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Special schurian vector space categories and l-hereditary right QF-2 rings

Let F be a division ring. We recall from [11], [13] that a *vector space category* $\boldsymbol{K}_F$ is an additive category $\boldsymbol{K}$ together with a faithful additive functor $|\cdot|$: $\boldsymbol{K} \to \operatorname{mod}(F)$ from $\boldsymbol{K}$ to the category of finite dimensional right vector spaces over F. The *subspace category* $\mathscr{U}(\boldsymbol{K}_F)$ of $\boldsymbol{K}_F$ is defined as follows. The objects of $\mathscr{U}(\boldsymbol{K}_F)$ are triples (U, X, φ), where U is a finite dimensional right vector space over F, X is an object in $\boldsymbol{K}$ and $\varphi\colon U \to |X|_F$ is an F-linear map. The map from (U, X, φ) into (U', X', φ') is a pair (u, h), where $u \in \operatorname{Hom}_F(U, U')$ and $h\colon X \to X'$ is a map in $\boldsymbol{K}$ such that $|h|\varphi = \varphi' u$.

The concepts of a vector space category over an algebraically closed field and a subspace category were introduced by Nazarova and Rojter [11], and were applied in the proof of the second Brauer–Thrall conjecture. An important role in these investigations play the vector space categories with only one-dimensional indecomposable objects because their subspace categories are close to categories of representations of partially ordered sets and therefore their representation type is known. They are also successfully applied by C. M. Ringel [14] in the investigation of one-relation finite dimensional algebras of tame type.

In the present paper we study special schurian vector space categories. We call $\boldsymbol{K}_F$ *special schurian* if $\boldsymbol{K}$ is a Krull–Schmidt category, $\boldsymbol{K}$ has only a finite number of pairwise non-isomorphic indecomposable objects and $\dim|X|_F = 1$ as well as $\operatorname{End}(X)$ is a division ring for any indecomposable object X in $\boldsymbol{K}$.

Following an idea of Drozd [5] we give a useful interpretation of the category $\mathscr{U}(\boldsymbol{K}_F)$, with $\boldsymbol{K}_F$ special schurian, in terms of l-hereditary modules over an l-hereditary right QF-2 ring.

We recall from [8] that a ring R is *l-hereditary* if every local one-sided ideal of R is projective. R is said to be *right QF-2 ring* if every indecomposable projective right ideal in R has a simple socle. A module M is said to be *l-hereditary* if every local submodule of M is projective (see [2], [3]). It is easy to see that if R is an l-hereditary right QF-2 artinian ring, then a right R-module N is l-hereditary if and only if $\operatorname{soc}(N)$ is projective.

In Section 1 we associate to any special schurian vector space category $\boldsymbol{K}_F$ and l-hereditary right QF-2 semiperfect ring $\boldsymbol{R}_{\boldsymbol{K}}$ and a full additive functor

$$\Phi\colon \mathcal{U}(\boldsymbol{K}_F) \to l\ \mathrm{her}(\boldsymbol{R}_{\boldsymbol{K}})$$

which establishes a representation equivalence between a cofinite subcategory of $\mathcal{U}(\boldsymbol{K}_F)$ and a cofinite subcategory of the category $l\ \mathrm{her}(\boldsymbol{R}_{\boldsymbol{K}})$ of finitely generated l-hereditary right $\boldsymbol{R}_{\boldsymbol{K}}$-modules.

In Section 2 we give an interpretation of the subspace category $\mathcal{U}(\boldsymbol{K}_F)$ in terms of Kleisli categories and we discuss its possible applications.

A part of results presented in Section 1 was announced in [17].

