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On uniformly convexifiable and B-convex Musielak–Orlicz spaces

Abstract. In this paper it is proved that every Musielak–Orlicz space is uniformly convexifiable and B-convex iff it is reflexive. This is a generalization of results of V. Akimovič [1] and of M. Denker and R. Kombrink [3].

0. Introduction. (T, Σ, μ) is a measure space with a non-negative, σ-finite and complete measure μ, $\Sigma_0 = \{A \in \Sigma\colon \mu(A) = 0\}$, $R_+ = [0, \infty)$, $R_+^e = [0, \infty]$. An Orlicz function Φ with parameter (from T) is a function defined on $T \times R_+$ with values in R_+^e, convex on R_+, continuous at zero, with $\Phi(t, 0) = 0$ and $\Phi(t, u) \to \infty$ as $u \to \infty$ for μ-almost every (μ-a.e.) $t \in T$ and μ-measurable on T for every $u \geqslant 0$. Henceforth, we write shortly "Orlicz function" instead of "Orlicz function with parameter". In the case of a purely atomic measure space (T, Σ, μ), where $T = N =$ the set of all positive integers, we write $\Phi_n(u)$ instead of $\Phi(n, u)$. We assume without loss of generality that $\mu(\{n\}) = 1$ for $n = 1, 2, \ldots$

The Musielak–Orlicz space L_Φ^μ [1] generated by an Orlicz function Φ and by a measure μ is the set of all real (complex)-valued and μ-measurable functions x defined on T (with usual identification $x = y$ iff $x(t) = y(t)$ for μ-a.e. $t \in T$) such that $\int_T \Phi(t, \lambda |x(t)|)\, d\mu < \infty$ for some $\lambda > 0$ depending on x. L_Φ^μ is a Banach space under the Luxemburg norm $\|x\|_\Phi = \inf\{u > 0\colon \int_T \Phi(t, |x(t)|/u)\, d\mu \leqslant 1\}$ (see e.g. [16]).

A normed linear space $(X, \|\cdot\|)$ is said to be *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\|(x+y)/2\| \leqslant 1 - \delta(\varepsilon) \quad \text{whenever } \|x-y\| \geqslant \varepsilon \text{ and } \|x\| = \|y\| = 1.$$

[1] In the case of the purely atomic measure such that $\mu(\{n\}) = 1$ for all $n \in N$ we write l_Φ instead of L_Φ^μ.

A normed linear space $(X, \|\cdot\|)$ is called *k, ε-convex* $(k \geqslant 2)$ if for each choice of $x_1, \ldots, x_k$ from the unit ball of X

$$\|x_1 \pm x_2 \pm \ldots \pm x_k\| \leqslant k(1-\varepsilon)$$

for some choice of the $+$ and $-$ signs (see [4]).

A normed linear space X is said to be *B-convex* (X satisfies property (B) in Beck's terminology, [2]) if X is k, ε-convex for some $k \geqslant 2$, $\varepsilon > 0$.

For every Orlicz function Φ we define its complementary function Φ^* by

$$\Phi^*(t, u) = \sup_{v>0} [uv - \Phi(t, v)] \quad \text{for every } t \in T,\ u \geqslant 0.$$

Φ^* is also an Orlicz function.

Denoting by $\varphi(t, \cdot)$ the right derivative of an Orlicz function $\Phi(t, \cdot)$ and defining the generalized inverse function φ^* of φ by

$$\varphi^*(t, s) = \sup [u > 0\colon\ \varphi(t, u) \leqslant s]; \qquad t \in T,\ s \geqslant 0,$$

we have $\Phi^*(t, u) = \int\limits_0^u \varphi^*(t, s)\,ds$ for $t \in T$, $u \geqslant 0$ (see [14]).

Let μ be an atomless measure and $l > 1$. Recall that an Orlicz function Φ satisfies the Δ_l-condition if there exist a constant $K > 0$, a non-negative function h with $\int\limits_T h(t)\,d\mu < \infty$ and a set $T_0 \in \Sigma_0$ such that

$$\Phi(t, lu) \leqslant K\Phi(t, u) + h(t) \quad \text{for every } t \in T \setminus T_0,\ u \geqslant 0.$$

Let μ be a purely atomic measure. We say that an Orlicz function $\Phi = (\Phi_n)$ satisfies the δ_l-condition if there exist positive numbers K, a and a sequence (c_n) with $c_n \geqslant 0$ and $\sum\limits_{n\geqslant 1} c_n < \infty$ such that for every $n \in N$ and $u \geqslant 0$ the condition $\Phi_n(u) \leqslant a$ implies

$$\Phi_n(lu) \leqslant K\Phi_n(u) + c_n.$$

It is not difficult to show that $\Phi = (\Phi_n)$ satisfies the δ_l-condition iff there are constants K, $\delta > 0$, and non-negative sequences (c_n), (d_n) such that $\Phi_n(d_n) = \delta$, $\sum\limits_{n\geqslant 1} \Phi_n(c_n) < \infty$ and

$$\Phi_n(lu) \leqslant K\Phi_n(u) \quad \text{if } u \in [c_n, d_n] \text{ for every } n \in N.$$

We say that an Orlicz function Φ satisfies the uniform Δ_l-condition if there exist a set $T_0 \in \Sigma_0$ and a constant $K > 0$ such that

$$\Phi(t, lu) \leqslant K\Phi(t, u) \quad \text{for every } t \in T \setminus T_0,\ u \geqslant 0.$$

In the case of a purely atomic measure μ, we write "uniform δ_l-condition" instead of "uniform Δ_l-condition".

Let μ be an atomless measure and Φ be an Orlicz function with right derivative φ on R_+ for μ-a.e. $t \in T$. We say that φ satisfies the Δ_l-condition if there are a constant $K > 0$, a set $T_0 \in \Sigma_0$, and a non-negative function f defined and μ-measurable on T such that $\int_T f(t)\varphi(t, f(t))\,d\mu < \infty$ and

$$\varphi(t, lu) \leqslant K\varphi(t, u) \qquad \text{for every } t \in T \setminus T_0,\ u \geqslant f(t).$$

Let μ be a purely atomic measure. Let $\Phi = (\Phi_n)$ be an Orlicz function and let φ_n denote the right derivative of Φ_n, $n = 1, 2, \ldots$ We say that $\varphi = (\varphi_n)$ fulfils the δ_l-condition if there are constants K, $\delta > 0$, non-negative sequences (c_n), (d_n) such that $\Phi_n(d_n) = \delta$, $\sum_{n \geqslant 1} c_n \varphi_n(c_n) < \infty$ and

$$\varphi_n(lu) \leqslant K\varphi_n(u) \qquad \text{if } u \in [c_n, d_n] \text{ for all } n \in N.$$

It is said that an Orlicz function $\Phi = (\Phi_n)$ satisfies the δ_l^0-condition (see [10], [17]) if there are constants K, $\delta > 0$, $m \in N$, a sequence $(c_n) \in l_1^0$ such that $\sum_{n \geqslant m} c_n < \infty$ and

$$\Phi_n(lu) \leqslant K\Phi_n(u) + c_n \qquad \text{if } \Phi_n(u) \leqslant \delta \text{ for all } n \geqslant m$$

(here l_1^0 denotes the space of all sequences (s_n) with $s_n \in [0, \infty]$ such that $\sum_{n \geqslant k} s_n < \infty$ for some $k \in N$).

