\delta \in \Delta$; hence $\mathscr{V}$ is locally finite.

7.2 Theorem. *In any topological space X the following are equivalent:*

i) $\mathrm{P}_1(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$,

ii) $\mathrm{P}_2(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$,

iii) $\mathrm{P}_3(\mathfrak{m})$,

iv) $\sigma\mathrm{LP}_1(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$.

Proof. i) $\to$ ii) by 6.1. ii) $\to$ iii) by 6.3. iii) $\to$ i) by 7.1, 6.8 and 5.9. i) $\to$ iv) by 6.1. iv) $\to$ ii) by 6.7.

7.3 THEOREM. *Let* $\mathscr{U} = \{U_\alpha |\ \alpha \in \Gamma\}$ *be an open covering of a space* X *with* $|\Gamma| \leqslant \mathfrak{m}$ *and suppose* $\mathscr{K} = \{K_\alpha |\ \alpha \in \Gamma\}$ *to be a point finite, closure preserving, closed covering such that* $K_\alpha \subseteq U_\alpha$ *for each* $\alpha \in \Gamma$. *Then there is an open covering* $\mathscr{V} = \{V_\lambda |\ \lambda \in \Lambda\}$ *such that* $|\Lambda| \leqslant \mathfrak{m}$ *and* $\mathscr{V} \leqslant {}^*\mathscr{U}$.

Proof. Let $x \in X$, since $\mathscr{K}$ is point finite $c(x, \Gamma, \mathscr{K})$ is finite for each $x \in X$. Let $\Lambda = \{c(x, \Gamma, \mathscr{K}) | x \in X\}$. Since $\Lambda \subseteq \Gamma^F$, $|\Lambda| \leqslant \mathfrak{m}$. For each $\lambda \in \Lambda$ let

$$V_\lambda = \bigcap \{U_\alpha |\ \alpha \in \lambda\} \cap (X \sim \bigcup \{K_\alpha |\ \alpha \notin \lambda\}).$$

Observe that if $\lambda = c(x, \Gamma, \mathscr{K})$, $x \in V_\lambda$. Since $\mathscr{K}$ is closure preserving, V_λ is open and thus $\mathscr{V} = \{V_\lambda |\ \lambda \in \Lambda\}$ is an open covering of X. Let $y \in X$ and suppose $y \in V_\lambda$; then $c(y, \Gamma, \mathscr{K}) \subseteq \lambda$. For, if $K_\alpha \cap V_\lambda \neq \emptyset$, then $\alpha \in \lambda$. Thus if $\beta \in c(y, \Gamma, \mathscr{K})$ and $y \in V_\lambda$ it follows that $\beta \in \lambda$ and hence $V_\lambda \subseteq U_\beta$. Thus $\mathrm{St}(y, \mathscr{V}) = \bigcup \{V_\lambda |\ y \in V_\lambda\} \subseteq U_\beta$ for each $\beta \in c(y, \Gamma, \mathscr{K})$. This shows that $\mathscr{V} \leqslant^* \mathscr{U}$.

7.4 COROLLARY. *Let* $\mathscr{U}$ *be an open covering of* X *with* $|\mathscr{U}| \leqslant \mathfrak{m}$ *and let* $\mathscr{K}$ *be a locally finite closed covering of* X *such that* $\mathscr{K} \leqslant \mathscr{U}$. *Then there is an open covering* $\mathscr{V}$ *of* X *such that* $\mathscr{V} \leqslant^* \mathscr{U}$ *and* $|\mathscr{V}| \leqslant \mathfrak{m}$.

Proof. Let $\mathscr{U} = \{U_\alpha |\ \alpha \in \Gamma\}$ with $|\Gamma| \leqslant \mathfrak{m}$. By 2.2, applied to $\mathscr{K}$, there is a locally finite closed covering $\mathscr{C} = \{C_\alpha |\ \alpha \in \Gamma\}$ such that $C_\alpha \subseteq U_\alpha$ for each $\alpha \in \Gamma$. Our conclusion now follows from 7.2.

7.5 COROLLARY. *If* X *is* $\mathrm{P}_3(\mathfrak{m})$, *then* X *is* $\mathrm{FN}_1(\mathfrak{m})$.

The proof of the following theorem is given essentially in [1] so we shall only outline it here.