Let R be an l-hereditary artinian ring and let $P_1, \ldots, P_n$ be a complete set of pairwise non-isomorphic indecomposable projective right ideals in R. If the ring $F_i = \operatorname{End}(P_i)$ is a division ring for any i, then we associate with R a *valued poset* $(\boldsymbol{I}_R, \boldsymbol{d})$, where $\boldsymbol{I}_R = \{1, \ldots, n\}$, $i < j \Leftrightarrow {}_iM_j = \operatorname{Hom}_R(P_j, P_i) \neq 0$ and $\boldsymbol{d} = (d_{ij})$ is the matrix with

$$d_{ij} = \dim({}_iM_j)_{F_j},\ d_{ji} = \dim_{F_i}({}_iM_j) \quad \text{for } i \neq j.$$

We will write

$$i \xrightarrow{(d_{ij}, d_{ji})} j$$

if $i < j$ and there is no k in $\boldsymbol{I}_R$ such that $i < k < j$. If $d_{ij} = d_{ji} = 1$ we write simply $i \to j$ (see [8]).

It is easy to prove that an indecomposable l-hereditary artinian ring R is a right QF-2 ring if and only if the valued poset of R has a unique maximal element m and $d_{mj} = 1$ for any j.

1. Main results. Let $\boldsymbol{K}_F$ be a special schurian vector space category with a faithful functor $|\cdot|\colon \boldsymbol{K} \to \operatorname{mod}(F)$. We fix a complete set $X_1, \ldots, X_n$ of pairwise non-isomorphic indecomposable objects in $\boldsymbol{K}$ and we put

$$F_{n+1} = F \quad \text{and} \quad F_i = \operatorname{End}(X_i) \quad \text{for } i = 1, \ldots, n.$$

For any $i, j \leqslant n$ the abelian group

$${}_iN_j = \operatorname{Hom}(X_j, X_i)$$

is an $F_i - F_j$-bimodule in a natural way. Since $\dim|X_i|_F = 1$ for all i and $F_1, \ldots, F_n$ are division rings then ${}_iN_j \neq 0$ implies ${}_jN_i = 0$. Therefore without loss of generality we can suppose that $i < j$ whenever ${}_iN_j \neq 0$.

We associate with K_F the triangular matrix ring

$$R_K = \begin{bmatrix} F_1 & {}_1N_2 & \cdots & {}_1N_n & {}_1N_{n+1} \\ & F_2 & \cdots & {}_2N_n & {}_2N_{n+1} \\ & & \ddots & \vdots & \vdots \\ & & & F_n & {}_nN_{n+1} \\ & 0 & & & F_{n+1} \end{bmatrix}$$

where ${}_iN_{n+1} = {}_{F_i}|X_i|_F$ and the multiplication is given by $F_i - F_k$-bilinear maps

$$c_{ijk}\colon {}_iN_j \otimes {}_jN_k \to {}_iN_k, \qquad \otimes = \otimes_{F_j},$$

defined by the formula

$$\begin{aligned} c_{ijk}(f \otimes g) &= fg && \text{for } k \leqslant n, \\ &= |f|(g) && \text{for } k = n+1. \end{aligned}$$

Since $\dim |X_i|_F = 1$ for $i = 1, \ldots, n$ then $c_{ijk}(f \otimes g) = 0$ if and only if either $f = 0$ or $g = 0$. Then by [8], Lemma 1, the ring R_K is l-hereditary. Moreover, it follows that the ring R_K is a right QF-2 semiperfect ring. We note that the ith row P_i of the matrix form of R_K is an indecomposable projective right ideal in R_K,

$$R_K = P_1 \oplus \ldots \oplus P_n \oplus P_{n+1}$$

and every simple projective right ideal in R_K is isomorphic to P_{n+1}.

We denote by $\mathscr{U}_0(K_F)$ (resp. by $l\ \mathrm{her}_0(R)$) the full subcategory of $\mathscr{U}(K_F)$ (resp. of $l\ \mathrm{her}(R)$) consisting of objects having no direct summands of the form $(F, 0, 0)$ (resp. P_{n+1}).

