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Fuzzy maximal ideals of pseudo MV-algebras

Abstract. The notion of fuzzy maximal ideals of a pseudo MV-algebra is introduced, and its characterizations are established.

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1. Introduction. Pseudo MV-algebras have been introduced independently by G. Georgescu and A. Iorgulescu in [4] and by J. Rachůnek in [7] (here called generalized MV-algebras or, in short, GMV-algebras) and they are a non-commutative generalization of MV-algebras. The notion of fuzzy sets, introduced by L. A. Zadeh in [9], can be applied to many mathematical branches. Recently, Y. B. Jun and A. Walendziak in [6] applied the concept to pseudo MV-algebras. They introduced the notions of fuzzy ideals and fuzzy implicative ideals in a pseudo MV-algebra, gave characterizations of them and provided conditions for a fuzzy set to be a fuzzy ideal. It is well known that in studying the structure of the general algebras, the maximal ideals and the prime ideals play an important role. In [2], the author of present paper introduced the notion of fuzzy prime ideals of pseudo MV-algebras and gave many interesting characterizations of it. In this paper we investigate fuzzy ideals and fuzzy implicative ideals in Section 3. We provided the homomorphic properties of them. Section 4 is devoted to introduce and characterize the notion of fuzzy maximal ideals of a pseudo MV-algebra. We obtain in this section the homomorphic properties of fuzzy maximal ideals. The relations among fuzzy maximal ideals, fuzzy prime ideals and fuzzy implicative ideals are established. For the convenience of the reader, in Section 2 we give the relevant material needed in sequel, thus making our exposition self-contained.

2. Preliminaries. Let $A = (A, \oplus, ^-, ^\sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. Set $x \cdot y = (y^- \oplus x^-)^\sim$ for any $x, y \in A$. We consider that the operation $\cdot$ has priority to the operation $\oplus$, i.e., we will write $x \oplus y \cdot z$ instead of $x \oplus (y \cdot z)$. The algebra A is called a *pseudo MV-algebra* if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(A2) $x \oplus 0 = 0 \oplus x = x$,
(A3) $x \oplus 1 = 1 \oplus x = 1$,
(A4) $1^{\sim} = 0, 1^{-} = 0$,
(A5) $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-}$,
(A6) $x \oplus x^{\sim} \cdot y = y \oplus y^{\sim} \cdot x = x \cdot y^{-} \oplus y = y \cdot x^{-} \oplus x$,
(A7) $x \cdot (x^{-} \oplus y) = (x \oplus y^{\sim}) \cdot y$,
(A8) $(x^{-})^{\sim} = x$.

If the addition $\oplus$ is commutative, then both unary operations $^{-}$ and $^{\sim}$ coincide and A can be considered as an MV-algebra.

Throughout this paper A will denote a pseudo MV-algebra. We will write $x^{=}$ instead of $(x^{-})^{-}$ and $x^{\approx}$ instead of $(x^{\sim})^{\sim}$. For any $x \in A$ and $n = 0, 1, 2, \ldots$ we put

$$\begin{aligned} 0x &= 0 \text{ and } (n+1)x = nx \oplus x, \\ x^{0} &= 1 \text{ and } x^{n+1} = x^{n} \cdot x. \end{aligned}$$

Proposition 2.1 (Georgescu and Iorgulescu [4]) *The following properties hold for any $x, y \in A$:*
(a) $(x^{\sim})^{-} = x$,
(b) $x \oplus x^{\sim} = 1, x^{-} \oplus x = 1$,
(c) $x \cdot x^{-} = 0, x^{\sim} \cdot x = 0$,
(d) $0^{-} = 0^{\sim} = 1$,
(e) $(x^{-})^{\approx} = x^{\sim}$,
(f) $(x \oplus y)^{-} = y^{-} \cdot x^{-}, (x \oplus y)^{\sim} = y^{\sim} \cdot x^{\sim}$,
(g) $x \oplus y = (y^{\sim} \cdot x^{\sim})^{-}$.

Proposition 2.2 (Georgescu and Iorgulescu [4]) *The following properties are equivalent for any $x, y \in A$:*
(a) $x^{-} \oplus y = 1$;
(b) $y \oplus x^{\sim} = 1$.

We define

$$x \leqslant y \Longleftrightarrow x^{-} \oplus y = 1. \tag{1}$$

As it is shown in [4], $(A, \leqslant)$ is a lattice in which the join $x \vee y$ and the meet $x \wedge y$ of any two elements x and y are given by:

$$\begin{aligned} x \vee y &= x \oplus x^{\sim} \cdot y = x \cdot y^{-} \oplus y, \\ x \wedge y &= x \cdot \big(x^{-} \oplus y\big) = (x \oplus y^{\sim}) \cdot y. \end{aligned}$$

Definition 2.3 A subset I of A is called an *ideal* of A if it satisfies:
(I1) $0 \in I$,
(I2) if $x, y \in I$, then $x \oplus y \in I$,
(I3) if $x \in I$, $y \in A$ and $y \leqslant x$, then $y \in I$.

Denote by $\mathcal{I}(A)$ the set of ideals of A.

Remark 2.4 Let $I \in \mathcal{I}(A)$. If $x, y \in I$, then $x \cdot y, x \wedge y, x \vee y \in I$.

Definition 2.5 Let I be a proper ideal of A (i.e., $I \neq A$). Then
(a) I is called *prime* if, for all $I_1, I_2 \in \mathcal{I}(A)$, $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.
(b) I is called *maximal* iff whenever J is an ideal such that $I \subseteq J \subseteq A$, then either $J = I$ or $J = A$.