Let μ be a mixed measure, i.e. $T = T_1 \cup T_2$; $T_1, T_2 \in \Sigma$, $T_1 \cap T_2 = \emptyset$, $\mu = \mu_1 + \mu_2$, where μ_1 is an atomless measure on $\Sigma \cap T_1$ and μ_2 is a purely atomic measure on $\Sigma \cap T_2$. Recall that Φ (φ) satisfies the (Δ_l, δ_l)-condition if the restrictions of Φ (φ) to the sets T_1 and T_2 satisfy the Δ_l-condition and δ_l-condition, respectively on T_1 and T_2.

Recall that an Orlicz function Φ is uniformly convex on R_+ (see [15]) if there exists a set $T_0 \in \Sigma_0$ such that

$$\forall\, 0 < a < 1\ \exists\, 0 < \delta(a) < 1\ \forall\, t \in T \setminus T_0\ \forall\, u \geqslant 0\ \forall\, 0 \leqslant b \leqslant a:$$
$$\Phi(t, (u + bu)/2) \leqslant (1 - \delta(a))\{\Phi(t, u) + \Phi(t, bu)\}/2.$$

It is known (see [1]) that an Orlicz function Φ is uniformly convex on R_+ iff for every $\varepsilon > 0$ there exists a constant $k_\varepsilon > 1$ such that

$$\varphi(t, (1+\varepsilon)u) \geqslant k_\varepsilon \varphi(t, u) \tag{0.1}$$

for every $t \in T \setminus T_0$, $u \geqslant 0$ ($\varphi(t, \cdot)$ denotes the right derivative of $\Phi(t, \cdot)$).

Now, we introduce a partial order "$\dashv$" in the class of all Orlicz functions. Let μ be an atomless measure and let Φ, Ψ be Orlicz functions. We say that Φ is non-stronger than Ψ (written $\Phi \dashv \Psi$) if there exist a constant $K > 0$, a set $T_0 \in \Sigma_0$, and a non-negative function $h \in L_1^\mu(T)$ such that

$$\Phi(t, Ku) \leqslant \Psi(t, u) + h(t) \qquad \text{for every } t \in T \setminus T_0,\ u \geqslant 0.$$

If μ is a purely atomic measure and $\Phi = (\Phi_n)$ and $\Psi = (\Psi_n)$ are Orlicz functions, then we say that Φ is non-stronger that Ψ (written $\Phi \dashv \Psi$) if there exist constants $K, \delta > 0$, and a non-negative sequence (c_n) with $\sum_{n\geq 1} c_n < \infty$ such that

$$\Phi_n(Ku) \leqslant \Psi_n(u) + c_n \quad \text{for all } n \in N \text{ if } \Psi_n(u) \leqslant \delta$$

(see [10]).

Let μ be a mixed measure, i.e. $T = T_1 \cup T_2$: $T_1, T_2 \in \Sigma$, $T_1 \cap T_2 = \emptyset$: $\mu = \mu_1 + \mu_2$, where μ_1 is an atomless measure on $\Sigma \cap T_1$ and μ_2 is a purely atomic measure on $\Sigma \cap T_2$. Let Φ and Ψ be Orlicz functions. Recall that Φ is non-stronger than Ψ if $\Phi/T_1 \dashv \Psi/T_1$ and $\Phi/T_2 \dashv \Psi/T_2$ (here Φ/T_1 denotes the restriction of Φ to the set T_1).

Let Φ and Ψ be Orlicz functions. Recall that Φ and Ψ are equivalent (written $\Phi \sim \Psi$) if $\Phi \dashv \Psi$ and $\Psi \dashv \Phi$.

Obviously, two equivalent Orlicz functions Φ and Ψ define the same Orlicz space and equivalent Luxemburg's norms $\|\cdot\|_\Phi$ and $\|\cdot\|_\Psi$.

THEOREM 0.1. *If μ is an atomless (a purely atomic or mixed) measure and Φ is an Orlicz function uniformly convex on R_+ and satisfying the Δ_2-condition (the uniform Δ_2-condition), then $(L^\mu_\Phi, \|\cdot\|_\Phi)$ is a uniformly convex space.*

For the proof of this theorem see [5] and [15].

Criteria for uniform convexity of Orlicz or Musielak–Orlicz spaces are contained in [6] and [8].

V. Akimovič [1] proved that if complementary Orlicz functions Φ and Φ^* without parameter satisfy the Δ_2-condition, then there exists an Orlicz function Φ_1 without parameter equivalent to Φ and such that the Orlicz space $(L_{\Phi_1}, \|\cdot\|_{\Phi_1})$ is uniformly convex.

M. Denker and R. Kombrink [3] proved in the case of a purely atomic measure μ such that $\mu(\{n\}) = 1$ for $n = 1, 2, \ldots$ and an Orlicz function Φ without parameter that l_φ is a B-convex space iff Φ and Φ^* satisfy the δ_2-condition (in this case the Δ_2-condition for small $u \geqslant 0$).

In this paper, modifying and complementing methods of V. Akimovič [1] and of M. Denker and R. Kombrink [3], we extend these results to Musielak–Orlicz spaces, generated by arbitrary non-negative, σ-finite and complete measure μ.

1. Results.

LEMMA 1.1. *If Φ is an Orlicz function and φ is its right derivative with the generalized inverse function φ^*, then the inequality*

$$\varphi(t, 2u) \leqslant k\varphi(t, u) \quad \textit{for some } k > 0, \textit{ all } u \geqslant 0 \textit{ and } t \in T$$

implies

$$\varphi^*(t, kv) \geqslant 2\varphi^*(t, v) \quad \textit{for each } v \geqslant 0 \textit{ and } t \in T.$$

Proof. By assumptions $\sup\{u>0\colon \varphi(t,u/2)\leqslant v\}\leqslant\sup\{u>0\colon \varphi(t,u)\leqslant kv\}$. So $2\varphi^*(t,v)=\sup\{2u\colon \varphi(t,u)\leqslant v\}=\sup\{u>0\colon \varphi(t,u/2)\leqslant v\}\leqslant\sup\{u>0\colon \varphi(t,u)\leqslant kv\}=\varphi^*(t,kv)$ for all $v\geqslant 0$ and $t\in T$.

Further, we shall consider the cases of atomless and of purely atomic measure μ, separately.

1.1. The case of an atomless measure.

LEMMA 1.1.1. *Let Φ and Ψ be Orlicz functions. Then*:

(i) *Φ satisfies the Δ_2-condition (the uniform Δ_2-condition) iff it satisfies the Δ_l-condition (the uniform Δ_l-condition) for every $l>1$;*

(ii) *if $\Phi \dashv \Psi$, then $\Psi^* \dashv \Phi^*$;*

(iii) *if $\Phi\sim\Psi$, then $\Phi^*\sim\Psi^*$;*

(iv) *if $\Phi\sim\Psi$ and Φ satisfies the Δ_l-condition ($l>1$), then Ψ satisfies it also.*

Proof. (i) This is proved in [5]. (ii) Easy proof is omitted. (iii) This follows from (ii).

(iv) By assumptions there are a set $T_0\in\Sigma_0$, positive constants K and k_1 and a non-negative function $h\in L_1^\mu(T)$ such that

$$\Phi(t,lu)\leqslant K\Phi(t,u)+h(t)$$

and

$$\Phi(t,k_1^{-1}u)-h(t)\leqslant\Psi(t,u)\leqslant\Phi(t,k_1u)+h(t)$$

for every $t\in T\setminus T_0$, $u\geqslant 0$. Applying the last inequalities and denoting by K_0 and h_0 the constant and function from the Δ_l-condition for $l=k_1^2$, respectively, we have for every $t\in T\setminus T_0$ and $u\geqslant 0$:

$$\begin{aligned}\Psi(t,lu)&\leqslant\Phi(t,lk_1u)+h(t)\leqslant K\Phi(t,k_1u)+2h(t)\\&\leqslant KK_0\Phi(t,k_1^{-1}u)+Kh_0(t)+2h(t)\leqslant KK_0\Psi(t,u)+Kh_0(t)+2h(t).\end{aligned}$$

Since the function $h_1(t)=Kh_0(t)+2h(t)$ belongs to $L_1^\mu(T)$, so the last inequality means that Ψ satisfies the Δ_l-condition.