Now we are able to formulate the main result of this paper:

THEOREM 1.1. *Let K_F be a special schurian vector space category such that the ring R_K associated to K_F is artinian. Then R_K is an l-hereditary right QF-2 ring and there exists a full and dense additive functor*

$$\Phi\colon \mathscr{U}(K_F) \to l\ \mathrm{her}_0(R_K)$$

with the following properties:

(a) *If A is an indecomposable object in $\mathscr{U}(K_F)$, then $\Phi(A) = 0$ if and only if A has one of the following forms $(F, 0, 0)$, $X_i = (|X_i|, X_i, id)$, $i = 1, \ldots, n$.*

(b) *If A and B are objects in $\mathscr{U}(K_F)$ having no summands of the form $X_1, \ldots, X_n$, then every isomorphism form $\Phi(A)$ into $\Phi(B)$ has the form $\Phi(h)$, where $h\colon A \to B$ is an isomorphism.*

We will define the functor Φ as a composition of two functors

$$\mathscr{U}(\boldsymbol{K}_F) \xrightarrow{\mathfrak{F}} \mathscr{V}(\boldsymbol{K}_F) \xrightarrow{\mathfrak{G}} l\ \mathrm{her}_0(\boldsymbol{R}_{\boldsymbol{K}}).$$

To do this, we need some preliminary results. First, given finite dimensional vector spaces V_i and V_j over F_i and F_j respectively we define a map

$$\theta\colon\ \mathrm{Hom}_{F_i}(V_i,\ V_j\otimes_j N_i) \to \mathrm{Hom}_F(\mathrm{V}_i\otimes|X_i|,\ V_j\otimes|X_j|)$$

as follows. Fix a basis $e_1, \ldots, e_r$ in V_j and $e'_1, \ldots, e'_m$ in V_i. If $u \in \mathrm{Hom}_{F_i}(V_i,\ V_j\otimes_j N_i)$ and $u(e'_j) = e_1\otimes k_{1j} + \ldots + e_r\otimes k_{rj}$ with $k_{sj}\in {}_jN_i$, then we define $\theta(u)$ by formula

$$\theta(u)(e'_j\otimes x_i) = e_1\otimes|k_{1j}|(x_i) + \ldots + e_r\otimes|k_{rj}|(x_i).$$

It is easy to see that the definition of θ does not depend on the choice of bases $e_1, \ldots, e_r$ and $e'_1, \ldots, e'_m$. Moreover, θ is natural with respect to linear maps $V_i \to V'_i$ and $V_j \to V'_j$.

Now given a vector space V_F over F we define F_j-linear maps

$$b_{ji}\colon\ \mathrm{Hom}_F({}_iN_{n+1},\ V)\otimes_i N_j \to \mathrm{Hom}_F({}_jN_{n+1},\ V)$$

by formula $b_{ji}(f\otimes g)x_j = f|g|(x_j)$, where $x_j\in {}_jN_{n+1} = |X_j|$. Since $\dim|X_i|_F = 1$ then there are an embedding of rings $F_i\subset F$ and an F_i-F-bimodule isomorphism ${}_iN_{n+1} \cong {}_{F_i}F_F$ for any $i = 1, \ldots, n$. Hence we derive an isomorphism $\mathrm{Hom}_F({}_iN_{n+1},\ V) \cong V_{F_i}$, where V_{F_i} is the vector space V considered as an F_i-space via the embedding $F_i\subset F$. Then b_{ji} together with the isomorphism above defines an F_j-linear map

$$\tilde{b}_{ji}\colon\ V_{F_i}\otimes_i N_j \to V_{F_j}.$$

The proof of the following simple lemma is left to the reader.

LEMMA 1.2. (1) $\tilde{b}_{ji}(\mathrm{v}\otimes x) = 0$ *if and only if either* $v = 0$ *or* $x = 0$.

(2) $\tilde{b}_{kj}(\tilde{b}_{ji}\otimes 1) = \tilde{b}_{ki}(1\otimes c_{ijk})$ *whenever* $i\leqslant j\leqslant k$ *in* $(\boldsymbol{I}_{\boldsymbol{R}_{\boldsymbol{K}}}, \boldsymbol{d})$.

(3) *For every F-linear map $g\colon V\to V'$ the diagram*

$$\begin{tikzcd} V\otimes_i N_j \arrow[r, "g\otimes 1"] \arrow[d, "\tilde{b}_{ji}"] & V'\otimes_i N_j \arrow[d, "\tilde{b}'_{ji}"] \\ V_{F_j} \arrow[r, "g"] & V'_{F_j} \end{tikzcd}$$

is commutative.