Proposition 2.6 (Georgescu and Iorgulescu [4]) *For $I \in \mathcal{I}(A)$, the following are equivalent:*
(a) *I is prime,*
(b) *if $x \wedge y \in I$, then $x \in I$ or $y \in I$.*

Proposition 2.7 (Walendziak [8]) *If $I \in \mathcal{I}(A)$ is maximal, then I is prime.*

Definition 2.8 An ideal I of A is called *normal* if it satisfies the condition:
(N) For all $x, y \in I$, $x \cdot y^{-} \in I \Longleftrightarrow y^{\sim} \cdot x \in I$.

Proposition 2.9 (Georgescu and Iorgulescu [4]) *Let I be a normal ideal of A. Then*

$$x \in I \Longleftrightarrow x^{\approx} \in I.$$

For every subset $W \subseteq A$, the smallest ideal of A which contains W, i.e., the intersection of all ideals $I \supseteq W$, is said to be the ideal *generated* by W, and will be denoted by $(W]$.

Proposition 2.10 (Georgescu and Iorgulescu [4]) *Let I be a normal ideal of A and $x \in A$. Then*

$$(I \cup \{x\}] = \{t \in A : t \leqslant y \oplus nx \ for\ some\ y \in I\ and\ n \in \mathbb{N}\}.$$

Following [4], for any normal ideal I of A, we define the congruence on A:

$$x \sim_I y \Longleftrightarrow x \cdot y^- \vee y \cdot x^- \in I.$$

We denote by x/I the congruence class of an element $x \in A$ and on the set $A/I = \{x/I : x \in A\}$ we define the operations:

$$x/I \oplus y/I = (x \oplus y)/I,\ (x/I)^- = \left(x^-\right)/I,\ (x/I)^\sim = (x^\sim)/I.$$

The resulting quotient algebra $A/I = (A/I, \oplus, ^-, ^\sim, 0/I, 1/I)$ becomes a pseudo MV-algebra, called *the quotient algebra of A by the normal ideal I*. Observe that for all $x, y \in A$,

$$\begin{aligned} x/I \cdot y/I &= (x \cdot y)/I, \\ x/I \vee y/I &= (x \vee y)/I, \\ x/I \wedge y/I &= (x \wedge y)/I. \end{aligned}$$

It is clear that:

$$x/I = y/I \Longleftrightarrow x \cdot y^- \vee y \cdot x^- \in I \Longleftrightarrow x^\sim \cdot y \vee y^\sim \cdot x \in I, \tag{2}$$

$$x/I = 0/I \Longleftrightarrow x \in I, \tag{3}$$

$$x/I = 1/I \Longleftrightarrow x^- \in I \Longleftrightarrow x^\sim \in I. \tag{4}$$

Definition 2.11 Let A and B be pseudo MV-algebras. A function $f : A \to B$ is a *homomorphism* if and only if it satisfies, for each $x, y \in A$, the following conditions:
(H1) $f(0) = 0$,
(H2) $f(x \oplus y) = f(x) \oplus f(y)$,
(H3) $f(x^-) = (f(x))^-$,
(H4) $f(x^\sim) = (f(x))^\sim$.

Remark 2.12 If $f : A \to B$ is a homomorphism, then, for each $x, y \in A$, we also have:
(a) $f(1) = 1$,
(b) $f(x \cdot y) = f(x) \cdot f(y)$,
(c) $f(x \vee y) = f(x) \vee f(y)$,
(d) $f(x \wedge y) = f(x) \wedge f(y)$.

We now review some fuzzy logic concepts. Let $\Gamma \subseteq [0,1]$. We define $\bigwedge \Gamma = \inf \Gamma$ and $\bigvee \Gamma = \sup \Gamma$. Obviously, if $\Gamma = \{\alpha, \beta\}$, then $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\alpha \vee \beta = \max\{\alpha, \beta\}$. Recall that a fuzzy set in A is a function $\mu : A \to [0,1]$. For any fuzzy sets μ and ν in A, we define

$$\mu \leqslant \nu \Longleftrightarrow \mu(x) \leqslant \nu(x) \text{ for all } x \in A.$$

Definition 2.13 Let A and B be any two sets, μ be any fuzzy set in A and $f : A \to B$ be any function. The fuzzy set ν in B defined by

$$\nu(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in B$, is called the *image* of μ under f and is denoted by $f(\mu)$.

Definition 2.14 Let A and B be any two sets, $f : A \to B$ be any function and ν be any fuzzy set in $f(A)$. The fuzzy set μ in A defined by

$$\mu(x) = \nu(f(x)) \text{ for all } x \in A$$

is called the *preimage* of ν under f and is denoted by $f^{-1}(\nu)$.

3. Fuzzy ideals. In this section we investigate fuzzy ideals and fuzzy implicative ideals of a pseudo MV-algebra. First, we recall from [6] the definition and some facts concerning fuzzy ideals.

Definition 3.1 A fuzzy set μ in a pseudo MV-algebra A is called a *fuzzy ideal* of A if it satisfies for all $x, y \in A$:
(d1) $\mu(x \oplus y) \geqslant \mu(x) \wedge \mu(y)$,
(d2) if $y \leqslant x$, then $\mu(y) \geqslant \mu(x)$.

It is easily seen that (d2) implies
(d3) $\mu(0) \geqslant \mu(x)$ for all $x \in A$.