7.6 THEOREM. *If* X *is* $\mathrm{E}_2(\mathfrak{m})$, *then* X *is* $\sigma\mathrm{DP}_2(\mathfrak{m})$.

Proof. Let $\mathscr{U}$ be an open covering of X such that $|\mathscr{U}| \leqslant \mathfrak{m}$. By the $\mathrm{E}_2(\mathfrak{m})$ property there is a sequence $(\hat{U}_n |\ n \in N^+)$ of symmetric neighborhoods of the diagonal such that $\mathscr{L}(\hat{U}_1) \leqslant \mathscr{U}$ and $\hat{U}_{n+1} \circ \hat{U}_{n+1} \subseteq \hat{U}_n$ for each $n \in N^+$. Define neighborhoods of the diagonal $\hat{W}_n$, $n \in N^+$, as follows: $\hat{W}_1 = \hat{U}_2$, $\hat{W}_{n+1} = \hat{U}_{n+2} \circ \hat{W}_n$ for $n \geqslant 1$. It is easily seen that $\hat{W}_n \subseteq \hat{U}_1$ and hence $\mathscr{L}(\hat{W}_n) \leqslant \mathscr{U}$ for each $n \in N^+$. Now let $<$ be a well ordering of the points of X. For each $n \in N^+$ and $x \in X$ let $\hat{W}_n^*(x) = \hat{W}_n[x] \sim \bigcup \{\hat{W}_{n+1}[y] |\ y < x\}$. If $\mathscr{W}_n = \{W_n^*(x) |\ x \in X\}$ and $\mathscr{W} = \bigcup \{\mathscr{W}_n |\ n \in N^+\}$ it is clear that $\mathscr{W} \leqslant \mathscr{U}$. To see that $\mathscr{W}$ is a covering of X and that $\mathscr{W}_n$ is discrete for each $n \in N^+$ use the argument in [1; 33 chap. v].

7.7 THEOREM. *If* X *is a topological space, then the following are equivalent* on X:

(1) $\mathrm{P}_1(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$, (2) $\mathrm{P}_2(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$, (3) $\mathrm{P}_3(\mathfrak{m})$, (4) $\sigma\mathrm{LP}_1(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$, (5) $\mathrm{FN}_1(\mathfrak{m})$, (6) $\mathrm{FN}_2(\mathfrak{m})$, (7) $\mathrm{FN}_3(\mathfrak{m})$, (8) $\mathrm{FN}_4(\mathfrak{m})$, (9) $\mathrm{E}_1(\mathfrak{m})$, (10) $\mathrm{E}_2(\mathfrak{m})$, (11) $\sigma\mathrm{LP}_2(\mathfrak{m}) \wedge \mathrm{CN}_2(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$, (12) $\sigma\mathrm{LP}_3(\mathfrak{m}) \wedge \mathrm{CN}_2(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$, (13) $\sigma\mathrm{DP}_1(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$, (14) $\sigma\mathrm{DP}_2(\mathfrak{m}) \wedge \mathrm{CN}_1(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$, (15) $\sigma\mathrm{DP}_2(\mathfrak{m}) \wedge$

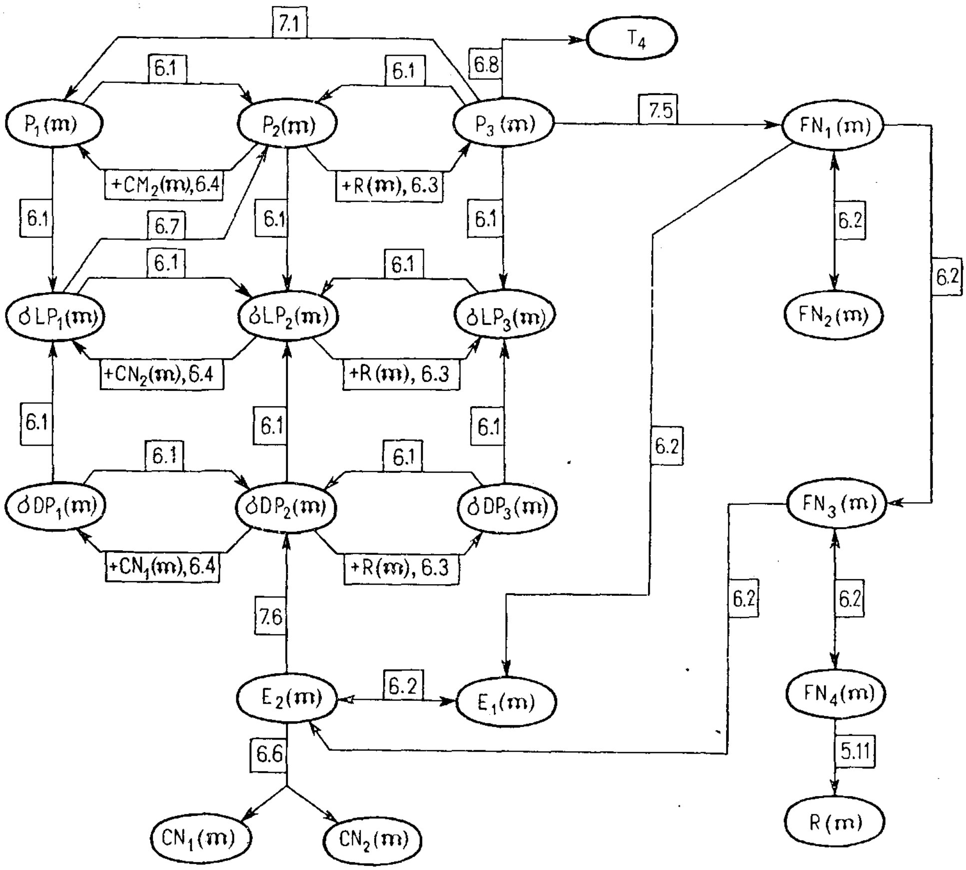

Fig. 1

$\wedge\ \mathrm{CN}_2(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$, (16) $\sigma\mathrm{DP}_3(\mathfrak{m}) \wedge \mathrm{CN}_1(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$, (17) $\sigma\mathrm{DP}_3(\mathfrak{m}) \wedge$ $\wedge\ \mathrm{CN}_2(\mathfrak{m}) \wedge \mathrm{R}(\mathfrak{m})$.