(4) *Let V be a vector space over F. If t_i: $V_i \to V$ is an F_i-linear map and $\bar{t}_i$: $V_i \otimes_i N_{n+1} \to V$ is the map adjoint to t_i, then the diagram*

$$\begin{tikzcd} V_i \otimes {}_iN_j \arrow[r, "t_i \otimes 1"] \arrow[d, "s"] & V \otimes {}_iN_j \arrow[r, "\tilde{b}_{ji}"] & V_{F_j} \\ \mathrm{Hom}_F({}_jN_{n+1}, V_j \otimes {}_iN_{n+1}) \arrow[rr, "\mathrm{Hom}(1, \bar{t}_i)"] & & \mathrm{Hom}_F({}_jN_{n+1}, V) \arrow[u, "\simeq"'] \end{tikzcd}$$

is commutative, where $s(v_i \otimes f)x_j = v_i \otimes |f|(x_j)$.

Let $\mathscr{V}(\boldsymbol{K}_F)$ be the category whose objects are systems $(V_i, t_i)_{i=1,\dots,n+1}$, where V_i is a finite dimensional vector space over F_i and

$$t_i\colon V_i \to V_{n+1} = V_F$$

is an F_i-linear map. Here we consider $V_{n+1} = V_F$ as a vector space over F_i via the embeddings $F_i \subset F$ which we fix throughout this section. A map from (V_i', t_i') to (V_i, t_i) is a system of F_i-linear maps g_{ij}: $V_j' \to V_i \otimes_i N_j$, $i \leqslant j$, such that for every j the following diagram is commutative

$$(*) \qquad \begin{tikzcd} \bigoplus\limits_{i \leqslant j} V_i \otimes {}_iN_j \arrow[r, "\oplus(t_i \otimes 1)"] & \bigoplus\limits_{i \leqslant j} V \otimes {}_iN_j \arrow[r, "\tilde{b}_{ji}"] & V \\ V_j' \arrow[u, "(g_{ij})"] \arrow[rr, "t_j'"] & & V' \arrow[u, "g"'] \end{tikzcd}$$

where we put $g = g_{n+1\,n+1}$. If we have two maps (g_{ij}'): $(V_i'', t_i'') \to (V_i', t_i')$ and (g_{ij}): $(V_i', t_i') \to (V_i, t_i)$ in $\mathscr{V}(\boldsymbol{K}_F)$, then we define their composition by the composed maps

$$V_j'' \xrightarrow{(g'_{ij})} \bigoplus_{i \leqslant j} V_i' \otimes_i N_j \xrightarrow{\oplus(g_{ij} \otimes 1)} \bigoplus_{k \leqslant i \leqslant j} V_k \otimes_k N_i \otimes_i N_j \xrightarrow{(1 \otimes c_{kij})} V_k \otimes_k N_j.$$

It is easy to see that $\mathscr{V}(\boldsymbol{K}_F)$ is an additive category.

Now we define a functor $\mathfrak{F}$: $\mathscr{U}(\boldsymbol{K}_F) \to \mathscr{V}(\boldsymbol{K}_F)$ as follows. Given an object (U, X, φ) in $\mathscr{U}(\boldsymbol{K}_F)$ we put $V_F = \mathrm{Coker}(U \xrightarrow{\varphi} |X|_F)$. Let $X = X_1^{s_1} \oplus \dots \oplus X_n^{s_n}$, where $X_j^{s_j}$ denotes the direct sum of s_j copies of X_j. Then we have the isomorphisms

$$|X|_F \cong |X_1^{s_1}|_F \oplus \dots \oplus |X_n^{s_n}|_F$$
$$\cong V_1 \otimes |X_1|_F \oplus \dots \oplus V_n \otimes |X_n|_F \cong V_1 \otimes_1 N_{n+1} \oplus \dots \oplus V_n \otimes_n N_{n+1},$$