Example 3.2 Let $A = \{(1, y) : y \geqslant 0\} \cup \{(2, y) : y \leqslant 0\}$, $\mathbf{0} = (1, 0)$, $\mathbf{1} = (2, 0)$. For any $(a, b), (c, d) \in A$, we define operations $\oplus, ^-, ^\sim$ as follows:

$$\begin{aligned} (a, b) \oplus (c, d) &= \begin{cases} (1, b + d) & \text{if } a = c = 1, \\ (2, ad + b) & \text{if } ac = 2 \text{ and } ad + b \leqslant 0, \\ (2, 0) & \text{in other cases,} \end{cases} \\ (a, b)^- &= \left(\frac{2}{a}, -\frac{2b}{a}\right), \\ (a, b)^\sim &= \left(\frac{2}{a}, -\frac{b}{a}\right). \end{aligned}$$

Then $A = (A, \oplus, ^-, ^\sim, \mathbf{0}, \mathbf{1})$ is a pseudo MV-algebra (see [1]). Let $A_1 = \{(1, y) : y > 0\}$ and $A_2 = \{(2, y) : y < 0\}$ and let $0 \leqslant \alpha_3 < \alpha_2 < \alpha_1 \leqslant 1$. We define a fuzzy set μ in A as follows:

$$\mu(x) = \begin{cases} \alpha_1 & \text{if } x = \mathbf{0}, \\ \alpha_2 & \text{if } x \in A_1, \\ \alpha_3 & \text{if } x \in A_2 \cup \{\mathbf{1}\}. \end{cases}$$

It is easily checked that μ satisfies (d1) and (d2). Thus μ is a fuzzy ideal of A.

Denote by $\mathfrak{FI}(A)$ the set of fuzzy ideals of A.

PROPOSITION 3.3 (JUN AND WALENDZIAK [6]) *Let μ be a fuzzy set in A. Then $\mu \in \mathfrak{FI}(A)$ if and only if it satisfies* (d1) *and*
(d4) $\mu(x \wedge y) \geqslant \mu(x)$ *for all* $x, y \in A$.

PROPOSITION 3.4 (JUN AND WALENDZIAK [6]) *Let $\mu \in \mathfrak{FI}(A)$. Then, for all $x, y \in A$, the following are true:*
(a) $\mu(x \cdot y) \geqslant \mu(x) \wedge \mu(y)$,
(b) $\mu(x \wedge y) \geqslant \mu(x) \wedge \mu(y)$,
(c) $\mu(x \vee y) = \mu(x) \wedge \mu(y)$,
(d) $\mu(x \oplus y) = \mu(x) \wedge \mu(y)$.

PROPOSITION 3.5 (JUN AND WALENDZIAK [6]) *Every fuzzy ideal μ of A satisfies the following two inequalities:*

$$\mu(y) \geqslant \mu(x) \wedge \mu\left(y \cdot x^{-}\right) \text{ for all } x, y \in A, \tag{5}$$

$$\mu(y) \geqslant \mu(x) \wedge \mu\left(x^{\sim} \cdot y\right) \text{ for all } x, y \in A. \tag{6}$$

PROPOSITION 3.6 (JUN AND WALENDZIAK [6]) *For a fuzzy set μ in A, the following are equivalent:*
(a) $\mu \in \mathfrak{FI}(A)$,
(b) μ *satisfies the conditions* (d3) *and* (5),
(c) μ *satisfies the conditions* (d3) *and* (6).

PROPOSITION 3.7 (JUN AND WALENDZIAK [6]) *Let μ be a fuzzy set in A. Then $\mu \in \mathfrak{FI}(A)$ if and only if its nonempty level subset*

$$U(\mu; \alpha) = \{x \in A : \mu(x) \geqslant \alpha\}$$

is an ideal of A for all $\alpha \in [0, 1]$.

EXAMPLE 3.8 Let A and μ be as in Example 3.2. One can easily check that for all $\alpha \in [0, 1]$ we have:

$$U(\mu; \alpha) = \begin{cases} \varnothing & \text{if } \alpha > \alpha_1, \\ \{\mathbf{0}\} & \text{if } \alpha_2 < \alpha \leqslant \alpha_1, \\ A_1 \cup \{\mathbf{0}\} & \text{if } \alpha_3 < \alpha \leqslant \alpha_2, \\ A & \text{if } \alpha \leqslant \alpha_3. \end{cases}$$

It is not difficult to see that $\{\mathbf{0}\}$, $A_1 \cup \{\mathbf{0}\}$ and A are all ideals of A. This is an another proof (by Proposition 3.7) that μ is a fuzzy ideal of A.

Now, we consider two special fuzzy sets in A. Let I be a subset of A. Define a fuzzy set μ_I in A by

$$\mu_I(x) = \begin{cases} \alpha & \text{if } x \in I, \\ \beta & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in [0,1]$ with $\alpha > \beta$. In particular, we have a fuzzy set χ_I which is the characteristic function of I:

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

We have the simple proposition.

PROPOSITION 3.9 *$I \in \mathfrak{I}(A)$ iff $\mu_I \in \mathfrak{FI}(A)$.*

COROLLARY 3.10 *$I \in \mathfrak{I}(A)$ iff $\chi_I \in \mathfrak{FI}(A)$.*

For an arbitrary fuzzy set μ in A, consider the set $A_\mu = \{x \in A : \mu(x) = \mu(0)\}$. We have the following simple proposition.

PROPOSITION 3.11 *If $\mu \in \mathfrak{FI}(A)$, then $A_\mu \in \mathfrak{I}(A)$.*

The following example shows that the converse of Proposition 3.11 does not hold.

EXAMPLE 3.12 Let A be as in Example 3.2. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0, \\ \frac{2}{3} & \text{if } x \neq 0. \end{cases}$$

Then $A_\mu = \{0\} \in \mathfrak{I}(A)$ but $\mu \notin \mathfrak{FI}(A)$.

Since $A_{\mu_I} = I$, we have the simple proposition.

PROPOSITION 3.13 *$\mu_I \in \mathfrak{FI}(A)$ iff $A_{\mu_I} \in \mathfrak{I}(A)$.*

The following two theorems give the homomorphic properties of fuzzy ideals.

THEOREM 3.14 *Let $f : A \to B$ be a surjective homomorphism and $\nu \in \mathfrak{FI}(B)$. Then $f^{-1}(\nu) \in \mathfrak{FI}(A)$.*

PROOF Let $x \in A$. Since $f(x) \in B$ and ν is a fuzzy ideal of B, we have $\nu(0) \geqslant \nu(f(x)) = \left(f^{-1}(\nu)\right)(x)$, but $\nu(0) = \nu(f(0)) = \left(f^{-1}(\nu)\right)(0)$. Thus $\left(f^{-1}(\nu)\right)(0) \geqslant \left(f^{-1}(\nu)\right)(x)$ for any $x \in A$, i.e., $f^{-1}(\nu)$ satisfies (d3).

