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Some generic properties in convex and non-convex optimization theory

0. Introduction. In this paper we consider minimization problems for classes of real valued lower semicontinuous functionals

$$g\colon X \to \boldsymbol{R}. \tag{0.1}$$

We study also problems $[f, p]$ of best approximation of the range of a continuous mapping

$$f\colon X \to E, \tag{0.2}$$

(E a Banach space) to a point $p \in E$. The hypotheses made on the domain X do not ensure in general the existence of solutions. This is the case if g is in $\mathscr{S}(X)$, the space of all functionals which are bounded from below and lower semicontinuous on X (here X is an open or closed set in a metric space) or if g is in $\mathscr{V}(X)$, the space of all $g \in \mathscr{S}(X)$ which are convex and coercive on a Banach space X. Under the metric $d(g_1, g_2) = \sup\limits_{X} |g_1(x) - g_2(x)|/(1 + |g_1(x) - g_2(x)|)$ such spaces are complete and so they are Baire spaces.

In such framework we study the "pathological" set of all functions for which the minimization problem is not well posed in the sense of Tihonov, (that is there is failure of existence or uniqueness or continuous dependence of solutions), and we prove that in the corresponding Baire space this set is small, namely of Baire first category. This implies in particular that for most g's, that is for all g in a dense G_δ-set, the minimization problem is actually well posed.

Similar problems are studied for other classes of convex and non-convex functionals and for problems of best approximation $[f, p]$.

This note develops certain ideas contained in [1], [3], [7]. For further developments, see Kenderov [5] and Penot [9].

Notations and some basic lemmas, useful in the study of optimization problems for mappings (0.2), are given in Section 1. In Section 2 (resp. 3) we consider generic properties concerning best approximation problems for mappings (0.2) defined on a closed (resp. open) set. Generic properties concerning the minimization of non-convex (resp. convex) functionals (0.1) are established in Section 4 (resp. 5). For a pathological example see Lions ([6], p. 95).

1. Notations and lemmas. Let A be a subset of a metric space $\mathscr{X}$. We denote by:

$\bar{A}$	the closure of A,
$\operatorname{int} A$	the interior of A,
∂A	the boundary of A,
$\operatorname{dist}(A, p)$	the infimum over A of the distances from $a \in A$ to a point $p \in \mathscr{X}$,
$\operatorname{diam} A$	the diameter of A,
$\boldsymbol{N}$	the set of positive integers,
$\boldsymbol{R}$	the set of real numbers,
$S_{\mathscr{X}}(x, r)$	the open ball in $\mathscr{X}$ with center at x and radius $r > 0$.

Let E be a real Banach space, with norm $\|\cdot\|$, and Y a complete metric space. Let X be a closed (or open) subset of Y with positive diameter and let C be a non-empty bounded closed convex body contained in E.

We denote by $\mathscr{M}(X, C)$ the set of all continuous mappings $f\colon X \to C$. This set, endowed with distance

$$d(f, g) = \sup_X \|f(x) - g(x)\|, \quad f, g \in \mathscr{M}(X, C),$$

is a complete metric space.

Occasionally, when the clarity is not affected, we shall write $\mathscr{M}$ instead of $\mathscr{M}(X, C)$. In the sequel analogous abbreviations will be introduced without comment.

For any $f \in \mathscr{M}(X, C)$ and $p \in \overline{E \setminus C}$, we set

$$\lambda_{f,p} = \inf_X \|f(x) - p\|, \quad \Omega_{f,p}(\sigma) = \{x \in X \mid \|f(x) - p\| \leqslant \lambda_{f,p} + \sigma\}, \quad \sigma > 0.$$

Note that, for each $\sigma > 0$, the set $\Omega_{f,p}(\sigma)$ is non-empty and $\Omega_{f,p}(\sigma') \subset \Omega_{f,p}(\sigma)$ whenever $0 < \sigma' < \sigma$. In addition, $\Omega_{f,p}(\sigma)$ is closed (in Y) if X is so.

For $f \in \mathscr{M}(X, C)$ and $p \in \overline{E \setminus C}$ we shall consider the problem $[f, p]$ of the best approximation of the range of f to p. More precisely, we wish to find an element $x_0 \in X$ such that

$$\|f(x_0) - p\| = \lambda_{f,p}.$$

Any such $x_0 \in X$ is called a *solution* of the *optimization* (or *best approximation*) *problem* $[f, p]$.

Definition 1.1. A sequence $\{x_n\} \subset X$ is called a *minimizing sequence* of problem $[f, p]$ if and only if $\lim_{n \to \infty} \|f(x_n) - p\| = \lambda_{f,p}$.

DEFINITION 1.2. Let $f \in \mathcal{M}(X, C)$ and $p \in \overline{E \setminus C}$. The problem $[f, p]$ is said to be *well posed* if and only if it has exactly one solution, say x_0, and, moreover, each minimizing sequence of $[f, p]$ converges to x_0.

The following characterizations of well posed problems will be very useful.

LEMMA 1.1. *Let X be a closed subset of Y. Let $f \in \mathcal{M}(X, C)$ and $p \in \overline{E \setminus C}$. Then the problem $[f, p]$ is well posed if and only if*

$$\inf_{\sigma > 0} \operatorname{diam} \Omega_{f,p}(\sigma) = 0. \tag{1.1}$$

Proof. This lemma is due to Furi and Vignoli [4] when $E = \boldsymbol{R}$. The same proof works in our case and, therefore, is omitted.