where $V_i = F_i^{s_i}$. Hence φ determines an epimorhism

$$\bar{t} = (\bar{t}_i)\colon\ \oplus V_i \otimes_i N_{n+1} \to V,$$

where each map $\bar{t}_i$ is adjoint to a uniquely determined F_i-linear map t_i: $V_i \to V_{F_i}$. We put

$$\mathfrak{F}(U, X, \varphi) = (V_i, t_i).$$

It is easy to check that an object (V_i, t_i) of $\mathscr{V}(\boldsymbol{K}_F)$ is in the image of $\mathfrak{F}$ if and only if the corresponding map $\bar{t} = (\bar{t}_i)$ is surjective. But this is the case if (V_i, t_i) has no direct summands of the form $(W_i, 0)$ with $W_{n+1} = F$ and $W_1 = \ldots = W_n = 0$.

It is also clear that a map $(U', X', \varphi') \to (U, X, \varphi)$ in $\mathscr{U}(\boldsymbol{K}_F)$ is uniquely determined by a system of maps $\bar{g}_{ij}$: $V_j' \otimes_j N_{n+1} \to V_i \otimes_i N_{n+1}$ such that the following diagram is commutative

$$\begin{tikzcd}
\oplus V_i \otimes_i N_{n+1} \arrow[r, "t"] & V \\
\oplus V_i' \otimes_i N_{n+1} \arrow[u, "(\bar{g}_{ij})"] \arrow[r, "t'"] & V' \arrow[u, "\bar{g}"']
\end{tikzcd}$$

or equivalently, for every j the diagram

(**)
$$\begin{tikzcd}
\bigoplus\limits_{i \leqslant j} V_i \otimes_i N_{n+1} \arrow[r, "(t_i)"] & V \\
V_j' \otimes_j N_{n+1} \arrow[u, "(\bar{g}_{ij})"] \arrow[r, "t_j"] & V' \arrow[u, "\bar{g}"']
\end{tikzcd}$$

is commutative. In view of the map

$$\theta\colon \operatorname{Hom}_{F_j}(V_j', V_i \otimes_i N_j) \to \operatorname{Hom}_F(V_j' \otimes_j N_{n+1}, V_i \otimes_i N_{n+1})$$

there are F_j-linear maps g_{ij}: $V_j' \to V_i \otimes_i N_j$ such that $\theta(g_{ij}) = (\bar{g}_{ij})$. Furthermore, diagram (**) is commutative if and only if diagram (*) is commutative (use Lemma 1.2 and the fact that θ is natural). We put $g_{n+1\,n+1} = g$ and $\mathfrak{F}(\bar{g}_{ij}) = (g_{ij})$. A simple computation shows that $\mathfrak{F}$ is an additive functor which is full and faithful. Then we have proved the following result.

PROPOSITION 1.3. *The functor* $\mathfrak{F}$: $\mathscr{U}_0(\boldsymbol{K}_F) \to \mathscr{V}(\boldsymbol{K}_F)$ *is full and faithful. Every indecomposable object in* $\mathscr{V}(\boldsymbol{K}_F)$ *except the simple object* $(W_i, 0)$ *with* $W_1 = \ldots = W_n = 0$ *and* $W_{n+1} = F$ *belongs to the image of* $\mathfrak{F}$.

For every $i = 1, \ldots, n$ we denote by $\boldsymbol{F}_i$ the simple object $(V_j^{(i)}, 0)$ in $\mathscr{V}(\boldsymbol{K}_F)$ with $V_i^{(i)} = F_i$ and $V_j^{(i)} = 0$ for $j \neq i$. The following lemma will be useful.

LEMMA 1.4. *If* (V_i, t_i) *is an indecomposable object in* $\mathscr{V}(\boldsymbol{K}_F)$, *then either* (V_i, t_i) *is isomorphic to some* $\boldsymbol{F}_j$ *or every map* t_i: $V_i \to V_{n+1} = V$ *is injective and* $(\operatorname{Im} t_i) \cap \sum_{j<i} \tilde{b}_{ij}(\operatorname{Im} t_j \otimes_j N_i) = 0$ *for* $i = 1, 2, \ldots, n$.