Now, let $y_A \in A$. Then, since ν satisfies (5), $\left(f^{-1}(\nu)\right)(y_A) = \nu(f(y_A)) \geqslant \nu(x_B) \wedge \nu\left(f(y_A) \cdot x_B^-\right)$ for any $x_B \in B$. Let x_A be an arbitrary preimage of x_B under f, i.e., $f(x_A) = x_B$. Then

$$\begin{aligned} \left(f^{-1}(\nu)\right)(y_A) &\geqslant \nu(x_B) \wedge \nu\left(f(y_A) \cdot x_B^-\right) \\ &= \nu(f(x_A)) \wedge \nu\left(f(y_A) \cdot (f(x_A))^-\right) \\ &= \nu(f(x_A)) \wedge \nu\left(f\left(y_A \cdot x_A^-\right)\right) \\ &= \left(f^{-1}(\nu)\right)(x_A) \wedge \left(f^{-1}(\nu)\right)\left(y_A \cdot x_A^-\right). \end{aligned}$$

Since x_B is an arbitrary element of B, the above inequality holds for any $x_A \in A$, i.e., $f^{-1}(\nu)$ satisfies (5). Hence, by Proposition 3.6, $f^{-1}(\nu)$ is a fuzzy ideal of A. ∎

Lemma 3.15 *Let $f : A \to B$ be a homomorphism, $\mu \in \mathfrak{FI}(A)$ and $\nu \in \mathfrak{FI}(B)$. Then:*
(a) *if μ is constant on* Ker$f$*, then $f^{-1}(f(\mu)) = \mu$,*
(b) *if f is surjective, then $f\left(f^{-1}(\nu)\right) = \nu$.*

Proof (a) Let $x \in A$, and $f(x) = y$. Hence

$$\left(f^{-1}(f(\mu))\right)(x) = (f(\mu))(f(x)) = (f(\mu))(y) = \sup_{t\in f^{-1}(y)}\mu(t).$$

For all $t \in f^{-1}(y)$, we have $f(t) = f(x)$. Hence $f(t \cdot x^-) = f(t) \cdot (f(x))^- = f(x) \cdot (f(x))^- = 0$ and, similarly, $f(x \cdot t^-) = 0$. Thus $t \cdot x^- \in \mathrm{Ker}f$ and $x \cdot t^- \in \mathrm{Ker}f$. Since μ is constant on Ker$f$, $\mu(t \cdot x^-) = \mu(x \cdot t^-) = \mu(0)$. Hence, by (5), $\mu(t) \geqslant \mu(x) \wedge \mu(t \cdot x^-) = \mu(x) \wedge \mu(0) = \mu(x)$ and, similarly, $\mu(x) \geqslant \mu(t)$. Hence $\mu(x) = \mu(t)$. Thus

$$\left(f^{-1}(f(\mu))\right)(x) = \sup_{t\in f^{-1}(y)}\mu(t) = \mu(x),$$

i.e., $f^{-1}(f(\mu)) = \mu$.

(b) Since f is surjective, for any $y \in B$ there is $x \in A$ such that $f(x) = y$. Next, $f^{-1}(\nu)$ is a fuzzy ideal of A which is constant on Ker$f$. Thus, by (a), we have

$$\begin{aligned} \left(f\left(f^{-1}(\nu)\right)\right)(y) &= \left(f\left(f^{-1}(\nu)\right)\right)(f(x)) = \left(f^{-1}\left(f\left(f^{-1}(\nu)\right)\right)\right)(x) \\ &= \left(f^{-1}(\nu)\right)(x) = \nu(f(x)) = \nu(y). \end{aligned}$$

Therefore $f\left(f^{-1}(\nu)\right) = \nu$. ∎

Theorem 3.16 *Let $f : A \to B$ be a surjective homomorphism and $\mu \in \mathfrak{FI}(A)$ be such that $A_\mu \supseteq$ Ker$f$. Then $f(\mu) \in \mathfrak{FI}(B)$.*

Proof Since μ is a fuzzy ideal of A and $0 \in f^{-1}(0)$, we have

$$(f(\mu))(0) = \sup_{t\in f^{-1}(0)}\mu(t) = \mu(0) \geqslant \mu(x)$$

for any $x \in A$. Hence

$$(f(\mu))(0) \geqslant \sup_{x \in f^{-1}(y)} \mu(x) = (f(\mu))(y)$$

for any $y \in B$. Thus $f(\mu)$ satisfies (d3).

Now, assume that

$$(f(\mu))(y_B) < (f(\mu))(x_B) \wedge (f(\mu))\left(y_B \cdot x_B^-\right)$$

for some $x_B, y_B \in B$. Since f is surjective, there are $x_A, y_A \in A$ such that $f(x_A) = x_B$ and $f(y_A) = y_B$. Thus

$$\left(f^{-1}(f(\mu))\right)(y_A) < \left(f^{-1}(f(\mu))\right)(x_A) \wedge \left(f^{-1}(f(\mu))\right)\left(y_A \cdot x_A^-\right).$$

Since $A_\mu \supseteq \mathrm{Ker} f$, μ is constant on $\mathrm{Ker} f$. Hence, by Lemma 3.15(a), we get

$$\mu(y_A) < \mu(x_A) \wedge \mu\left(y_A \cdot x_A^-\right)$$

which is a contradiction with a fact that μ is a fuzzy ideal. Thus $f(\mu)$ satisfies (5) and hence, by Proposition 3.6, it is a fuzzy ideal of B. ■

Now, we investigate fuzzy implicative ideals of a pseudo MV-algebra. First, we give the definition and some characterizations of a fuzzy implicative ideal (see [6]).