Remark 1.1 Suppose that X is open. Then the sets

$$X_n = X \setminus \bigcup_{x \in \partial X} S_Y(x, r/n), \qquad n \in \boldsymbol{N},$$

are closed and non-empty, provided $r > 0$ is sufficiently small, and satisfy

$$X_1 \subset X_2 \subset \ldots \qquad \text{and} \qquad \bigcup_{n=1}^{\infty} X_n = X.$$

LEMMA 1.2. *Let X be an open subset of Y. Let $f \in \mathcal{M}(X, C)$ and $p \in \overline{E \setminus C}$. Then the problem $[f, p]$ is well posed if and only if* (1.1) *is satisfied and, in addition,*

$$\Omega_{f,p}(\sigma_0) \subset X_{n_0} \tag{1.2}$$

for some $\sigma_0 > 0$ and $n_0 \in \boldsymbol{N}$.

Proof. Indeed, if we assume that (1.1) and (1.2) are satisfied ((1.2) for convenient $\sigma_0 > 0$ and $n_0 \in \boldsymbol{N}$), then a straightforward application of Cantor's theorem gives that the intersection of all closed sets $\Omega_{f,p}(\sigma)$, $0 < \sigma \leqslant \sigma_0$, consists of a unique point $\tilde{x} \in X$. Clearly $\tilde{x}$ is the unique solution of problem $[f, p]$ and any minimizing sequence converges to $\tilde{x}$. Thus $[f, p]$ is well posed.

Conversely, suppose that the problem $[f, p]$ is well posed and denote by $x_0 \in X$ its unique solution. If $\operatorname{diam} \Omega_{f,p}(\sigma)$ does not vanish as $\sigma \to 0$ there is some $\beta > 0$ and a sequence $\{x_1, x_1', x_2, x_2', \ldots\}$ (x_n, $x_n' \in \Omega_{f,p}(1/n)$) such that for each $n \in \boldsymbol{N}$ the distance of x_n and x_n' is greater than β. Since this sequence is minimizing and non-convergent, we obtain a contradiction. Thus (1.1) is true. To complete the proof, let $\delta > 0$ and $n_0 \in \boldsymbol{N}$ be such that $S_Y(x_0, \delta) \subset X_{n_0}$. Since $x_0 \in \Omega_{f,p}(\sigma)$ and (1.1) is satisfied, for some $\sigma_0 > 0$ we have

$$\Omega_{f,p}(\sigma_0) \subset S_Y(x_0, \delta) \subset X_{n_0},$$

that is (1.2) is true and the proof is complete.

Remark 1.2. If the best approximation problem $[f, p]$ is well posed, then the corresponding solution x_0 depends continuously upon f and p, that is: if, for $n \to \infty$, $f_n \to f$ and $p_n \to p$ and if the problem $[f_n, p_n]$ has a solution x_n, then $x_n \to x_0$ as $n \to \infty$.

Proof. Indeed, given $\varepsilon > 0$ let $n_0 \in N$ be such that $d(f_n, f) \leqslant \varepsilon/4$ and $\|p_n - p\| \leqslant \varepsilon/4$ for $n \geqslant n_0$. From

$$\|f(x_n) - p\| \leqslant \|f(x_n) - f_n(x_n)\| + \|f_n(x_n) - p_n\| + \|p_n - p\|$$

we obtain

$$\|f(x_n) - p\| \leqslant \lambda_{f_n, p_n} + \tfrac{1}{2}\varepsilon \leqslant \lambda_{f,p} + \varepsilon, \qquad n \geqslant n_0,$$

being $|\lambda_{f,p} - \lambda_{f_n, p_n}| \leqslant \varepsilon/2$. Thus $\{x_n\}$ is a minimizing sequence of problem $[f, p]$ and therefore it must converge to x_0. This completes the proof.

Remark 1.3. When $p \in f(X)$, the problem $[f, p]$ has a solution but this is not necessarily unique and so the problem $[f, p]$ is not well posed, in general. As a trivial example consider $f(x) \equiv \sin x$ and $p = 0$. This same example shows that, if g is a continuous and bounded function which is sufficiently near to f, then also the problem $[g, 0]$ is not well posed. In order to avoid situations of this type we have restricted our attention to problems of best approximation $[f, p]$ with $f \in \mathcal{M}(X, C)$, hence with range contained in a fixed set $C \subset E$ and with $p \in \overline{E \setminus C}$. (See also Remark 2.1 and Sections 4 and 5.)

2. Well posed best approximation problems for mappings with closed domain. Throughout this section we assume that X is a closed subset of Y with positive diameter and we study problems of best approximation $[f, p]$, where $f \in \mathcal{M} = \mathcal{M}(X, C)$ and $p \in \overline{E \setminus C}$.

THEOREM 2.1. *Let X, $\mathcal{M}$, and p be as above. Let $\mathcal{M}_0$ be the set of all $f \in \mathcal{M}$ such that the problem $[f, p]$ is well posed. Then $\mathcal{M}_0$ is a dense G_δ subset of $\mathcal{M}$.*

Proof. In order to prove the theorem it is sufficient to show that $\mathcal{M}_0$ is the countable intersection of open dense subsets of $\mathcal{M}$.

To this end, set

$$\mathcal{M}_k = \{f \in \mathcal{M} \mid \inf_{\sigma > 0} \operatorname{diam} \Omega_{f,p}(\sigma) < 1/k\}, \qquad k \in N, \tag{2.1}$$

and observe that by Lemma 1.1

$$\mathcal{M}_0 = \bigcap_{k=1}^{\infty} \mathcal{M}_k.$$

Thus the theorem is proved if each $\mathcal{M}_k$ is shown to be open and dense in $\mathcal{M}$.