Proof. It is clear that the object $(K_i, 0)$ with $K_{n+1} = 0$ and $K_i = \operatorname{Ker} t_i$, $i \leqslant n$, is a direct summand of (V_i, t_i). Hence $t_1, \ldots, t_n$ are injective provided (V_i, t_i) is indecomposable but not isomorphic to some $\boldsymbol{F}_j$, $j = 1, \ldots, n$. Without loss of generality we can suppose that the t_i are inclusions.

Assume that $L_i = V_i \cap \sum_{j<i} \tilde{b}_{ij}(V_j \otimes_j N_i) \neq 0$ for a certain i and consider the object $\boldsymbol{L} = (L_k, 0)$ in $V(\boldsymbol{K}_F)$ with $L_k = 0$ for $k \neq i$. By our assumption we have a commutative diagram

$$\begin{tikzcd}
\bigoplus\limits_{j<i} V_j \otimes_j N_i \arrow[rr, "\oplus(t_j \otimes 1)"] & & \bigoplus\limits_{j<i} V \otimes_j N_i \arrow[d, "(\tilde{b}_{ij})"] \\
L_i \arrow[u, "(r_{ji})"] \arrow[rr, hook] \arrow[d, hook, "w"] & & V \\
V_i \arrow[urr, "t_i"'] & &
\end{tikzcd}$$

which yields the commutative diagram

$$\begin{tikzcd}
\bigoplus\limits_{j\leqslant i} V_j \otimes_j N_i \arrow[r, "\oplus(t_j \otimes 1)"] & \bigoplus\limits_{j\leqslant i} V \otimes_j N_i \arrow[r, "(\tilde{b}_{ij})"] & V \\
L_i \arrow[u, "(r_{ji})"] \arrow[rr] & & 0 \arrow[u]
\end{tikzcd}$$

with $r_{ii} = -w$. Then we have defined a split embedding $\boldsymbol{L} \to (V_j, t_j)$ and we get a contradiction. The lemma is proved.

Now we are going to define the functor $\mathfrak{G}\colon \mathscr{V}(\boldsymbol{K}_F) \to l\,\mathrm{her}(\boldsymbol{R}_K)$. First we recall from [3], and [15] (Sec. 3) that modules M in $l\,\mathrm{her}(\boldsymbol{R}_K)$ can be identified with families of F_i-modules M_i, $i = 1, \ldots, n+1$, together with F_j-linear maps ${}_j\varphi_i\colon M_i \otimes_i N_j \to M_j$ satisfying the following conditions:

1° ${}_i\varphi_i\colon M_i \otimes F_i \to M_i$ defines the structure of F_i-module on M_i,

2° ${}_k\varphi_i(1 \otimes c_{ijk}) = {}_k\varphi_j({}_j\varphi_i \otimes 1)$ for $i \leqslant j \leqslant k$,

3° ${}_j\varphi_i(-\otimes x)\colon M_i \to M_j$ is injective for any non-zero x in ${}_iN_j$.

The relationship between M and $(M_i, {}_j\varphi_i)$ is the following. We consider the module M in $l\,\mathrm{her}(\boldsymbol{R}_K)$ as a contravariant additive functor from the category consisting of finitely generated projective right $\boldsymbol{R}_K$-modules to the category of abelian groups. Obviously every such functor is uniquely determined by its values on the modules $P_1, \ldots, P_n, P_{n+1}$. We take for M_i the value of M on P_i and given a map $x \in {}_iN_j \cong \operatorname{Hom}_{\boldsymbol{R}_K}(P_j, P_i)$ we take for ${}_j\varphi_i(x)\colon M_i \to M_j$ the map $M(x)$ induced by x.