Definition 3.17 Let μ be a fuzzy ideal of A. We say that μ is *fuzzy implicative* if it satisfies:
for all $x, y, z \in A$, $\mu(x \cdot y) \geqslant \mu(x \cdot y \cdot z^-) \wedge \mu(z \cdot y)$ or equivalently
for all $x, y, z \in A$, $\mu(x \cdot y) \geqslant \mu(x \cdot y \cdot z) \wedge \mu(z^\sim \cdot y)$.

Proposition 3.18 (Jun and Walendziak [6]) *Let μ be a fuzzy ideal of A. Then the following are equivalent:*
(a) *μ is a fuzzy implicative ideal of A,*
(b) *for all $x \in A$, if $x^2 = 0$, then $\mu(x) = \mu(0)$,*
(c) *for all $x \in A$, $\mu(x \wedge x^-) = \mu(0)$,*
(d) *for all $x \in A$, $\mu(x \wedge x^\sim) = \mu(0)$.*

Example 3.19 Let A and $\alpha_1, \alpha_3 \in [0,1]$ be as in Example 3.2. Let ν be a fuzzy set in A defined by

$$\nu(x) = \begin{cases} \alpha_1 & \text{if } x \in A_1 \cup \{\mathbf{0}\}, \\ \alpha_3 & \text{if } x \in A_2 \cup \{\mathbf{1}\}. \end{cases}$$

Then it is easy to show that ν is a fuzzy ideal of A. In fact, it is a fuzzy implicative ideal of A. Indeed, observe that for all $x \in A$, $x \wedge x^- \in A_1 \cup \{\mathbf{0}\}$. So $\nu(x \wedge x^-) = \alpha_1 = \nu(\mathbf{0})$ and, by Proposition 3.18, ν is a fuzzy implicative ideal of A.

Theorem 3.20 (Implicative extension property for fuzzy ideals) *Let μ be a fuzzy implicative ideal of A and ν any fuzzy ideal of A such that $\mu \leqslant \nu$ and $\mu(0) = \nu(0)$. Then ν is a fuzzy implicative ideal of A.*

Proof Let $x \in A$ be such that $x^2 = 0$. Then, by Proposition 3.18, $\nu(x) \geqslant \mu(x) = \mu(0) = \nu(0)$. Hence $\nu(x) = \nu(0)$ and, again by Proposition 3.18, ν is a fuzzy implicative ideal of A. ■

Let $0 \leqslant t < 1$ be a real number. If $\alpha \in [0,1]$, then α^t shall mean the positive root. Let $\mu : A \to [0,1]$ be a fuzzy set in A. We define $\mu^t : A \to [0,1]$ by $\mu^t(x) = (\mu(x))^t$ for all $x \in A$. It is easily verified that if μ is a fuzzy ideal of A, then so is μ^t, and if $\mu(0) = 1$, then $A_{\mu^t} = A_\mu$.

Theorem 3.21 *Let μ be a fuzzy implicative ideal of A such that $\mu(0) = 1$. Then for every $0 \leqslant t < 1$, μ^t is a fuzzy implicative ideal of A.*

Proof We have that $\mu^t(0) = (\mu(0))^t = 1 = \mu(0)$ and $\mu \leqslant \mu^t$. This means, by Theorem 3.20, that μ^t is a fuzzy implicative ideal of A. ■

Next two theorems express the homomorphic properties of fuzzy implicative ideals.

Theorem 3.22 *Let $f : A \to B$ be a surjective homomorphism and ν be a fuzzy implicative ideal of B. Then $f^{-1}(\nu)$ is a fuzzy implicative ideal of A.*

Proof By Theorem 3.14, $f^{-1}(\nu) \in \mathfrak{FI}(A)$. Let $x \in A$. Then

$$\begin{aligned} \left(f^{-1}(\nu)\right)\left(x \wedge x^-\right) &= \nu\left(f\left(x \wedge x^-\right)\right) = \nu\left(f(x) \wedge (f(x))^-\right) = \\ \nu(0) &= \nu(f(0)) = \left(f^{-1}(\nu)\right)(0). \end{aligned}$$

From Proposition 3.18 it follows that $f^{-1}(\nu)$ is a fuzzy implicative ideal of A. ■

Theorem 3.23 *Let $f : A \to B$ be a surjective homomorphism and μ be a fuzzy implicative ideal of A such that $A_\mu \supseteq \mathrm{Ker} f$. Then $f(\mu)$ is a fuzzy implicative ideal of B.*

Proof By Theorem 3.16, $f(\mu) \in \mathfrak{FI}(B)$. Let $y \in B$. Since f is surjective, there is $x \in A$ such that $f(x) = y$. Since μ is a fuzzy implicative ideal of A, $\mu(x \wedge x^-) = \mu(0)$ by Proposition 3.18. Then

$$(f(\mu))(y \wedge y^-) = \mu(0) = (f(\mu))(f(0))$$

and applying again Proposition 3.18, $f(\mu)$ is a fuzzy implicative ideal of B. ■

Definition 3.24 A fuzzy ideal μ of A is said to be *fuzzy normal* if it satisfies:

$$\mu(x \cdot y^{-}) = \mu(0) \Longleftrightarrow \mu(y^{\sim} \cdot x) = \mu(0)$$

for all $x, y \in A$.

We immediately have the following proposition.

Proposition 3.25 *Let $\mu \in \mathfrak{FI}(A)$. Then μ is a fuzzy normal ideal of A if and only if A_μ is a normal ideal of A.*

To the end of this section we give the definition and some facts concerning fuzzy prime ideals of pseudo MV-algebras (see [2] for details).

Definition 3.26 A fuzzy ideal μ of A is said to be *fuzzy prime* if it is non-constant and satisfies:

$$\mu(x \wedge y) = \mu(x) \vee \mu(y)$$

for all $x, y \in A$.