$\mathcal{M}_k$ is open in $\mathcal{M}$. Indeed, suppose that $f \in \mathcal{M}_k$. Then there is $\sigma_0 > 0$

such that

$$\operatorname{diam} \Omega_{f,p}(\sigma) < 1/k \quad \text{if} \quad 0 < \sigma \leqslant \sigma_0.$$

We claim that any $g \in S_{\mathcal{M}}(f, \sigma)$, where $0 < \sigma < \sigma_0/3$, is contained in $\mathcal{M}_k$. For this purpose it is sufficient to verify that

$$\Omega_{g,p}(\sigma) \subset \Omega_{f,p}(3\sigma) \quad \text{for} \quad \text{every } g \in S_{\mathcal{M}}(f, \sigma). \tag{2.2}$$

In fact the latter inclusion implies

$$\operatorname{diam} \Omega_{g,p}(\sigma) \leqslant \operatorname{diam} \Omega_{f,p}(\sigma_0) < 1/k,$$

and so $g \in \mathcal{M}_k$. To prove (2.2), let $g \in S_{\mathcal{M}}(f, \sigma)$ and $x \in \Omega_{g,p}(\sigma)$. We have

$$\|f(x) - p\| \leqslant \|f(x) - g(x)\| + \|g(x) - p\| \leqslant \lambda_{g,p} + 2\sigma$$

and, since $|\lambda_{g,p} - \lambda_{f,p}| \leqslant \sigma$,

$$\|f(x) - p\| \leqslant \lambda_{f,p} + 3\sigma.$$

This implies that $x \in \Omega_{f,p}(3\sigma)$ and completes the proof that $\mathcal{M}_k$ is open.

$\mathcal{M}_k$ is dense in $\mathcal{M}$. To see this, consider any $f \in \mathcal{M}$ and fix $\varepsilon > 0$ such that $\varepsilon < 2 \operatorname{diam} C$.

Take a point $q \in \operatorname{int} C$ and a number t such that $1 - \varepsilon/(2 \operatorname{diam} C) < t < 1$ and define

$$\tilde{f}(x) = q + t(f(x) - q), \qquad x \in X.$$

It is evident that $\tilde{f} \in \mathcal{M}$ and $d(\tilde{f}, f) \leqslant \varepsilon/2$. Next, we choose $0 < \delta < \varepsilon/2$ such that $S_E(0, \delta) \subset (1-t)(C-q)$ and we observe that

$$\tilde{f}(X) + S_E(0, \delta) \subset C. \tag{2.3}$$

Now, let us fix any point $x_0 \in X$ satisfying

$$\|\tilde{f}(x_0) - p\| \leqslant \lambda_{\tilde{f},p} + \delta/2. \tag{2.4}$$

From (2.3) it follows that $\|\tilde{f}(x_0) - p\| \geqslant \delta$. By Dugundji's theorem ([2], p. 188) there is a continuous mapping $h\colon X \to E$ such that $\|h(x)\| \leqslant \delta$, $x \in X$,

$$h(x_0) = -\delta \frac{\tilde{f}(x_0) - p}{\|\tilde{f}(x_0) - p\|}, \tag{2.5}$$

$$h(x) = 0 \quad \text{for } x \in X \setminus S_X(x_0, (1/4k)). \tag{2.6}$$

(Taking a larger k, if necessary, we can always assume that the last set is non-empty.) Hence we define

$$g(x) = \tilde{f}(x) + h(x), \qquad x \in X,$$

and we note that $g \in \mathcal{M}$ and $d(g,f) < \varepsilon$, thus $g \in S_{\mathcal{M}}(f, \varepsilon)$. Furthermore, as we shall see,

$$\inf_{\sigma > 0} \operatorname{diam} \Omega_{g,p}(\sigma) \leqslant 1/(2k) \tag{2.7}$$

and so $g \in \mathcal{M}_k$. In fact,

$$\begin{aligned} \|g(x_0)-p\| &= \left\| \tilde{f}(x_0) - \delta \frac{\tilde{f}(x_0)-p}{\|\tilde{f}(x_0)-p\|} - p \right\| \\ &= \left\| (\tilde{f}(x_0)-p)\left(1 - \frac{\delta}{\|\tilde{f}(x_0)-p\|}\right) \right\| \\ &= \|\tilde{f}(x_0)-p\| - \delta \end{aligned}$$

and, by virtue of (2.4),

$$\|g(x_0)-p\| \leqslant \lambda_{\tilde{f},p} - \delta/2$$

from which

$$\lambda_{g,p} \leqslant \lambda_{\tilde{f},p} - \delta/2. \tag{2.8}$$

On the other hand, if $x \in X \setminus S_X(x_0, 1/(4k))$, we have

$$\|g(x)-p\| = \|\tilde{f}(x)-p\| \geqslant \lambda_{\tilde{f},p}. \tag{2.9}$$

Let $0 < \sigma_0 < \delta/2$. By virtue of (2.8), for each $x \in \Omega_{g,p}(\sigma_0)$ we have

$$\|g(x)-p\| \leqslant \lambda_{g,p} + \sigma_0 \leqslant \lambda_{\tilde{f},p} - \tfrac{1}{2}\delta + \sigma_0 < \lambda_{\tilde{f},p}.$$

From this and (2.9) it follows that

$$\Omega_{g,p}(\sigma_0) \subset S_X(x_0, 1/(4k))$$

and so (2.7) is true. Then $g \in \mathcal{M}_k$ and $\mathcal{M}_k$ is dense in $\mathcal{M}$. This completes the proof.

Using the preceding argument we can prove the following

Remark 2.1. Let X be a closed subset of Y. Let $f\colon X \to E$ be continuous and bounded and let $p \in E$ satisfy $\operatorname{dist}(f(X), p) > 0$. Then there is $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \leqslant \varepsilon_0$ and for almost all (in the sense of the Baire category) continuous and bounded mappings $\varphi\colon X \to E$ such that $d(\varphi, f) \leqslant \varepsilon$, the problem of best approximation $[\varphi, p]$ is well posed.