LEMMA 1.1.2. *For every Orlicz function Φ satisfying the Δ_l-condition ($l>1$) there exists an Orlicz function Φ_1 equivalent to Φ and satisfying the uniform Δ_l-condition.*

Proof. We may assume without loss of generality that $l=2$. First, we shall prove that the Δ_2-condition is equivalent to the following one

$$(\Delta^2)\quad \exists\, K>0\ \exists\, f\colon T\to R_+ \text{ with } \int_T\Phi(t,f(t))\,d\mu<\infty\ \exists\, T_0\in\Sigma_0\ \forall\, t\in T\setminus T_0$$

$$\forall\, u\geqslant f(t)\colon\ \Phi(t,2u)\leqslant K\Phi(t,u).$$

Assume that Φ satisfies the Δ_2-condition. Let $f(t) = \sup\{u \geq 0\colon \Phi(t, u) \leq h(t)\}$. Since $f(t) = \sup\{u \in Q_+\colon \Phi(t, u) \leq h(t)\}$, where Q_+ is the set of all non-negative rational numbers, so it is a μ-measurable function. We have $\Phi(t, f(t)) = h(t)$ and so

$$\Phi(t, 2u) \leq K\Phi(t, u) + h(t) \leq (K+1)\Phi(t, u)$$

for every $t \in T \setminus T_0$ and $u \geq f(t)$, i.e., Φ satisfies the Δ^2-condition.

Conversely, let us assume that Φ satisfies the Δ^2-condition. Then denoting $\Phi(t, f(t)) = h(t)$, we have for every $t \in T \setminus T_0$ and $u \geq 0$:

$$\Phi(t, 2u) \leq K\Phi(t, u) + Kh(t).$$

Since $\int_T Kh(t)\,d\mu < \infty$, so Φ satisfies the Δ_2-condition.

Now, we define the Orlicz function Φ_1 by

$$\Phi_1(t, u) = \begin{cases} \frac{1}{2} \cdot \dfrac{\Phi(t, f(t))}{f^2(t)}\, u^2 & \text{for } 0 \leq u < f(t), \\ \Phi(t, u) - \frac{1}{2}\Phi(t, f(t)) & \text{for } u \geq f(t). \end{cases}$$

Since $\Phi(t, f(t))/f(t) \leq \varphi(t, f(t))$ for $f(t) > 0$, where $\varphi(t, \cdot)$ denotes the right derivative of $\Phi(t, \cdot)$, so Φ_1 is an Orlicz function. It is obvious that

$$\Phi_1(t, u) \leq \Phi(t, u) + \tfrac{1}{2}\Phi(t, f(t))$$

and

$$\Phi(t, u) \leq \Phi_1(t, u) + \Phi(t, f(t))$$

for every $t \in T$, $u \geq 0$. Since $\int_T \Phi(t, f(t))\,d\mu < \infty$, so $\Phi_1 \sim \Phi$. In order to prove that Φ satisfies the uniform Δ_2-condition we shall consider three cases for $f(t) > 0$.

(I) $0 \leq u \leq f(t)/2$. Then $\Phi_1(t, 2u) = 4\Phi_1(t, u)$.

(II) $f(t)/2 \leq u \leq f(t)$. Then we have

$$\begin{aligned} \Phi_1(t, 2u) &= \Phi(t, 2u) - \tfrac{1}{2}\Phi(t, f(t)) \leq \Phi(t, 2f(t)) - \tfrac{1}{2}\Phi(t, f(t)) \\ &\leq (K - \tfrac{1}{2})\Phi(t, f(t)) = (K - \tfrac{1}{2}) \frac{\Phi(t, f(t))}{f^2(t)} f^2(t) \\ &\leq 4(K - \tfrac{1}{2}) \frac{\Phi(t, f(t))}{f^2(t)} u^2 = 8(K - \tfrac{1}{2})\Phi_1(t, u). \end{aligned}$$

(III) $u \geqslant f(t)$. We have

$$\begin{aligned}\Phi_1(t, 2u) &= \Phi(t, 2u) - \tfrac{1}{2}\Phi(t, f(t)) \leqslant K\Phi(t, u) - \tfrac{1}{2}\Phi(t, f(t)) \\ &\leqslant K\Phi(t, u) - \tfrac{1}{2}\Phi(t, f(t)) + K\Phi(t, f(t)) - \frac{2K}{2}\Phi(t, f(t)) \\ &\leqslant 2K\Phi(t, u) - \frac{2K+1}{2}\Phi(t, f(t)) \leqslant 2K\Phi_1(t, u).\end{aligned}$$

Taking into account that $K \geqslant 2$, we have $\Phi_1(t, 2u) \leqslant 8K\Phi_1(t, u)$ for every $t \in T \setminus T_0$, $u \geqslant 0$, and so the proof is finished.

COROLLARY 1.1.3. *If Φ and Φ^* are complementary Orlicz functions without parameter satisfying the Δ_l-condition for large $u \geqslant 0$* (see [14]), *then there exists an Orlicz function Φ_1 equivalent to Φ such that Φ_1 and Φ_1^* satisfy the Δ_l-condition for all $u \geqslant 0$ (i.e., the uniform Δ_l-condition)*(2).

LEMMA 1.1.4. *Let Φ be an Orlicz function and let φ be its right derivative. Then Φ satisfies the Δ_l-condition (uniform Δ_l-condition) if and only if φ satisfies the Δ_l-condition (uniform Δ_l-condition).*

Proof. We shall prove only the case of the Δ_2-condition. Let Φ satisfy the Δ_2-condition. Then there exists a constant $K > 0$, a set $T_0 \in \Sigma_0$ and a μ-measurable function $f\colon T \to R_+^e$ such that $\int_T \Phi(t, f(t))\,d\mu < \infty$ and

$$\Phi(t, 2u) \leqslant K\Phi(t, u)$$

for every $t \in T \setminus T_0$ and $u \geqslant f(t)$. We have for μ-a.e. $t \in T$ and all $u \geqslant 0$

$$\tfrac{1}{2}u\varphi(t, \tfrac{1}{2}u) \leqslant \Phi(t, u) \leqslant u\varphi(t, u).$$

Thus, we have for μ-a.e. $t \in T$ and all $u \geqslant f(t)$

$$\varphi(t, 2u) \leqslant \frac{\Phi(t, 4u)}{2u} \leqslant \frac{K^2\Phi(t, u)}{2u} \leqslant \frac{K^2\varphi(t, u)}{2}.$$

Moreover, we have for μ-a.e. $t \in T$, $f(t)\varphi(t, f(t)) \leqslant \Phi(t, 2f(t)) \leqslant K\Phi(t, f(t))$, so $\int_T f(t)\varphi(t, f(t))\,d\mu < \infty$. Thus φ satisfies the Δ_2-condition. Conversely, let φ satisfy the Δ_2-condition. Then there exist a constant $K > 0$, a set $T_0 \in \Sigma_0$ and a non-negative and μ-measurable function f defined on T such that $\int_T f(t)\varphi(t, f(t))\,d\mu < \infty$ and

(2) The function Φ_1 may depend on parameter.

$$\varphi(t, 2u) \leqslant K\varphi(t, u)$$

for every $t \in T \setminus T_0$ and $u \geqslant f(t)$. We have

$$\tfrac{1}{2}\Phi(t, 2u) - \tfrac{1}{2}\Phi(t, 2f(t)) = \int_{f(t)}^{u} \varphi(t, 2v)\,dv \leqslant K \int_{f(t)}^{u} \varphi(t, v)\,dv$$
$$= K\{\Phi(t, u) - \Phi(t, f(t))\}.$$

Hence, we have for every $t \in T \setminus T_0$ and $u \geqslant f(t)$

$$\Phi(t, 2u) \leqslant 2K\Phi(t, u) + \Phi(t, 2f(t)).$$

Since $\Phi(t, 2f(t)) \leqslant 2f(t)\varphi(t, 2f(t)) \leqslant 2Kf(t)\varphi(t, f(t))$, so $\int_T \Phi(t, 2f(t))\,d\mu < \infty$, and this means that Φ satisfies the Δ_2-condition.