Now given an object (V_i, t_i) in $\mathscr{V}(\boldsymbol{K}_F)$ we define $\mathfrak{G}(V_i, t_i) = (M_i, {}_j\varphi_i)$ as follows. We take for M_i the image of the composed map

$$\bigoplus_{j\leqslant i} V_j\otimes_j N_i \xrightarrow{\oplus(t_j\otimes 1)} \bigoplus_{j\leqslant i} V\otimes_j N_i \xrightarrow{(\tilde{b}_{ij})} V,$$

where $V = V_{n+1}$. It follows from Lemma 1.2 that for $i \leqslant k$ we have a unique factorization

$$\begin{tikzcd}
M_i \otimes_i N_k \arrow[r] \arrow[d, "{}_k\varphi_i"] & V \otimes_i N_k \arrow[d, "\tilde{b}_{ki}"] \\
M_k \arrow[r] & V
\end{tikzcd}$$

such that conditions 1°–3° are satisfied. We recall that $M_{n+1} = V_{n+1} = V$.

Now suppose (g_{ij}): $(V_i', t_i') \to (V_i, t_i)$ is a map in $\mathscr{V}(\boldsymbol{K}_F)$ and let $\mathfrak{G}(V_i', t_i') = (M_i', {}_j\varphi_i')$ with $M'_{n+1} = V'_{n+1} = V'$. We denote by g the map $g_{n+1\,n+1}$: $V' \to V$. By Lemma 1.2 the commutative diagram $(*)$ yields the following commutative diagram

$$\begin{tikzcd}
\bigoplus_{k\leqslant j} V_k\otimes_k N_i \arrow[rrd, "\oplus(t_k\otimes 1)"] & & \\
(\bigoplus_{k\leqslant j} V_k\otimes_k N_j)\otimes_j N_i \arrow[u, "\oplus(1\otimes c_{kji})"] \arrow[r, "\oplus(t_k\otimes 1)"] & \bigoplus_{k\leqslant j} V\otimes_k N_j\otimes_j N_i \arrow[r, "\oplus(1\otimes c_{kji})"] \arrow[d, "(\tilde{b}_{jk}\otimes 1)"] & \bigoplus_{k\leqslant j} V\otimes_k N_i \arrow[d, "(\tilde{b}_{ik})"] \\
 & V\otimes_j N_i \arrow[r, "\tilde{b}_{ij}"] & V \\
V_j'\otimes_j N_i \arrow[uu, "(g_{kj})\otimes 1"] \arrow[r, "t_j'\otimes 1"] & V'\otimes_j N_i \arrow[u, "g\otimes 1"] \arrow[r, "\tilde{b}'_{ij}"] & V' \arrow[u, "g"]
\end{tikzcd}$$

It follows that $g(M_i') \subset M_i$ and therefore we get a family of F_i-linear maps g_i: $M_i' \to M_i$ such that $g_{jj}\varphi_i' = {}_j\varphi_i(g_i\otimes 1)$. We put $\mathfrak{G}(g_{ij}) = (g_i)$. It is clear the we have defined an additive functor $\mathfrak{G}$.

Now we are able to prove an important result which together with Proposition 1.3 yields our Theorem 1.1. A special case of it was announced in [17], Theorem 2.4.

THEOREM 1.5. (1) *The functor* $\mathfrak{G}$: $\mathscr{V}(\boldsymbol{K}_F) \to l\,\mathrm{her}(\boldsymbol{R}_{\boldsymbol{K}})$ *is full and dense.*

(2) *If A and $\boldsymbol{B}$ are indecomposable objects in $\mathscr{V}(\boldsymbol{K}_F)$ having no direct summands isomorphic to the simple objects $\boldsymbol{F}_1, \ldots, \boldsymbol{F}_n$, then every isomorphism from $\mathfrak{G}(A)$ to $\mathfrak{G}(\boldsymbol{B})$ has the form $\mathfrak{G}(\boldsymbol{g})$, where $\boldsymbol{g}$: $A \to \boldsymbol{B}$ is an isomorphism.*

(3) *If A is an indecomposable object in $\mathscr{V}(\boldsymbol{K}_F)$, then $\mathfrak{G}(A) = 0$ if and only if A is isomorphic with some $\boldsymbol{F}_i$, $i = 1, \ldots, n$.*