THEOREM. 2.2. *Let X, $\mathcal{M}$, and $\mathcal{M}_0$ be as in Theorem 2.1. In addition, suppose that X is connected. Then the set $\mathcal{M} \setminus \mathcal{M}_0$ is dense in $\mathcal{M}$.*

Proof. We want to prove that for every $f \in \mathcal{M}$ and any $\varepsilon > 0$ there is a $g \in \mathcal{M} \setminus \mathcal{M}_0$ such that $d(g,f) < \varepsilon$.

As in the proof of Theorem 2.1 we take a convenient δ, $0 < \delta < \varepsilon/2$, and we construct a mapping $\tilde{f} \in \mathcal{M}$ with $d(\tilde{f}, f) \leqslant \varepsilon/2$ satisfying (2.3). Suppose that $\tilde{f} \in \mathcal{M}_0$ (otherwise the statement is trivially satisfied). Then there is a unique point $x_0 \in X$ such that

$$\|\tilde{f}(x_0) - p\| = \lambda_{\tilde{f},p}.$$

By Dugundji's theorem ([2], p. 188) there is a continuous mapping $h\colon X \to E$ such that $\|h(x)\| \leqslant \delta$, $x \in X$, and

$$h(x) = \lambda_{\tilde{f},p} \frac{\tilde{f}(x) - p}{\|\tilde{f}(x) - p\|} - \tilde{f}(x) + p \quad \text{for each } x \in \Omega_{\tilde{f},p}(\delta).$$

Now define $g(x) = \tilde{f}(x) + h(x)$, $x \in X$, and observe that $d(g, f) < \varepsilon$.

Since, for each $x \in X \setminus \Omega_{\tilde{f},p}(\delta)$, we have

$$\|g(x) - p\| \geqslant \|\tilde{f}(x) - p\| - \|h(x)\| > \lambda_{\tilde{f},p} + \delta - \delta = \lambda_{\tilde{f},p}$$

while, for each $x \in \Omega_{\tilde{f},p}(\delta)$,

$$\|g(x) - p\| = \lambda_{\tilde{f},p}$$

we can conclude that every point in the set $\Omega_{\tilde{f},p}(\delta)$ is a solution of problem $[g, p]$. Moreover, $\Omega_{\tilde{f},p}(\delta)$ contains more that one point, being X connected, and so the problem $[g, p]$ is not well posed. This completes the proof.

Let X be a closed subset of Y with positive diameter and let D be a non-empty closed subset of $\overline{E \setminus C}$. Set $\mathfrak{M} = \mathcal{M} \times D$, where $\mathcal{M} = \mathcal{M}(X, C)$, and observe that $\mathfrak{M}$, endowed with metric

$$\max\{d(f, g), \|x - y\|\}, \quad (f, x), (g, y) \in \mathfrak{M},$$

is a complete metric space.

Lemma 2.1. *Suppose that the problem $[g, q]$, $(g, q) \in \mathfrak{M}$, is well posed. Then, for every $\varepsilon > 0$ there exists $\delta_{g,q}(\varepsilon) > 0$ such that*

$$\operatorname{diam} \Omega_{f,p}(\delta_{g,q}(\varepsilon)) \leqslant \varepsilon \quad \textit{for each } (f, p) \in S_{\mathfrak{M}}((g, q), \delta_{g,q}(\varepsilon)).$$

Proof. Let x_0 be the unique solution of problem $[g, q]$. Let $\varepsilon > 0$. By Lemma 1.1 there is $\sigma_0 > 0$ such that

$$\Omega_{g,q}(\sigma) \subset S_X(x_0, \varepsilon/2) \quad \text{for every } 0 < \sigma \leqslant \sigma_0. \tag{2.10}$$

Let $0 < \delta < \sigma_0/5$. Let $(f, p) \in S_{\mathfrak{M}}((g, q), \delta)$. Then, for each $x \in \Omega_{f,p}(\delta)$, we have

$$\begin{aligned} \|g(x) - q\| &\leqslant \|g(x) - f(x)\| + \|f(x) - p\| + \|p - q\| \\ &< \delta + \lambda_{f,p} + \delta + \delta \leqslant \lambda_{g,q} + 5\delta \end{aligned}$$

and so $x \in \Omega_{g,q}(5\delta)$. Hence $\Omega_{f,p}(\delta) \subset \Omega_{g,q}(\sigma_0)$. Then, setting $\delta_{g,q}(\varepsilon) = \delta$, from the last inclusion by virtue of (2.10), we obtain the statement. This completes the proof.

Lemma 2.2. *Suppose that D is a separable and closed subset of $\overline{E \setminus C}$. Denote by $\mathfrak{M}_0$ the set of all $(f, p) \in \mathfrak{M}$ such that the problem $[f, p]$ is well posed. Then $\mathfrak{M}_0$ contains a dense G_δ subset of $\mathfrak{M}$.*

Proof. Let $\Sigma = \{q_1, q_2, \ldots\}$ be a dense subset of D. By Theorem 2.1, for each $q_k \in \Sigma$ there is a dense G_δ subset $\mathcal{M}_k$ of $\mathcal{M}$ such that the problem $[f, q_k]$ is well posed for each $f \in \mathcal{M}_k$. Set

$$\hat{\mathcal{M}} = \bigcap_{k=1}^{\infty} \mathcal{M}_k.$$

Obviously $\hat{\mathcal{M}}$ is a dense G_δ subset of $\mathcal{M}$ and, for every $f \in \hat{\mathcal{M}}$ and every $q \in \Sigma$, the problem $[f, q]$ is well posed.