LEMMA 1.1.5. *Let Φ and Φ^* be complementary Orlicz functions and let Φ^* satisfy the Δ_2-condition. Then there exists an Orlicz function Φ_2 uniformly convex on R_+ and equivalent to Φ.*

Proof. By Lemma 1.1.2 there exists an Orlicz function $\Phi_1^* \sim \Phi^*$ satisfying the uniform Δ_2-condition. Thus the right derivative φ_1^* of Φ_1^* also satisfies the uniform Δ_2-condition. Hence, by Lemma 1.1, we obtain

(1.1.1) $$\varphi_1(t, lv) \geqslant 2\varphi_1(t, v) \quad \text{for each } v \geqslant 0 \text{ and } t \in T \setminus T_0,$$

where $\mu(T_0) = 0$ and l is an absolute constant $\geqslant 2$[3]. Now, let us denote

(1.1.2) $$u_k = l^k, \quad k = 0, \pm 1, \pm 2, \pm \ldots$$

and define the function φ_2: $\varphi_2(t, \cdot)$-continuous, $\varphi_2(t, 0) = 0$, $\varphi_2(t, u_k) = \varphi_1(t, u_k)$ and $\varphi_2(t, \cdot)$-linear between points u_k and u_{k+1}, $k = 0, \pm 1, \pm 2, \pm \ldots$ We can prove that it follows from the last conditions that for every $\varepsilon > 0$ there exists a constant $k_\varepsilon > 1$ such that

(1.1.3) $$\varphi_2(t, (1+\varepsilon)u) \geqslant k_\varepsilon \varphi_2(t, u) \quad \text{for every } u \geqslant 0 \text{ and } t \in T \setminus T_0.$$

We may assume without loss of generality that $1+\varepsilon \leqslant l$. First, we shall prove inequality (1.1.3) for $u \geqslant 1$. Let $t \in T \setminus T_0$ and $u \geqslant 1$. There exists $k \in N \cup \{0\}$ such that $u_k \leqslant u < u_{k+1}$. Since $1+\varepsilon \leqslant l$, so $u_k \leqslant (1+\varepsilon)u < u_{k+2}$. We shall consider two cases: (I) $u_k \leqslant (1+\varepsilon)u < u_{k+1}$ and (II) $u_{k+1} \leqslant (1+\varepsilon)u < u_{k+2}$. Denote by $\varphi_{2,k}$ a function identical with φ_2 on the interval $[u_k, u_{k+1}]$. We have

(1.1.4) $$\varphi_{2,k}(t, u) = a_k(t)u + b_k(t),$$

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[3] By φ_1 we denote the generalized inverse function of φ_1^*.

more precisely

$$(1.1.5) \qquad \varphi_{2,k}(t, u) = \varphi_1(t, u_k) + \frac{\varphi_1(t, u_{k+1}) - \varphi_1(t, u_k)}{u_{k+1} - u_k}(u - u_k).$$

(Ia) $b_k(t) \geqslant 0$. Let us consider the function

$$f_{k,\varepsilon}(t, u) = \varphi_{2,k}\big(t, (1+\varepsilon)u\big)/\varphi_{2,k}(t, u).$$

We have $[f_{k,\varepsilon}(t, u)]'_u = a_k(t)b_k(t)\varepsilon/\big(a_k(t)u + b_k(t)\big)^2$. Since $a_k(t) \geqslant 0$ for every $t \in T$, so $[f_{k,\varepsilon}(t, u)]'_u \geqslant 0$. Thus the function $f_{k,\varepsilon}(t, u)$ is non-decreasing with respect to the variable u for every fixed $t \in T$, and by (1.1.1)

$$f_{k,\varepsilon}(t, u) \geqslant f_{k,\varepsilon}(t, u_k) = \frac{\varphi_1(t, u_k) + \dfrac{\varphi_1(t, u_{k+1}) - \varphi_1(t, u_k)}{u_{k+1} - u_k}\varepsilon u_k}{\varphi_1(t, u_k)}$$

$$= 1 + \frac{\varphi_1(t, u_{k+1}) - \varphi_1(t, u_k)}{\varphi_1(t, u_k)} \frac{\varepsilon u_k}{u_{k+1} - u_k} \geqslant 1 + \frac{\varepsilon}{l-1} > 1.$$

(Ib) $b_k(t) < 0$. Then, we have $[f_{k,\varepsilon}(t, u)]'_u < 0$ and so $f_{k,\varepsilon}(t, u)$ is a decreasing function with respect to the variable u for every fixed $t \in T$. Hence, we have

$$f_{k,\varepsilon}(t, u) \geqslant f_{k,\varepsilon}(t, u_{k+1}) = \frac{\varphi_1(t, u_{k+1}) + \dfrac{\varphi_1(t, u_{k+1}) - \varphi_1(t, u_k)}{u_{k+1} - u_k}\varepsilon u_{k+1}}{\varphi_1(t, u_{k+1})}$$

$$\geqslant 1 + \varepsilon l/2(l-1) > 1.$$

Denoting $k_\varepsilon = \min[1+\varepsilon/(l-1),\ 1+\varepsilon l/2(l-1)] = 1+\varepsilon/(l-1)$, we get the inequality (1.1.3) in the case I.

(IIa) Let $u_{k+1} \leqslant (1+\varepsilon)u < u_{k+2}$ and $u_k \leqslant \sqrt{1+\varepsilon}\cdot u < u_{k+1}$. Then whole interval $[u, \sqrt{1+\varepsilon}\cdot u]$ is contained in the interval $[u_k, u_{k+1})$. Let $\varepsilon_1 > 0$ be such that $\sqrt{1+\varepsilon} = 1+\varepsilon_1$. Then, by the case (I), we have

$$\varphi_2\big(t, (1+\varepsilon)u\big)/\varphi_2(t, u) \geqslant \varphi_2\big(t, (1+\varepsilon_1)u\big)/\varphi_2(t, u) \geqslant k_{\varepsilon_1} > 1.$$

(IIb) Let $\sqrt{1+\varepsilon}\cdot u \in [u_{k+1}, u_{k+2})$ and $\sqrt{1+\varepsilon} = 1+\varepsilon_1$. Denoting $\sqrt{1+\varepsilon}\cdot u = v$, we have $[v, (1+\varepsilon_1)v] \subset [u_{k+1}, u_{k+2})$ and so

$$\frac{\varphi_2\big(t, (1+\varepsilon)u\big)}{\varphi_2(t, u)} \geqslant \frac{\varphi_2\big(t, (1+\varepsilon)u\big)}{\varphi_2(t, \sqrt{1+\varepsilon}\cdot u)} = \frac{\varphi_2\big(t, (1+\varepsilon_1)v\big)}{\varphi_2(t, v)} = f_{k+1,\varepsilon_1}(t, v) \geqslant k_{\varepsilon_1} > 1.$$

Thus, inequality (1.1.3) is proved for $u \geqslant 1$. The proof of this inequality for $0 \leqslant u < 1$ is analogous to the proof in the case $u \geqslant 1$ and so it is omitted here. The Orlicz functions Φ_1 and Φ_2 are equivalent, because for μ-a.e. $t \in T$

and for every $u \geqslant 0$

$$\varphi_1(t, l^{-1}u) \leqslant \varphi_2(t, u) \leqslant \varphi_1(t, lu).$$

Since Φ_1 is equivalent to Φ, so Φ_2 is also equivalent to Φ. This finishes the proof.