Proposition 3.27 (Dymek [2]) *Let μ be a non-constant fuzzy ideal of A. Then the following are equivalent:*
(a) *μ is a fuzzy prime ideal of A,*
(b) *for all $x, y \in A$, if $\mu(x \wedge y) = \mu(0)$, then $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$,*
(c) *for all $x, y \in A$, $\mu(x \cdot y^{-}) = \mu(0)$ or $\mu(y \cdot x^{-}) = \mu(0)$,*
(d) *for all $x, y \in A$, $\mu(x^{\sim} \cdot y) = \mu(0)$ or $\mu(y^{\sim} \cdot x) = \mu(0)$.*

Proposition 3.28 (Dymek [2]) *Let $\mu \in \mathfrak{FI}(A)$. Then μ is a fuzzy prime ideal of A if and only if A_μ is a prime ideal of A.*

4. Fuzzy maximal ideals. In this section we define the notion of a fuzzy maximal ideal of a pseudo MV-algebra and investigate its properties.

Definition 4.1 A fuzzy ideal μ of A is called *fuzzy maximal* iff A_μ is a maximal ideal of A.

Example 4.2 Let A and μ be as in Example 3.2 and ν be as in Example 3.19. Then, since $A_\nu = A_1 \cup \{\mathbf{0}\}$ is a maximal ideal of A and $A_\mu = \{\mathbf{0}\}$ is not the one, we obtain that ν is a fuzzy maximal ideal of A and μ is not the one.

By Proposition 3.13, we have the following theorem.

Theorem 4.3 *An ideal I of A is maximal if and only if μ_I is fuzzy maximal.*

COROLLARY 4.4 *An ideal I of A is maximal if and only if χ_I is fuzzy maximal.*

THEOREM 4.5 *If μ is a fuzzy maximal ideal of A, then μ has exactly two values.*

PROOF Assume that μ is a fuzzy maximal ideal of A. Then A_μ is a maximal ideal of A. Since A_μ is proper, $|\mathrm{Im}\mu| > 1$. If $|\mathrm{Im}\mu| > 2$, then there are $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Im}\mu$, where $0 \leqslant \alpha_1 < \alpha_2 < \alpha_3 = \mu(0)$. Hence $A_\mu = U(\mu; \alpha_3) \subset U(\mu; \alpha_2) \subset U(\mu; \alpha_1) \subseteq A$. But an ideal $U(\mu; \alpha_2) \neq A_\mu$ and $U(\mu; \alpha_2) \neq A$. This is a contradiction. Therefore μ has exactly two values. ■

THEOREM 4.6 *Let μ be a non-constant fuzzy set in A. Then μ is a fuzzy maximal ideal of A if and only if for each $a \in [0,1]$, $U(\mu; \alpha) = \varnothing$ or $U(\mu; \alpha)$ is a maximal ideal of A if it is proper.*

PROOF Let μ be a fuzzy maximal ideal of A and $U(\mu; \alpha) \neq \varnothing$ for some $\alpha \in [0,1]$. Then, by Theorem 4.5, μ has exactly two values. Hence $U(\mu; \alpha) = A$ or $U(\mu; \alpha) = A_\mu$. Assume that $U(\mu; \alpha) \neq A$. Then $U(\mu; \alpha) = A_\mu$ is a maximal ideal of A.

Conversely, we know that A_μ is nonempty. Since μ is non-constant, A_μ is proper. Thus $A_\mu = U(\mu; \mu(0))$ is a maximal ideal of A. Therefore μ is a fuzzy maximal ideal of A. ■

Now, recall that a pseudo MV-algebra A is *locally finite* if and only if for any $x \neq 0$ there exists $n \in \mathbb{N}$ such that $nx = 1$. Recall also that a pseudo MV-algebra A is *simple* if and only if there is no non-trivial proper ideal of A.

PROPOSITION 4.7 (DYMEK AND WALENDZIAK [3]) *A pseudo MV-algebra A is locally finite if and only if it is simple.*

PROPOSITION 4.8 (DVUREČENSKIJ [1]) *A normal ideal I of a pseudo MV-algebra A is maximal if and only if A/I is a simple pseudo MV-algebra.*

THEOREM 4.9 *Let μ be a fuzzy normal ideal of A. Then the following are equivalent:*
(a) *μ is fuzzy maximal,*
(b) *for all $x \in A$, if $\mu(x) < \mu(0)$, then there is $n \in \mathbb{N}$ such that $\mu((x^-)^n) = \mu(0)$,*
(c) *for all $x \in A$, if $\mu(x) < \mu(0)$, then there is $n \in \mathbb{N}$ such that $\mu((x^\sim)^n) = \mu(0)$,*
(d) *A/A_μ is a locally finite pseudo MV-algebra,*
(e) *A/A_μ is a simple pseudo MV-algebra.*

PROOF (a) $\Rightarrow$ (b): Assume that μ is a fuzzy maximal ideal of A. Let $x \in A$ and $\mu(x) < \mu(0)$. Then $x \notin A_\mu$. Take $I = (A_\mu \cup \{x\}]$. From Proposition 3.25 we have that A_μ is a normal ideal of A. Hence, by Proposition 2.10, $I = \{t \in A : t \leqslant y \oplus nx$ for some $y \in A_\mu$ and $n \in \mathbb{N}\}$. Since $x \in I - A_\mu$ and A_μ is a maximal ideal of A, we obtain $I = A$. Thus $1 \in I$, i.e.,

$$1 = y \oplus nx \text{ for some } y \in A_\mu \text{ and } n \in \mathbb{N}.$$

From Axiom (A8) we have $y \oplus ((nx)^-)^\sim = 1$ and hence $(nx)^- \leqslant y$ by (1) and Proposition 2.2. Therefore $(x^-)^n \leqslant y$ by Proposition 2.1(f). Since $y \in A_\mu$ and $(x^-)^n \leqslant y$, we have $(x^-)^n \in A_\mu$. Thus $\mu((x^-)^n) = \mu(0)$.