Define

$$\mathfrak{M}_* = \bigcap_{k=1}^{\infty} \bigcup_{(g,q) \in \hat{\mathcal{M}} \times \Sigma} S_{\mathfrak{M}}\big((g, q), \delta_{g,q}(1/k)\big),$$

where $\delta_{g,q}(1/k)$ corresponds to $[g, q]$ and k, according to Lemma 2.1. We observe that $\mathfrak{M}_*$ is a dense G_δ subset of $\mathfrak{M}$. To finish the proof it is enough to verify that $\mathfrak{M}_* \subset \mathfrak{M}_0$.

In fact, let $(f, p) \in \mathfrak{M}_*$. Then, for every $k \in N$ there exists $(g_k, q_k) \in \hat{\mathcal{M}} \times \Sigma$ such that

$$(f, p) \in S_{\mathfrak{M}}\big((g_k, q_k), \delta_{g_k,q_k}(1/k)\big).$$

By Lemma 2.1

$$\operatorname{diam} \Omega_{f,p}\big(\delta_{g_k,q_k}(1/k)\big) \leqslant 1/k,$$

which implies

$$\inf_{\sigma > 0} \operatorname{diam} \Omega_{f,p}(\sigma) = 0.$$

Then, by Lemma 1.1, problem $[f, p]$ is well posed and so $(f, p) \in \mathfrak{M}_0$. This completes the proof.

As an immediate consequence of Lemma 2.2 and a theorem of Kuratowski and Ulam ([8], p. 56) we have the following

THEOREM 2.3. *Let X be a closed subset of Y. Let D be a separable and closed subset of $\overline{E \setminus C}$. Then there exists a dense G_δ subset $\mathcal{M}_0$ of $\mathcal{M}(X, C)$ such that for every $f \in \mathcal{M}_0$ the problem $[f, p]$ is well posed for each $p \in D_f$, where D_f is a dense G_δ subset of D (depending on f).*

3. Well posed best approximation problems for mappings with open domain. In this section X denotes a non-empty open subset of Y and $\{X_n\}$ the sequence of closed sets, associated with X, defined in Remark 1.1.

We shall study problems of best approximation $[f, p]$, where $f \in \mathcal{M} = \mathcal{M}(X, C)$ and $p \in \overline{E \setminus C}$. Let us define

$$\tilde{\mathcal{M}} = \{f \in \mathcal{M} \mid \Omega_{f,p}(\sigma) \subset X_n \text{ for some } \sigma > 0 \text{ and some } n \in N\}. \tag{3.1}$$

Note that for each $f \in \tilde{\mathcal{M}}$ there is $\sigma_f > 0$ such that $\Omega_{f,p}(\sigma)$ is closed (in Y) for $0 < \sigma \leqslant \sigma_f$.

LEMMA 3.1. *Let X be an open subset of Y. Let $p \in \overline{E \setminus C}$. Then $\tilde{\mathcal{M}}$ is an open dense subset of $\mathcal{M} = \mathcal{M}(X, C)$.*

Proof. First of all we show that $\tilde{\mathcal{M}}$ is open. Let $f \in \tilde{\mathcal{M}}$. Let $\sigma_0 > 0$ and $n_0 \in N$ be such that $\Omega_{f,p}(\sigma_0) \subset X_{n_0}$. We claim that for $0 < \sigma < \sigma_0/3$, the ball $S_{\mathcal{M}}(f, \sigma)$ is contained in $\tilde{\mathcal{M}}$. Indeed, let $x \in \Omega_{g,p}(\sigma)$, where $g \in S_{\mathcal{M}}(f, \sigma)$. We have

$$\|f(x)-p\| \leqslant \|f(x)-g(x)\|+\|g(x)-p\| < \lambda_{g,p}+2\sigma \leqslant \lambda_{f,p}+3\sigma,$$

that is $x \in \Omega_{f,p}(3\sigma)$. Hence $\Omega_{g,p}(\sigma) \subset \Omega_{f,p}(\sigma_0) \subset X_{n_0}$, for every $g \in S_{\mathcal{M}}(f, \sigma)$, and so $\tilde{\mathcal{M}}$ is open.

In order to prove that $\tilde{\mathcal{M}}$ is dense in $\mathcal{M}$, consider any ball $S_{\mathcal{M}}(f, \varepsilon)$, where $f \in \mathcal{M}$ and $0 < \varepsilon < 2 \operatorname{diam} C$.

Let $q \in \operatorname{int} C$. Let t be such that $1-\varepsilon/(2 \operatorname{diam} C) < t < 1$. Define $\tilde{f}$ and x_0 as in Theorem 2.1. Let $\tau > 0$ be such that $S_Y(x_0, 2\tau) \subset X$.

By Dugundji's theorem ([2], p. 188) there is a continuous mapping $h\colon X \to E$ satisfying (2.5), (2.6) (for each $x \in X \setminus S_Y(x_0, r)$), and such that $\|h(x)\| \leqslant \delta$, $x \in X$ $(0 < \delta < \varepsilon/2)$. Then, setting $g(x) = \tilde{f}(x)+h(x)$, $x \in X$, we have $g \in \mathcal{M}$ and $d(g,f) < \varepsilon$. As in the proof of Theorem 2.1 one can verify that

$$\Omega_{g,p}(\sigma_0) \subset S_Y(x_0, \tau) \quad \text{if} \quad 0 < \sigma_0 < \delta/2.$$

Let n_0 be large enough so that $S_Y(x_0, \tau) \subset X_{n_0}$. Then we have

$$\Omega_{g,p}(\sigma_0) \subset X_{n_0}$$

which implies that $g \in \tilde{\mathcal{M}}$. This completes the proof.