Remark 1.1.6. If Φ is an Orlicz function uniformly convex on R_+, then its complementary function Φ^* satisfies the uniform Δ_2-condition (see [15]). This is in some sense a converse of the last lemma.

THEOREM 1.1.7. *Let Φ and Φ^* be complementary Orlicz functions satisfying the Δ_2-condition. Then there exists an Orlicz function Φ_2 equivalent to Φ and such that the space $(L_{\Phi_2}, \|\cdot\|_{\Phi_2})$ is uniformly convex.*

Proof. By the last lemma there exists an Orlicz function Φ_2 uniformly convex on R_+ and equivalent to Φ. By Lemma 1.1.1 (iv), Φ_2 satisfies the Δ_2-condition. Applying Theorem 0.1 we get the desired result.

1.2. The case of a purely atomic measure.

LEMMA 1.2.1. *An Orlicz function $\Phi = (\Phi_n)$ satisfies the δ_l-condition with $l > 1$ iff $\varphi = (\varphi_n)$ satisfies the δ_l-condition.*

Proof. We may assume that Φ satisfies the δ_{2l}-condition (see [11]). Then there exist constants $K, \delta > 0$, non-negative sequences (c_n), (d_n) such that $\Phi_n(d_n) = \delta$, $\sum_{n\geqslant 1} \Phi_n(c_n) < \infty$ and

$$\Phi_n(2lu) \leqslant K\Phi_n(u) \quad \text{for all } u \in [c_n, d_n],\ n \in N. \tag{1.2.1}$$

We have $\Phi_n(u) \leqslant u\varphi_n(u)$ and $\Phi_n(2lu) \geqslant lu\varphi_n(lu)$ for all $u \geqslant 0$, $n \in N$. Hence and from (1.2.1) we get $\sum_{n\geqslant 1} c_n \varphi_n(c_n) \leqslant K \sum_{n\geqslant 1} \Phi_n(c_n) < \infty$ and $\varphi_n(lu) \leqslant (K/l) \times$ $\times \varphi_n(u)$ for $u \in [c_n, d_n]$, $n \in N$. Thus $\varphi = (\varphi_n)$ satisfies the δ_l-condition.

Conversely, assume the δ_l-condition of $\varphi = (\varphi_n)$ be satisfied. Then $\varphi_n(lu) \leqslant K\varphi_n(u)$ for all $u \in [c_n, d_n]$, $n \in N$, where $\Phi_n(d_n) = \delta$ and $\sum_{n\geqslant 1} c_n \varphi_n(c_n)$ $< \infty$. Integrating this inequality we obtain

$$l^{-1}\Phi_n(lu) - l^{-1}\Phi_n(lc_n) = \int_{c_n}^{u} \varphi_n(lt)\,dt \leqslant K \int_{c_n}^{u} \varphi_n(t)\,dt = K\Phi_n(u) - K\Phi_n(c_n)$$

for each $u \in [c_n, d_n]$, $n \in N$. Then

$$\Phi_n(lu) \leqslant lK\Phi_n(u) + \Phi_n(lc_n) \quad \text{for all } u \in [c_n, d_n],\ n \in N.$$

Moreover, $\sum_{n\geqslant 1} \Phi_n(lc_n) \leqslant l \sum_{n\geqslant 1} c_n \varphi_n(lc_n) \leqslant lK \sum_{n\geqslant 1} c_n \varphi_n(c_n) < \infty$. Therefore Φ $= (\Phi_n)$ satisfies the δ_l-condition.

LEMMA 1.2.2. *Let $\Phi = (\Phi_n)$ and $\Psi = (\Psi_n)$ be Orlicz functions. If $\Phi \sim \Psi$ and Φ satisfies the δ_2-condition, then Ψ satisfies the δ_2^0-condition.*

Proof. By assumptions, there exist constants $K, K_1 > 1$, $\delta > 0$, and sequences $c = (c_n)$, $d = (d_n)$, $e = (e_n)$ with non-negative elements and belonging to l_1 such that

$$\Psi_n(K^{-1}u) \leqslant \Phi_n(u) + c_n, \tag{1.2.2}$$

$$\Phi_n(K^{-1}u) \leqslant \Psi_n(u) + d_n, \tag{1.2.3}$$

$$\Phi_n(2K^2 u) \leqslant K_1 \Phi_n(u) + e_n, \tag{1.2.4}$$

if $\Phi_n(u) \leqslant \delta$, $\Psi_n(u) \leqslant \delta$, $n \in N$.

Let $\delta_0 = \delta/3K_1$. If $\Psi_n(u) \leqslant \delta_0$, then $\Phi_n(K^{-1}u) \leqslant \delta/2K_1$ by (1.2.3) and $\Phi_n(2Ku) = \Phi_n(2K^2K^{-1}u) \leqslant K_1 \Phi_n(K^{-1}u) + e_n$ by (1.2.4) for sufficiently large $n \in N$. Then $\Psi_n(u) \leqslant \delta_0$ implies

$$\Psi_n(2u) \leqslant \Phi_n(2Ku) + c_n \leqslant K_1 \Phi_n(K^{-1}u) + e_n + c_n \leqslant K_1 \Psi_n(u) + K_1 d_n + e_n + c_n$$

for sufficiently large $n \in N$, by (1.2.2), (1.2.4) and (1.2.3). Thus $\Psi = (\Psi_n)$ fulfils the δ_2^0-condition.

Remark. The last two lemmas will stay valid if we replace the δ_l-condition and δ_2-condition by the δ_l^0-condition and δ_2^0-condition, respectively.

LEMMA 1.2.3. *If $\Phi = (\Phi_n)$ and $\Psi = (\Psi_n)$ are equivalent Orlicz functions, then $\Phi^* = (\Phi_n^*)$ and $\Psi^* = (\Psi_n^*)$ are equivalent too.*

Proof. It is enough to show that if $\Phi \dashv \Psi$, i.e. $\Phi_n(ku) \leqslant \Psi_n(u) + c_n$ if $\Psi_n(u) \leqslant \delta$, $n \in N$, for some constants $k, \delta > 0$ and a non-negative sequence (c_n), where $\sum_{n \geqslant 1} c_n < \infty$, then $\Psi^* \dashv \Phi^*$. Let us denote by φ_n and ψ_n the right derivative of Φ_n and of Ψ_n, respectively.