The proof is easily seen from the diagram of implications given in Fig. 1.

8. Miscellaneous results. In this paragraph we add results and remarks amplifying some of the earlier parts of the paper.

The general nature of the $\mathrm{R}(\mathfrak{m})$ property is not clear to the author. However, in case $\mathfrak{m} = \aleph_0$, we have the following theorem.

8.1 THEOREM. *If X is a T_4 space, then the following are equivalent*:

i) $\mathrm{R}(\aleph_0)$,

ii) $\mathrm{P}_1(\aleph_0)$.

Proof ii) $\to$ i) follows immediately from 5.9.

i) $\to$ ii) Let $\mathscr{U} = \{U_n \mid n \in N^+\}$ be an open covering of X. By the $\mathrm{R}(\aleph_0)$ property we can find an open covering $\mathscr{V} = \{V_n \mid n \in N^+\}$ such that

$\mathscr{V}^{-} \leqslant \mathscr{U}$. For each $n \in N^{+}$ choose $m(n) \in N^{+}$ such that $V_n^{-} \subseteq U_{m(n)}$ and let $W_n = U_{m(n)}$. Then $\mathscr{W} = \{W_n | \ n \in N^{+}\}$ is an open covering of X such that $V_n^{-} \subseteq W_n$ for each $n \in N^{+}$. By Theorem 3 of [4] there is a locally finite open covering $\mathscr{S}$ of X such that $\mathscr{S} \leqslant \mathscr{W}$. Since $\mathscr{W} \leqslant \mathscr{U}$, it follows that $\mathscr{S} \leqslant \mathscr{U}$ and hence X is $P_1(\aleph_0)$. (Note that the T_1 hypothesis in Theorem 3 of [4] is superfluous.)

Observe, that it follows immediately from the definitions, that all the properties defined in paragraph 4, except $R(\mathfrak{m})$, $FN_1(\mathfrak{m})$ and $E_1(\mathfrak{m})$ are hereditary in the following sense: if $\mathfrak{n}$ is an infinite cardinal such that $\mathfrak{n} \leqslant \mathfrak{m}$ and the property holds for $\mathfrak{m}$ then it holds for $\mathfrak{n}$. In fact, however, it follows from 7.7 that $FN_1(\mathfrak{m})$ and $E_1(\mathfrak{m})$ are hereditary in the above sense. This implies for example that a space is fully normal iff it is $FN_1(\mathfrak{m})$ for all $\mathfrak{m}$. The point here is that in a fully normal space the star refining covering can always be taken so that its cardinality is not greater than the covering which it refines. It is clear that $R(\mathfrak{m})$ is hereditary if X is $P_1(\mathfrak{m})$. We do not know if $R(\mathfrak{m})$ is hereditary in general.

We make one final observation. One may also define a property $P_4(\mathfrak{m})$ for spaces as follows:

$P_4(\mathfrak{m})$: For any open covering $\mathscr{U}$ of X with $|\mathscr{U}| \leqslant \mathfrak{m}$ there is a closed, point finite, closure preserving covering $\mathscr{V}$ of X such that $\mathscr{V} \leqslant \mathscr{U}$.

8.2 THEOREM. *In a space X, $P_4(\mathfrak{m})$ is equivalent to* $P_3(\mathfrak{m})$.

Proof. It is clear that $P_3(\mathfrak{m})$ implies $P_4(\mathfrak{m})$. Now suppose that $P_4(\mathfrak{m})$ holds. Let $\mathscr{U} = \{U_\alpha | \ \alpha \in \Gamma\}$ be an open covering of X with $|\Gamma| \leqslant \mathfrak{m}$. By hypothesis there is a closed, point finite, closure preserving covering $\mathscr{V}$ of X with $\mathscr{V} \leqslant \mathscr{U}$. By 2.3 there is a closed, point finite, closure preserving covering $\mathscr{K} = \{K_\alpha | \ \alpha \in \Gamma\}$ such that $K_\alpha \subseteq U_\alpha$ for each $\alpha \in \Gamma$. By 7.3 there is an open covering $\mathscr{W}$ of X with $|\mathscr{W}| \leqslant \mathfrak{m}$ and $\mathscr{W} \leqslant^{*} \mathscr{U}$. Thus $P_4(\mathfrak{m})$ implies $FN_1(\mathfrak{m})$. By 7.7 $FN_1(\mathfrak{m})$ is equivalent to $P_3(\mathfrak{m})$. Thus $P_4(\mathfrak{m})$ is equivalent to $P_3(\mathfrak{m})$.

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