THEOREM 3.1. *Let X be an open subset of Y. Let $p \in \overline{E \setminus C}$. Let $\mathcal{M}_0$ be the set of all $f \in \mathcal{M} = \mathcal{M}(X, C)$ such that the problem $[f, p]$ is well posed. Then $\mathcal{M}_0$ is a dense G_δ subset of $\mathcal{M}$.*

Proof. Using the argument of Theorem 2.1 one can prove that each set $\mathcal{M}_k$ defined by (2.1) is open and dense in $\mathcal{M}$. By Lemma 3.1 the same is true for $\tilde{\mathcal{M}}$. Since, by Lemma 1.2,

$$\mathcal{M}_0 = \tilde{\mathcal{M}} \cap \Big(\bigcap_{k=1}^{\infty} \mathcal{M}_k\Big)$$

we can conclude that $\mathcal{M}_0$ is a dense G_δ subset of $\mathcal{M}$. This completes the proof.

Remark 3.1. The statement of Remark 2.1 remains true also when X is supposed to be open.

4. Well posed minimization problems for nonconvex functionals. In this section we study minimization problems for lower semicontinuous (non-convex) functionals defined on a subset of a complete metric space Y.

Let X be a non-empty subset of Y with positive diameter. Denote by $\mathscr{F} = \mathscr{F}(X)$ the set of all functionals $f\colon X \to \boldsymbol{R}$ which are bounded from below. For $f, g \in \mathscr{F}$ we put

$$d_0(f, g) = \sup_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|},$$

$$\lambda_f = \inf_X f(x), \qquad \Omega_f(\sigma) = \{x \in X \mid f(x) \leqslant \lambda_f + \sigma\}, \qquad \sigma > 0.$$

Observe that d_0 is a distance on $\mathscr{F}$; moreover, if X is closed and f lower semicontinuous, the set $\Omega_f(\sigma)$ is closed (in Y). For any $f \in \mathscr{F}$ we want to establish whether f has minimum on X. We call this a *minimization problem.* We say that a minimization problem is *well posed* if there exists exactly one point $x_0 \in X$ such that $f(x_0) = \lambda_f$ and, moreover, any minimizing sequence converges to x_0.

Let X be a closed (or open) subset of Y with positive diameter. We denote by $\mathscr{S}(X)$ the set of all $f \in \mathscr{F}(X)$ such that f is lower semicontinuous. Note that $\mathscr{S}(X)$ becomes a complete metric space under the distance d_0.

LEMMA 4.1. *Let X be a non-empty closed (resp. an open) subset of Y. Let $f \in \mathscr{S}(X)$. Then the minimization problem for f is well posed if and only if*

$$\inf_{\sigma > 0} \operatorname{diam} \Omega_f(\sigma) = 0 \tag{4.1}$$

(*resp. if and only if* (4.1) *is satisfied and, in addition, $\Omega_f(\sigma_0) \subset X_{n_0}$ for some $\sigma_0 > 0$ and $n_0 \in \boldsymbol{N}$, where the sequence $\{X_n\}$ corresponds to X according to Remark* 1.1).

Proof. This lemma, when X is closed, is established in [7]; when X is open is proved as Lemma 1.2.

THEOREM 4.1. *Let X be a non-empty open subset of Y. Let $\mathscr{S}_0$ be the set of all $f \in \mathscr{S} = \mathscr{S}(X)$ for which the minimization problem is well posed. Then $\mathscr{S}_0$ is a dense G_δ subset of $\mathscr{S}$.*

Proof. Let $\{X_n\}$ be the sequence of closed sets, corresponding to X, defined in Remark 1.1.

Denote by $\tilde{\mathscr{S}}$ the set of all $f \in \mathscr{S}$ such that $\Omega_f(\sigma) \subset X_n$ for some $\sigma > 0$ and some $n \in \boldsymbol{N}$.

$\tilde{\mathscr{S}}$ is an open dense subset of $\mathscr{S}$. The fact that $\tilde{\mathscr{S}}$ is open is proved using the argument of Lemma 3.1. To see that $\tilde{\mathscr{S}}$ is dense, fix any $f \in \mathscr{S}$ and $\varepsilon > 0$. Choose $x_0 \in \Omega_f(\varepsilon/3)$ and define

$$g(x) = \begin{cases} f(x_0) - \frac{2}{3}\varepsilon, & \text{if } x = x_0, \\ f(x), & \text{if } x \neq x_0. \end{cases}$$

Obviously, $g \in \tilde{\mathscr{S}}$ and $d_0(g, f) < \varepsilon$, thus the density is proved.

By a similar method one can show that the sets $\mathscr{S}_k = \{f \in \mathscr{S} \mid \inf_{\sigma > 0} \operatorname{diam} \Omega_f(\sigma) < 1/k\}$, $k \in \boldsymbol{N}$, are open and dense in $\mathscr{S}$.

By Lemma 4.1, $\mathscr{S}_0 = \tilde{\mathscr{F}} \cap (\bigcap_{k=1}^{\infty} \mathscr{S}_k)$ and so $\mathscr{S}_0$ is a dense G_δ subset of $\mathscr{S}$. This completes the proof.

Let X be a closed (or open) subset of Y with positive diameter. Denote by $\mathscr{C}(X)$ the set of all $f \in \mathscr{F}(X)$ which are continuous. Observe that $\mathscr{C}(X)$ is made into a complete metric space under the distance d_0.

THEOREM 4.2. *Let X be a non-empty open subset of Y. Let $\mathscr{C}_0$ be the set of all $f \in \mathscr{C}(X)$ for which the minimization problem is well posed. Then $\mathscr{C}_0$ is a dense G_δ subset of $\mathscr{C}(X)$.*

Proof. The proof can be carried out using the preceding technique.

Remark 4.1. When X is closed, corresponding results have been previously established by Lucchetti and Patrone [7].