Let N_1 be a subset of N such that for all $n \in N_1$ there exists $u_n > 0$ such that $\Psi_n(u_n) < \delta$ and $\Psi_n(u) = \infty$ for all $u > u_n$. Then we have $\Phi_n(ku) \leqslant \Psi_n(u) + c_n$ for all $u \geqslant 0$ and $n \in N_1$. Hence

$$\Psi_n^*(v) = \sup_{u \geqslant 0} [uv - \Psi_n(u)] \leqslant \sup_{u \geqslant 0} [uv - \Phi_n(ku)] + c_n = \Phi_n^*(k^{-1}v) + c_n$$

for all $v \geqslant 0$. Therefore

$$\Psi_n^*(kv) \leqslant \Phi_n^*(v) + c_n \quad \text{for each } v \geqslant 0,\ n \in N_1. \tag{1.2.5}$$

For all $n \in N_2 = N \setminus N_1$ there exists $u_n > 0$ such that $\Psi_n(u_n) = \delta$ and

$$\Phi_n(ku) \leqslant \Psi_n(u) + c_n \quad \text{for every } u \leqslant u_n,\ n \in N_2. \tag{1.2.6}$$

Let $\bar{\Phi}_n(u) = \Phi_n(ku)$ and $\bar{\varphi}_n$ be the right derivative of $\bar{\Phi}_n$ and $\bar{\Phi}_n^*$ be the complementary function to $\bar{\Phi}_n$. We have

$$uv = \Psi_n(u) + \Psi_n^*(v) \quad \text{for each } u \geqslant 0,\ v \in [\lim_{w \to u^-} \psi_n(w), \lim_{w \to u^+} \psi_n(w)] \tag{1.2.7}$$

(see [14]), and

(1.2.8) $$uv \leqslant \bar{\Phi}_n(u)+\bar{\Phi}_n^*(v) \quad \text{for all } u,\, v \geqslant 0,$$

by the Young's inequality (see [14]). Thus, by (1.2.5), (1.2.7) and (1.2.8) it is obtained

(1.2.9) $$\Psi_n^*(v) \leqslant \bar{\Phi}_n^*(v)+c_n \quad \text{for each } v \leqslant \psi_n(u_n),\ n \in N_2.$$

Since

$$\tfrac{1}{2}u\bar{\varphi}_n(\tfrac{1}{2}u) \leqslant \bar{\Phi}_n(u) \leqslant \Psi_n(u)+c_n \leqslant u\psi_n(u)+c_n \quad \text{for all } u \leqslant u_n,\ n \in N,$$

so

(1.2.10) $$\tfrac{1}{2}u_n\,\varphi_n(\tfrac{1}{2}u_n) \leqslant u_n\psi_n(u_n)+c_n \quad \text{for all } n \in N.$$

But $\bar{\Phi}_n^*(v) \geqslant \frac{1}{2}u_n v-\frac{1}{2}u_n\bar{\varphi}_n(\frac{1}{2}u_n)$ for all $n \in N_2$ and $v \geqslant 0$ by the definition of conjugate function. Then, by (1.2.10)

(1.2.11) $$\bar{\Phi}_n^*(v) \geqslant \tfrac{1}{2}u_n v-u_n\psi_n(u_n)-c_n \quad \text{for all } n \in N_2,\ v \geqslant 0.$$

Let $\alpha > 0$ be arbitrary. If $\bar{\Phi}_n^*(v) \leqslant \alpha$ and $n \in N_2$, then by (1.2.11)

$$\frac{v}{\psi_n(u_n)} \leqslant \frac{2(c_n+\alpha)}{u_n\psi_n(u_n)}+2 \quad \text{for all } v \geqslant 0.$$

However, putting $u = u_n$, $v = \psi_n(u_n)$ in (1.2.7), we have

$$u_n\psi_n(u_n) = \delta+\Psi_n^*\big(\psi_n(u_n)\big) \geqslant \delta \quad \text{for each } n \in N_2,$$

because $\Psi_n(u_n) = \delta$. Therefore

$$\frac{v}{\psi_n(u_n)} \leqslant \frac{2\max\limits_n(c_n+\alpha)}{\delta}+2 \quad \text{for all } n \in N_2,\ v \geqslant 0.$$

Denoting $2\max\limits_n(c_n+\alpha)/\delta+2 = k_1$, we get $v/k_1 \leqslant \psi_n(u_n)$ if $\bar{\Phi}_n^*(v) \leqslant \alpha$. Now, applying inequality (1.2.9) and the fact $\bar{\Phi}_n^*(v) = \Phi_n^*(k^{-1}v)$ it is obtained

(1.2.12) $$\Psi_n^*(kk_1^{-1}v) \leqslant \Phi_n^*(v)+c_n \quad \text{if } \Phi_n^*(v) \leqslant \alpha,\ n \in N_2.$$

Combining (1.2.5) and (1.2.12), we obtain $\Psi^* \dashv \Phi^*$, which finishes the proof of this lemma.

LEMMA 1.2.4. *If an Orlicz function $\Phi = (\Phi_n)$ satisfies the δ_2^0-condition, then there exists an Orlicz function $\bar{\Phi} = (\bar{\Phi}_n)$ equivalent to Φ and satisfying the uniform δ_2-condition. If we additionally suppose*

(*) $$\varphi_n(lu) \geqslant r\varphi_n(u) \quad \textit{for some } l,\, r > 1 \textit{ and all } u \in R_+,\ n \in N,$$

then the function $\bar{\varphi} = (\bar{\varphi}_n)$ fulfils also condition (*) with some l_1, $r_1 > 1$.

Proof. If Φ satisfies the δ_2^0-condition, then it satisfies also the δ_l^0-

condition for all $l > 1$ (see [11]). Then there are k, $\delta > 0$, $m \in N$, (c_n), (d_n) with non-negative terms and such that (see Lemma 1.2.1)

(1.2.13) $$\varphi_n(lu) \leqslant k\varphi_n(u) \quad \text{for all } u \in [c_n, d_n],\ n > m,$$

where $\sum_{n>m} c_n \varphi_n(c_n) < \infty$, $\Phi_n(d_n) = \delta$. We can assume $lc_n \leqslant d_n$ for $n > m$. Let us define a new sequence $\bar{\varphi} = (\bar{\varphi}_n)$ as follows:

(1.2.14) $$\bar{\varphi}_n(u) = \begin{cases} \frac{1}{l^i k^i}\varphi_n(l^i u) & \text{for } u \in [c_n/l^i, c_n/l^{i-1}),\ i = 1, 2, \ldots, \\ \varphi_n(u) & \text{for } u \in [c_n, d_n), \\ l^i k^i \varphi_n(u/l^i) & \text{for } u \in [l^{i-1} d_n, l^i d_n),\ i = 1, 2, \ldots \end{cases}$$

for $n > m$ and $\bar{\varphi}_n(u) = u$ for $n \leqslant m$, $u \geqslant 0$. Then

(1.2.15) $$\bar{\varphi}_n(lu) = l\bar{\varphi}_n(u) \quad \text{for } n \leqslant m \text{ and } u \geqslant 0.$$

It is not difficult to show that $\bar{\varphi}_n(u)$ is non-decreasing. If u, $lu \in [c_n, d_n)$, then $\bar{\varphi}_n(u) = \varphi_n(u)$ and $\bar{\varphi}_n(lu) = \varphi_n(lu)$. Hence

(1.2.16) $$r\bar{\varphi}_n(u) \leqslant \bar{\varphi}_n(lu) \leqslant k\bar{\varphi}_n(u)$$

if u, $lu \in [c_n, d_n)$, by (1.2.13). If $u \in [c_n, d_n)$ and $lu \in [d_n, ld_n)$, then

(1.2.17) $$\bar{\varphi}_n(lu) = lk\varphi_n\left(\frac{1}{l}\, lu\right) = lk\bar{\varphi}_n(u).$$

Now, let $u \in [l^{i-1} d_n, l^i d_n)$ for some $i \in N$. Then

(1.2.18) $$\bar{\varphi}_n(lu) = l^{i+1} k^{i+1} \varphi_n\left(\frac{1}{l^{i+1}}\, lu\right) = lkl^i k^i \varphi_n\left(\frac{1}{l^i}\, u\right) = lk\bar{\varphi}_n(u).$$

Similar equality will be satisfied if $u \in [c_n/l^i, c_n/l^{i-1})$ for some $i \in N$. Therefore