(b) $\Rightarrow$ (c): Let $x \in A$ and $\mu(x) < \mu(0)$. Then $(x^-)^n \in A_\mu$. Hence, by Proposition 2.9, $((x^-)^n)^\approx \in A_\mu$. Now, applying Proposition 2.1(e,f) we get

$$(x^\sim)^n = (nx)^\sim = ((nx)^-)^\approx = ((x^-)^n)^\approx.$$

Thus $(x^\sim)^n \in A_\mu$. Therefore $\mu((x^\sim)^n) = \mu(0)$.

(c) $\Rightarrow$ (d): Let $x/A_\mu \neq 0/A_\mu$. From (3) it follows that $x \notin A_\mu$. Then, by (c), $\mu((x^\sim)^n) = \mu(0)$ for some $n \in \mathbb{N}$, i.e., $(x^\sim)^n \in A_\mu$ for some $n \in \mathbb{N}$. Hence $0/A_\mu = (x^\sim)^n/A_\mu = (x^\sim/A_\mu)^n = ((x/A_\mu)^\sim)^n$. Thus

$$1/A_\mu = (0^-/A_\mu) = (0/A_\mu)^- = [((x/A_\mu)^\sim)^n]^- = n(x/A_\mu)$$

by Proposition 2.1(g). Therefore A/A_μ is locally finite.

(d) $\Rightarrow$ (e): See Proposition 4.7.

(e) $\Rightarrow$ (a): See Proposition 4.8. ∎

Remark 4.10 Theorem 4.9 implies Theorem 3.28 of [5].

The following two theorems give the homomorphic properties of fuzzy maximal ideals.

Theorem 4.11 *Let $f : A \to B$ be a surjective homomorphism and ν be a fuzzy maximal ideal of B. Then $f^{-1}(\nu)$ is a fuzzy maximal ideal of A.*

Proof By Theorem 3.14, $f^{-1}(\nu)$ is a fuzzy ideal of A. Now, we prove that $f^{-1}(\nu)$ is fuzzy maximal. Observe that $A_{f^{-1}(\nu)}$ is proper. Indeed, if $A_{f^{-1}(\nu)} = A$, then $f^{-1}(\nu)$ is constant. Since f is surjective, for any $y \in B$ there is $x \in A$ such that $f(x) = y$. Thus $\nu(y) = \nu(f(x)) = \left(f^{-1}(\nu)\right)(x)$ and so ν is constant. This is a contradiction, because ν is fuzzy maximal. Therefore $A_{f^{-1}(\nu)}$ is proper.

Now, let J be an ideal of A such that $A_{f^{-1}(\nu)} \subseteq J \subseteq A$. We prove that $J = A_{f^{-1}(\nu)}$ or $J = A$.
First, it is not difficult to see that if f is surjective and $J \supseteq \mathrm{Ker} f$, then $f(J)$ is an ideal of B.
Next, we prove that $B_\nu \subseteq f(J)$. Let $y \in B_\nu$. Then $\nu(y) = \nu(0) = \nu(f(0))$. Since f is surjective, there exists $x \in A$ such that $f(x) = y$. Hence $\nu(f(x)) = \nu(f(0))$, i.e., $\left(f^{-1}(\nu)\right)(x) = \left(f^{-1}(\nu)\right)(0)$. Thus $x \in A_{f^{-1}(\nu)} \subseteq J$ and so $y = f(x) \in f(J)$. It follows $B_\nu \subseteq f(J)$.
Since B_ν is maximal, we have $f(J) = B_\nu$ or $f(J) = B$. Suppose, first, that $f(J) = B_\nu$. Let $x \in J$. Then $f(x) \in f(J) = B_\nu$ and so $\nu(f(x)) = \nu(0) = \nu(f(0))$. Thus $\left(f^{-1}(\nu)\right)(x) = \left(f^{-1}(\nu)\right)(0)$, i.e., $x \in A_{f^{-1}(\nu)}$. Hence $J \subseteq A_{f^{-1}(\nu)}$ and so $J = A_{f^{-1}(\nu)}$. Suppose, now, that $f(J) = B$. Let $x \in A$. Then $f(x) \in B = f(J)$. Hence $f(x) = f(x_1)$, where $x_1 \in J$. It follows $f\left(x \cdot x_1^-\right) = f(x) \cdot (f(x_1))^- = 0$ and so $x \cdot x_1^- \in \mathrm{Ker} f \subseteq J$. We have $x \leqslant x \vee x_1 = x \cdot x_1^- \oplus x_1 \in J$. Hence $x \in J$, i.e., $A \subseteq J$. So $J = A$. Therefore $f^{-1}(\nu)$ is a fuzzy maximal ideal of A. ∎

THEOREM 4.12 *Let $f : A \to B$ be a surjective homomorphism and μ be a fuzzy maximal ideal of A such that $A_\mu \supseteq \mathrm{Ker}f$. Then $f(\mu)$ is a fuzzy maximal ideal of B.*

PROOF First, we prove that $B_{f(\mu)}$ is proper. Suppose that $B_{f(\mu)} = B$. Let $x \in A$. Then $f(x) \in B = B_{f(\mu)}$. Thus $(f(\mu))(f(x)) = (f(\mu))(0) = (f(\mu))(f(0))$ and so $\left(f^{-1}(f(\mu))\right)(x) = \left(f^{-1}(f(\mu))\right)(0)$. Since $A_\mu \supseteq \mathrm{Ker}f$, μ is constant on $\mathrm{Ker}f$. So, by Lemma 3.15(a), $\mu(x) = \mu(0)$. This means $x \in A_\mu$. Hence $A_\mu = A$, which is a contradiction, because μ is a fuzzy maximal ideal of A. Therefore $B_{f(\mu)}$ is proper.