We observe also that if the minimization problem for $f \in \mathscr{S}(X)$ is well posed, then the solution depends continuously upon f, that is: if $f_n \to f$, $f_n \in \mathscr{S}(X)$, and if the minimization problem for f_n has a solution x_n, then $x_n \to x_0$ as $n \to \infty$.

5. Well posed minimization problems for convex functionals. Let F be a real Banach space with norm $\|\cdot\|$. In this section we study minimization problems for lower semicontinuous convex functionals $f\colon X \to \mathbf{R}$, with domain $X \subset F$. We consider both cases, X a closed convex bounded subset of F and $X = F$.

Let X be a closed convex and bounded subset of F with positive diameter. We denote by $\mathscr{K} = \mathscr{K}(X)$ the set of all functionals $f \in \mathscr{F} = \mathscr{F}(X)$ such that f is convex and lower semicontinuous on X. Under the distance d_0 the set $\mathscr{K}$ becomes a complete metric space.

In the proof of Theorem 5.1 we use the following lemma. The proof, quite elementary, is omitted.

LEMMA 5.1. *Let $\Phi = \{\varphi \in \mathscr{K} \mid \varphi(x) \leqslant \beta(x),\ x \in X\}$, where $\beta \in \mathscr{F}$. Then Φ is non-empty and the functional $g\colon X \to \mathbf{R}$ defined by $g(x) = \sup_{\varphi \in \Phi} \varphi(x)$, $x \in X$, is in $\mathscr{K}$.*

THEOREM 5.1. *Let X and $\mathscr{K}$ be as above. Let $\mathscr{K}_0$ be the set of all $f \in \mathscr{K}$ for which the minimization problem is well posed. Then $\mathscr{K}_0$ is a dense G_δ subset of $\mathscr{K}$.*

Proof. For every $k \in N$ we define

$$\mathscr{K}_k = \{f \in \mathscr{K} \mid \inf_{\sigma > 0} \operatorname{diam} \Omega_f(\sigma) < 1/k\}. \tag{5.1}$$

Since, by virtue of Lemma 4.1, $\mathscr{K}_0 = \bigcap_{k=1}^{\infty} \mathscr{K}_k$, the theorem is established if we show that each set $\mathscr{K}_k$ is open and dense in $\mathscr{K}$.

The fact that $\mathscr{K}_k$ is open is proved using the argument of Theorem 2.1.

To see the denseness, consider any $f \in \mathscr{K}$ and $\varepsilon > 0$. Then, choose any $x_0 \in \Omega_f(\varepsilon/4)$ and define

$$\beta(x) = \begin{cases} f(x_0) - \frac{3}{4}\varepsilon, & \text{if } x = x_0, \\ f(x), & \text{if } x \neq x_0. \end{cases} \tag{5.2}$$

Observe that β is bounded from below. Also, define

$$\alpha(x) = f(x_0) - \tfrac{3}{4}\varepsilon + \frac{\varepsilon}{4}\frac{\|x - x_0\|}{r}, \qquad x \in X,$$

where $r > 0$ is the radius of a ball, with center at x_0, which contains the (bounded) set X. It is clear that $\alpha \in \mathscr{K}$ and for this functional, the minimization problem is well posed. Moreover, for each $x \neq x_0$,

$$\alpha(x) \leqslant f(x_0) - \tfrac{1}{2}\varepsilon \leqslant \lambda_f + \tfrac{1}{4}\varepsilon - \tfrac{1}{2}\varepsilon < f(x) = \beta(x)$$

and, since $\alpha(x_0) = \beta(x_0)$, we have $\alpha(x) \leqslant \beta(x)$ for each $x \in X$, thus $\alpha \in \Phi$.

With the above choice of β, define g as in Lemma 5.1. Observe that $g \in \mathscr{K}$. Moreover,

$$\alpha(x) \leqslant g(x), \qquad x \in X, \qquad \lambda_\alpha = f(x_0) - \tfrac{3}{4}\varepsilon = \lambda_g. \tag{5.3}$$

These imply $\Omega_g(\sigma) \subset \Omega_\alpha(\sigma)$, $\sigma > 0$, from which it follows that also for g the minimization problem is well posed. Thus $g \in \mathscr{K}_k$.

Since the functional defined by $f(x) - \frac{3}{4}\varepsilon$, $x \in X$, is in Φ and $f(x) \geqslant \beta(x) \geqslant g(x)$, we have

$$f(x) \geqslant g(x) \geqslant f(x) - \tfrac{3}{4}\varepsilon, \qquad x \in X.$$

Hence $|f(x) - g(x)| \leqslant \frac{3}{4}\varepsilon$, $x \in X$, and so $d_0(g, f) < \varepsilon$. This completes the proof.

Remark 5.1. Theorem 5.1 is false when X is unbounded. To see why is so, take in Theorem 5.1 $X = \boldsymbol{R}$ and $f \in \mathscr{K}$ such that $f(x) \equiv 0$, $x \in X$. Then for this f and for each $g \in \mathscr{K}$ satisfying $d_0(g, f) \leqslant \varepsilon$ $(0 < \varepsilon < 1)$ the minimization problem is not well posed since each such g is a constant function.

We consider now the case in which the functionals $f\colon X \to \boldsymbol{R}$ are defined on the set $X = F$. Denote by $\mathscr{V} = \mathscr{V}(X)$ the set of all functionals $f \in \mathscr{F}(X)$ such that f is lower semicontinuous convex and *coercive*, that is

$$\lim_{\|x\| \to +\infty} f(x) = +\infty.$$

Under the distance d_0 the set $\mathscr{V}$ is a complete metric space.

Note that if $f \in \mathscr{F}(X)$ is coercive then, for each $\sigma > 0$, the set $\Omega_f(\sigma)$ is bounded.