(1.2.19) $$r\bar{\varphi}_n(u) \leqslant \bar{\varphi}_n(lu) \leqslant lk\bar{\varphi}_n(u) \quad \text{for all } u \geqslant 0,\ n \in N,$$

by (1.2.16), (1.2.17) and (1.2.18). Let

$$\bar{\Phi}_n(u) = \int_0^u \bar{\varphi}_n(t)\,dt \quad \text{for } n \in N,\ u \geqslant 0.$$

We shall prove that $\bar{\Phi}$ is equivalent to Φ. Indeed, for $n > m$ and $u \in [c_n, d_n)$ we have

$$\bar{\Phi}_n(u) = \int_0^{c_n} \bar{\varphi}_n(t)\,dt + \int_{c_n}^{u} \varphi_n(t)\,dt = \bar{\Phi}_n(c_n) + \Phi_n(u) - \Phi_n(c_n).$$

But $\bar{\Phi}_n(u) \leqslant \bar{\Phi}_n(c_n)$ for $0 \leqslant u \leqslant c_n$, so

(1.2.20) $$\bar{\Phi}_n(u) \leqslant \Phi_n(u) + \bar{\Phi}_n(c_n) \quad \text{for } 0 \leqslant u \leqslant d_n,\ n > m.$$

Similarly, we get the following inequality

(1.2.21) $$\Phi_n(u) \leqslant \bar{\Phi}_n(u)+\Phi_n(c_n) \quad \text{for } 0 \leqslant u \leqslant d_n,\ n > m.$$

Note that $0 < \bar{\Phi}_n(u) < \infty$ for all $u > 0$, $n \in N$, and $\delta_1 = \inf_{n > n_1} \bar{\Phi}_n(d_n) > 0$ for some $n_1 \in N$, by (1.2.21). So there exists a sequence (d'_n) such that $\bar{\Phi}_n(d'_n) = \delta_1$ for all $n \in N$. Let $0 < d''_n \leqslant d'_n$ be such that $\Phi_n(d''_n) < \infty$ for $n \leqslant n_1$. Put $\delta_2 = \min\{\delta_1, \inf_{n \leqslant n_1} \bar{\Phi}_n(d''_n)\}$ and $e_n = \Phi_n(c_n)$ for $n > n_1$ and $e_n = \Phi_n(d''_n)$ for $n \leqslant n_1$. Then $\Phi_n(u) \leqslant \bar{\Phi}_n(u)+e_n$ if $\bar{\Phi}_n(u) \leqslant \delta_2$ for all $n \in N$. So $\Phi \dashv \bar{\Phi}$, because $\sum_{n > n_1} \Phi_n(c_n) \leqslant \sum_{n > n_1} c_n \varphi_n(c_n) < \infty$, by the assumption. Let $d_n = \sup\{u > 0\colon \Phi_n(u) \leqslant \delta\}$ for $n < m$. We have $\bar{\Phi}_n(u) \leqslant \bar{\Phi}_n(d_n)$ for $0 \leqslant u \leqslant d_n$. Hence and from (1.2.20) it is seen that $\bar{\Phi} \dashv \Phi$ if $\sum_{n > m} \bar{\Phi}_n(c_n) < \infty$. But, by the definition of $\bar{\Phi}_n$ and by (1.2.13), we get

$$\sum_{n>m} \bar{\Phi}_n(c_n) = \sum_{n>m} \sum_{i \geqslant 1} \int_{l^{-i}c_n}^{l^{-i+1}c_n} l^{-i} k^{-i} \varphi_n(l^i t)\,dt$$
$$= \sum_{n>m} c_n \varphi_n(lc_n) \sum_{i \geqslant 1} l^{-i} k^{-i} (l-1)/l^i < \infty.$$

By (1.2.19) it is evident that $\bar{\Phi} = (\bar{\Phi}_n)$ and $\bar{\varphi} = (\bar{\varphi}_n)$ satisfies the thesis of our lemma.

Remark. If $\Phi = (\Phi_n)$ is an Orlicz function satisfying the δ_2^0-condition, then the function $\bar{\Phi} = (\bar{\Phi}_n)$ defined by: $\bar{\Phi}_n(u) = u$ for all $n < m$ and $u \geqslant 0$; $\bar{\Phi}_n(u) = \frac{1}{2}(\Phi_n(u)/c_n^2)u^2$ for $0 \leqslant u \leqslant c_n$, $n \geqslant m$; $\bar{\Phi}_n(u) = \Phi_n(u) - \frac{1}{2}\Phi_n(c_n)$ for $c_n \leqslant u \leqslant d_n$, $n \geqslant m$ and $\bar{\Phi}_n(u) = \left[(\Phi_n(2d_n) - \frac{1}{2}\Phi_n(c_n))/d_n^2\right]u^2 - [\Phi_n(2d_n) - \Phi_n(d_n)]$ for $u \geqslant d_n$, $n \geqslant m$, where c_n and d_n are such that $\Phi_n(d_n) = \delta$, $\sum_{n=m}^{\infty} \Phi_n(c_n) < \infty$, is an Orlicz function satisfying the uniform δ_2-condition and equivalent to Φ.

THEOREM 1.2.5. *Let $\Phi = (\Phi_n)$ and $\Phi^* = (\Phi_n^*)$ be complementary Orlicz functions satisfying the δ_2^0-condition. Then there exists an Orlicz function $\bar{\Phi}$ equivalent to Φ and such that $(l_{\bar{\Phi}}, \|\cdot\|_{\bar{\Phi}})$ is a uniformly convex space.*

Proof. Since Φ^* satisfies the δ_2^0-condition, so there exists an Orlicz function $\tilde{\Phi}^*$ equivalent to Φ^* satisfying the uniform δ_2-condition (see the previous lemma). Thus there exists a constant $k > 0$ such that $\tilde{\varphi}_n^*(2s) \leqslant k\tilde{\varphi}_n^*(s)$ for every $s \geqslant 0$, where $\tilde{\varphi}_n^*$ denotes the right derivative of $\tilde{\Phi}_n^*$ (see Lemma 1.2.1). By Lemma 1.1 we have

(1.2.22) $$\tilde{\varphi}_n(ku) \geqslant 2\tilde{\varphi}_n(u) \quad \text{for every } u \geqslant 0,$$

where $\tilde{\varphi}_n(u) = \sup\{s \geqslant 0\colon \tilde{\varphi}_n^*(s) \leqslant u\}$. Let

$$\tilde{\Phi}_n(u) = \int_0^u \tilde{\varphi}_n(v)\,dv \quad \text{for every } u \geqslant 0,\ n \in N.$$

Then $\tilde{\Phi} = (\tilde{\Phi}_n)$ and $\tilde{\Phi}^* = (\tilde{\Phi}_n^*)$ are complementary Orlicz functions. Since $\tilde{\Phi}^* \sim \Phi^*$, we get $\tilde{\Phi} \sim \Phi$ (see Lemma 1.2.3).

We shall find an Orlicz function $\bar{\Phi}$ equivalent to $\tilde{\Phi}$ and satisfying the thesis of our theorem. Since Φ satisfies the δ_2^0-condition and $\tilde{\Phi} \sim \Phi$, by Lemma 1.2.2, $\tilde{\Phi}$ satisfies the δ_2^0-condition. Applying Lemma 1.2.4 we shall find an Orlicz function $\hat{\Phi} \sim \tilde{\Phi}$, $\hat{\Phi}$ satisfying the uniform δ_2-condition, and such that there exist $l,\ r > 1$ satisfying for every $u \geqslant 0$ the inequality

$$\hat{\varphi}_n(lu) \geqslant r\hat{\varphi}_n(u), \tag{1.2.23}$$

where $\hat{\varphi}_n$ is the right derivative of $\hat{\Phi}_n$.