Now, take an ideal J of B such that $B_{f(\mu)} \subseteq J \subseteq B$. We prove that $J = B_{f(\mu)}$ or $J = B$.

Note that $A_\mu \subseteq f^{-1}(J)$. Indeed, let $x \in A_\mu$. Then $\mu(x) = \mu(0)$ and, by Lemma 3.15(a), $(f(\mu))(f(x)) = \left(f^{-1}(f(\mu))\right)(x) = \left(f^{-1}(f(\mu))\right)(0) = (f(\mu))(f(0)) = (f(\mu))(0)$. Thus $f(x) \in B_{f(\mu)} \subseteq J$ and so $x \in f^{-1}(J)$. Hence $A_\mu \subseteq f^{-1}(J)$. Since A_μ is a maximal ideal of A and $f^{-1}(J)$ is an ideal of A, we have $f^{-1}(J) = A_\mu$ or $f^{-1}(J) = A$. Suppose that $f^{-1}(J) = A_\mu$. Let $y \in J$. Then $y = f(x)$, where $x \in f^{-1}(J) = A_\mu$. Hence $\mu(x) = \mu(0)$ and so, again by Lemma 3.15(a), $\left(f^{-1}(f(\mu))\right)(x) = \left(f^{-1}(f(\mu))\right)(0)$, i.e., $(f(\mu))(y) = (f(\mu))(0)$. Thus $y \in B_{f(\mu)}$. It means that $J \subseteq B_{f(\mu)}$, i.e., $J = B_{f(\mu)}$. Suppose, now, that $f^{-1}(J) = A$. Let $y \in B$. Since f is surjective, $y = f(x)$, where $x \in A = f^{-1}(J)$. Hence $y = f(x) \in J$. It means that $B \subseteq J$, i.e., $J = B$. Thus $B_{f(\mu)}$ is a maximal ideal of B. Therefore $f(\mu)$ is a fuzzy maximal ideal of B. ∎

THEOREM 4.13 *If μ is a fuzzy maximal ideal of A, then μ is a fuzzy prime ideal of A.*

PROOF Assume that μ is a fuzzy maximal ideal of A. Then A_μ is a maximal ideal of A. By Proposition 2.7, A_μ is a prime ideal of A. Therefore, by Proposition 3.28, μ is a fuzzy prime ideal of A. ∎

Next example shows that the converse of the Theorem 4.13 does not hold.

EXAMPLE 4.14 Let A and μ be as in Example 3.2. Then $A_\mu = \{\mathbf{0}\}$ is a prime ideal of A and is not maximal. Therefore μ is a fuzzy prime ideal of A and is not fuzzy maximal.

THEOREM 4.15 *Let μ be a non-constant fuzzy ideal of A. Then the following are equivalent:*
(a) *μ is fuzzy maximal and fuzzy implicative,*
(b) *μ is fuzzy prime and fuzzy implicative,*
(c) *for all $x \in A$, $\mu(x) = \mu(0)$ or $\mu(x^-) = \mu(0)$,*
(d) *for all $x \in A$, $\mu(x) = \mu(0)$ or $\mu(x^\sim) = \mu(0)$,*
(e) *for all $x \in A$, $\mu(x) = \mu(0)$ or $\mu(x^-) = \mu(0)$ or $\mu(x^\sim) = \mu(0)$.*

PROOF (a) $\Rightarrow$ (b): Follows from Theorem 4.13.

(b) $\Rightarrow$ (c): Let $x \in A$. Since μ is fuzzy implicative, we have, by Proposition 3.18, $\mu(x \wedge x^-) = \mu(0)$. Since μ is fuzzy prime, we obtain, by Proposition 3.27, $\mu(x) = \mu(0)$ or $\mu(x^-) = \mu(0)$.

(c) $\Rightarrow$ (d): Let $x \in A$ and suppose that $\mu(x) \neq \mu(0)$. Then $\mu\left((x^\sim)^-\right) = \mu(x) \neq \mu(0)$. Thus, by (c), $\mu(x^\sim) = \mu(0)$.

(d) $\Rightarrow$ (e): Obvious.

(e) $\Rightarrow$ (a): Let $x \in A$. Suppose that $\mu(x \wedge x^-) \neq \mu(0)$. Since $x \wedge x^- \leqslant x$ and $x \wedge x^- \leqslant x^-$, we have $\mu(x \wedge x^-) \geqslant \mu(x)$ and $\mu(x \wedge x^-) \geqslant \mu(x^-)$. It follows that $\mu(x) \neq \mu(0)$ and $\mu(x^-) \neq \mu(0)$. Thus, by (e), $\mu(x^\sim) = \mu(0)$ and since $\mu(x \wedge x^\sim) \geqslant \mu(x^\sim)$, we conclude $\mu(x \wedge x^\sim) = \mu(0)$. Therefore, by Proposition 3.18, μ is a fuzzy implicative ideal of A.

Note that μ is also a fuzzy maximal ideal of A, i.e., A_μ is a maximal ideal of A. Let J be an ideal of A such that $A_\mu \subset J$. For every $y \in J - A_\mu$, we have that $\mu(y) \neq \mu(0)$. Hence, by (e), $\mu(y^-) = \mu(0)$ or $\mu(y^\sim) = \mu(0)$, i.e., $y^- \in A_\mu \subset J$ or $y^\sim \in A_\mu \subset J$. It follows that $y^- \oplus y \in J$ or $y \oplus y^\sim \in J$. But $y^- \oplus y = y \oplus y^\sim = 1$. Thus $1 \in J$ and so $J = A$. ∎

Corollary 4.16 *Let μ be a non-constant fuzzy implicative ideal of A. Then μ is fuzzy prime if and only if μ is fuzzy maximal.*

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