The following lemma is elementary and is stated without proof.

LEMMA 5.2. *Let* $\Phi = \{\varphi \in \mathscr{V} \mid \varphi(x) \leqslant \beta(x),\ x \in X\}$, *where* $\beta \in \mathscr{F}(X)$. *Then, if the set* Φ *is non-empty, the functional* $g\colon X \to \boldsymbol{R}$ *defined by* $g(x) = \sup_{\varphi \in \Phi} \varphi(x)$, $x \in X$, *is in* $\mathscr{V}$.

THEOREM 5.2. *Let $X = F$. Let $\mathscr{V}_0$ be the set of all functionals $f \in \mathscr{V} = \mathscr{V}(X)$ for which the minimization problem is well posed. Then $\mathscr{V}_0$ is a dense G_δ subset of $\mathscr{V}$.*

Proof. As in the proof of Theorem 5.1 we introduce the sets $\mathscr{V}_k$ defined by (5.1), where $\mathscr{K}$ is replaced by $\mathscr{V}$. The openness of $\mathscr{V}_k$ is proved as in Theorem 5.1. In order to establish the denseness we fix any $f \in \mathscr{V}$, $\varepsilon > 0$ and a point $x_0 \in \Omega_f(\varepsilon/4)$. Since f is coercive there exists $r_0 > 0$ such that

$$\Omega_f(\tfrac{3}{4}\varepsilon) \subset S_X(x_0, r_0).$$

Let $r > r_0$. Then,

$$\|x - x_0\| \geqslant r \quad \text{implies} \quad f(x) > \lambda_f + \tfrac{3}{4}\varepsilon. \tag{5.4}$$

Now we define

$$\begin{aligned} \tilde{f}(x) &= \max\{f(x), \lambda_f + \tfrac{1}{4}\varepsilon\}, \qquad x \in X, \\ \gamma(x) &= \tilde{f}(x_0) + \tfrac{1}{4}\varepsilon \frac{\|x - x_0\|}{r}, \qquad x \in X, \end{aligned} \tag{5.5}$$

and we observe that $\tilde{f}$ and γ are in $\mathscr{V}$. Furthermore, the minimization problem for γ is well posed.

We claim that

$$\|x - x_0\| > r \quad \text{implies} \quad \gamma(x) \leqslant \tilde{f}(x). \tag{5.6}$$

Otherwise, there is $x_1 \in X$ with $\|x_1 - x_0\| > r$ such that $\gamma(x_1) > \tilde{f}(x_1)$. Let $\xi = tx_0 + (1-t)x_1$, $0 < t < 1$, be the unique point of the line segment of end points x_0 and x_1 satisfying $\|\xi - x_0\| = r$. Then, by the choice of x_0 and (5.4), we have

$$\gamma(\xi) = \tilde{f}(x_0) + \tfrac{1}{4}\varepsilon \leqslant \lambda_f + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon < f(\xi) = \tilde{f}(\xi).$$

On the other hand by the definition of γ and the convexity of $\tilde{f}$ we obtain

$$\gamma(\xi) = t\gamma(x_0) + (1-t)\gamma(x_1) > t\tilde{f}(x_0) + (1-t)\tilde{f}(x_1) \geqslant \tilde{f}(\xi).$$

This is a contradiction and so (5.6) is proved.

Next, we set

$$\alpha(x) = \gamma(x) - \tfrac{1}{4}\varepsilon + f(x_0) - \gamma(x_0), \qquad x \in X.$$

Clearly $\alpha \in \mathscr{V}$ and, moreover,

$$\alpha(x) \leqslant f(x), \qquad x \in X. \tag{5.7}$$

To see this, first we observe that $f(x_0) - \gamma(x_0) \leqslant 0$. Hence, if $\|x - x_0\| > r$, by virtue of (5.6) and (5.4) we have

$$\alpha(x) < \gamma(x) \leqslant \tilde{f}(x) = f(x)$$

while, if $\|x-x_0\| \leqslant r$,

$$\alpha(x) \leqslant \gamma(x) - \tfrac{3}{4}\varepsilon \leqslant \tilde{f}(x_0) + \tfrac{1}{4}\varepsilon - \tfrac{3}{4}\varepsilon = \lambda_f + \tfrac{2}{4}\varepsilon - \tfrac{3}{4}\varepsilon < f(x)$$

thus, also (5.7) is true.

Therefore, it follows that $\alpha(x) \leqslant \beta(x)$ for each $x \in X$, where β is defined by (5.2). With this choice of β we define the set Φ and the functional g as in Lemma 5.2. Since $\alpha \in \Phi$, by Lemma 5.2 we have that $g \in \mathscr{V}$. By construction g satisfies (5.3). Then from (5.3) it follows that $\Omega_g(\sigma) \subset \Omega_\alpha(\sigma)$, $\sigma > 0$, thus since for α the minimization problem is well posed, the same is true for g. This shows that $g \in \mathscr{V}_k$. The conclusion is similar to that of Theorem 5.1. This completes the proof.

Remark 5.2. It is easy to see that the functional $\tilde{f}$ defined by (5.5) has infinite points of minimum. From this and the construction of $\tilde{f}$ it follows that each functional $f \in \mathscr{V} = \mathscr{V}(X)$ can be approximated (in $\mathscr{V}$) by one, namely $\tilde{f}$, for which the minimization problem is not well posed. This shows that the set $\mathscr{V} \setminus \mathscr{V}_0$ is dense in $\mathscr{V}$.

Notice that, if X is a reflexive Banach space, for each $f \in \mathscr{V}$ the minimization problem has always a solution, though this is not necessarily unique. In a non-reflexive Banach space even existence can fail but, in view of Theorem 5.2, this occurs rather exceptionally (in the sense of the Baire category).

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