Let

$$u_k = l^k, \qquad k \in Z$$

and a function $\bar{\varphi}_n$ be defined in the following way: $\bar{\varphi}_n(u)$ is continuous on R_+, $\bar{\varphi}_n(0) = 0$, $\bar{\varphi}_n(u_k) = \hat{\varphi}_n(u_k)$ and $\bar{\varphi}_n$ is linear on each interval $[u_k, u_{k+1}]$. We shall prove that

$$\bar{\varphi}_n\big((1+\varepsilon)u\big) \geqslant k_\varepsilon\, \bar{\varphi}_n(u)$$

for arbitrary $\varepsilon > 0$, some constant $k_\varepsilon > 1$ and all $n \in N$, $u \in R_+$. The proof of this formula will be omitted, because it is analogous to that of inequality (1.1.3). Let us note only that the roles of the parameter t, the function $\varphi_2(t, \cdot)$ and condition (1.1.1) in the proof of (1.1.3) are played here by n, $\bar{\varphi}_n$ and inequality (1.1.23), respectively. Thus

$$k_\varepsilon = 1 + [(r-1)]\,\varepsilon/(l-1)\min(1,\ l/r),$$

where l, r are constants from (1.2.23).

Hence it follows that $\bar{\Phi} = (\bar{\Phi}_n)$, where $\bar{\Phi}_n(u) = \int_0^u \bar{\varphi}_n(v)\,dv$ for all $n \in N$, $u \geqslant 0$, is an Orlicz function uniformly convex on R_+.

Now, we shall prove that the function $\bar{\Phi}$ is equivalent to Φ and that $\bar{\Phi}$ satisfies the uniform δ_2-condition.

Let $u > 0$ be arbitrary. There exists an index $k \in Z$ such that $v_k < u \leqslant v_{k+1}$. Then, we have for any $n \in N$

$$\begin{aligned}\hat{\varphi}_n(l^{-1}u) &\leqslant \hat{\varphi}_n(l^{-1}u_{k+1}) = \hat{\varphi}_n(u_k) = \bar{\varphi}_n(u_k) \leqslant \bar{\varphi}_n(u) \leqslant \bar{\varphi}_n(u_{k+1})\\ &= \hat{\varphi}_n(u_{k+1}) = \hat{\varphi}_n(lu_k) \leqslant \hat{\varphi}_n(lu).\end{aligned}$$

Hence, we have for any $n \in N$, $u \geqslant 0$:

$$\hat{\varphi}_n(l^{-1}u) \leqslant \bar{\varphi}_n(u) \leqslant \hat{\varphi}_n(lu).$$

This implies that $\bar{\Phi} \sim \hat{\Phi}$. Since $\hat{\Phi} \sim \Phi$, so $\bar{\Phi} \sim \Phi$. Moreover, the Orlicz function $\hat{\Phi}$ satisfies the uniform δ_2-condition. Hence and from the last inequalities follows that $\bar{\Phi}$ also satisfies the uniform δ_2-condition.

Now, applying Theorem 0.1, we obtain that the space $(l_{\bar{\Phi}}, \|\cdot\|_{\bar{\Phi}})$ is uniformly convex, and the proof is finished.

COROLLARY 1.2.6. *It follows from Theorems* 1.1.7 *and* 1.2.5 *that if* (T, Σ, μ) *is a mixed measure space and* Φ, Φ^* *are complementary Orlicz functions satisfying the* (Δ_2, δ_2)*-condition, then there exists an Orlicz function* Φ_2 *such that* $\Phi_2 \sim \Phi$ *and* $(L_{\Phi_2}, \|\cdot\|_{\Phi_2})$ *is a uniformly convex space.*

Remark 1.2.7. If Φ and Φ^* are complementary Orlicz functions and Φ^* satisfies the δ_2-condition ((Δ_2, δ_2)-condition) in the case of a purely atomic (a mixed) measure μ, then there exists an Orlicz function $\Phi_2 \sim \Phi$, Φ_2 uniformly convex on R_+.

This follows from the proof of the last theorem and from Lemma 1.1.5.

1.3. *B*-convexity.

THEOREM 1.3.1. *Let* Φ *and* Φ^* *be complementary Orlicz functions. Then the spaces* $(L^\mu_\Phi, \|\cdot\|_\Phi)$ *and* $(L^\mu_{\Phi^*}, \|\cdot\|_{\Phi^*})$ *are B-convex if and only if* Φ *and* Φ^* *satisfy the* Δ_2*-condition* (δ_2*-condition*) [(Δ_2, δ_2)*-condition*], *respectively, in the case of an atomless* (*a purely atomic*) [*a mixed*] *measure* μ.

Proof. We prove this theorem only in the case of an atomless measure μ. The proof in other two cases is analogous. Let Φ and Φ^* satisfy the Δ_2-condition. Then there exist Orlicz functions Φ_2 and Φ_3 equivalent to Φ and Φ^*, respectively, and such that the spaces $(L^\mu_\Phi, \|\cdot\|_{\Phi_2})$ and $(L^\mu_\Phi, \|\cdot\|_{\Phi_3})$ are uniformly convex and thus they are B-convex (see [4] Example 3 (ii), p. 118). Since the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_{\Phi^*}$ are equivalent to the norms $\|\cdot\|_{\Phi_2}$ and $\|\cdot\|_{\Phi_3}$, respectively, so by Theorem 5, p. 129, [4], the spaces $(L^\mu_\Phi, \|\cdot\|_\Phi)$ and $(L^\mu_{\Phi^*}, \|\cdot\|_{\Phi^*})$ are B-convex.

Conversely, let us suppose Φ or Φ^* does not satisfy the Δ_2-condition. We shall prove that the space $(L^\mu_\Phi, \|\cdot\|_\Phi)$ is not B-convex. For, we shall consider two cases.

(I) Φ does not satisfy the Δ_2-condition. Then (see [7])[(4)] the space $(L^\mu_\Phi, \|\cdot\|_\Phi)$ contains an isometric copy of l^∞. Since the space l^∞ is not B-convex (see [4], Example 3 (iv)), $(L^\mu_\Phi, \|\cdot\|_\Phi)$ is not B-convex too.

(II) Φ satisfies and Φ^* does not satisfy the Δ_2-condition. Then the dual space $(L^\mu_\Phi)^*$ of L^μ_Φ is isomorphic to the space $L^\mu_{\Phi^*}$. Since, by the case (I) the space $L^\mu_{\Phi^*}$ is not B-convex, so (see [4], Theorem 3, p. 127) the space L^μ_Φ is not B-convex too.

COROLLARY. 1.3.2. *Let* Φ *be an Orlicz function. The following conditions are equivalent*:

(i) Φ *and* Φ^* *satisfy the* Δ_2*-condition* (δ_2*-condition*) [(Δ_2, δ_2)*-condition*] *in the case of an atomless* (*a purely atomic*) [*a mixed*] *measure* μ, *respectively,*

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(4) In the case of a purely atomic measure μ, see [9].

(ii) L^{μ}_{Φ} *is reflexive,*

(iii) L^{μ}_{Φ} *is uniformly convexifiable,*

(iv) L^{μ}_{Φ} *is B-convex.*

Proof. The space L^{μ}_{Φ} is reflexive iff Φ and Φ^* satisfy the Δ_2-condition (δ_2-condition) [(Δ_2, δ_2)-condition] in the case of an atomless (a purely atomic) [a mixed] measure space (T, Σ, μ). This follows from the results of papers [7], [9] and [12]. Next, applying the results of this paper, we obtain our corollary.

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