Proof. In order to prove that $\mathfrak{G}$ is dense take a module $M = (M_i, {}_j\varphi_i)$ in $l\,\mathrm{her}(\boldsymbol{R}_K)$ and define an object (V_i, t_i) in $\mathscr{V}(\boldsymbol{K}_F)$ such that $\mathfrak{G}(V_i, t_i) = M$. We put $V_{n+1} = M_{n+1}$. For any i we denote by s_i the composed monomorphism $M_i \xrightarrow{{}_{n+1}\bar{\varphi}_i} \mathrm{Hom}_F({}_iN_{n+1}, M_{n+1}) \xrightarrow{\simeq} M_{n+1}$, where ${}_{n+1}\bar{\varphi}_i$ is the map adjoint to ${}_{n+1}\varphi_i$ and the right-hand map is the isomorphism induced by our fixed bimodule isomorphism ${}_iN_{n+1} \cong {}_{F_i}F_F$. A simple calculation shows that the equality ${}_{n+1}\varphi_i(1\otimes c_{ijn+1}) = {}_{n+1}\varphi_j({}_j\varphi_i\otimes 1)$ yields the commutative diagram

$$\begin{tikzcd}
M_i \otimes {}_iN_j \arrow[r, "s_i\otimes 1"] \arrow[d, "{}_j\varphi_i"] & M_{n+1}\otimes {}_iN_j \arrow[d, "\tilde{b}_{ji}"] \\
M_j \arrow[r, "s_j"] & M_{n+1}
\end{tikzcd}$$

Since $s_1, \ldots, s_n$ are injective then without loss of generality we can suppose they are inclusions.

If i is a minimal element in the poset $(\boldsymbol{I}_{\boldsymbol{R}_K}, \boldsymbol{d})$, then we put $V_i = M_i$. If i is arbitrary and V_j are defined for $j < i$ (with respect to the order in $\boldsymbol{I}_{\boldsymbol{R}_K}$) we take for V_i an F_i-subspace of M_{n+1} such that

$$M_i = V_i \oplus \sum_{j<i} \tilde{b}_{ij}(V_j \otimes {}_jN_i).$$

Then we have defined V_i for any $i = 1, \ldots, n+1$ and we take for t_i: $V_i \to V_{n+1}$ the inclusions. From the definition of $\mathfrak{G}$ immediately follows that $\mathfrak{G}(V_i, t_i) = M$ and therefore $\mathfrak{G}$ is dense.

Now let (g_i): $(M_i', {}_j\varphi_i') \to (M_i, {}_j\varphi_i)$ be a map in $l\,\mathrm{her}(\boldsymbol{R}_K)$ and suppose $\mathfrak{G}(V_i', t_i') = (M_i', {}_j\varphi_i')$, $\mathfrak{G}(V_i, t_i) = (M_i, {}_j\varphi_i)$. We will define F_j-linear maps g_{ij}: $V_j' \to V_i \otimes {}_iN_j$ for any $i \leqslant j \leqslant n+1$ in a such a way that for any j diagram $(*)$ is commutative. By the definition of $\mathfrak{G}$ we know that $M_{n+1} = V_{n+1}$, $M_{n+1}' = V_{n+1}'$.

We put $g_{n+1\,n+1} = g_{n+1}$. For any j we consider the following diagram

$$\begin{tikzcd}
\bigoplus\limits_{i\leqslant j} V_i \otimes {}_iN_j \arrow[r, "\oplus(t_i\otimes 1)"] & V_{n+1}\otimes {}_iN_j \arrow[r, "\tilde{b}_{ji}"] & M_j \\
V_j' \arrow[u, "(g_{ij})"] \arrow[rr, dashed, "t_j'"] & & M_j' \arrow[u, "g_j"']
\end{tikzcd}$$

We know from the definition of the functor $\mathfrak{G}$ that the horizontal composed map in the diagram is surjective. Hence there exists a map (g_{ij}) making the diagram commutative. It follows that the linear maps g_{ij} are such that for any j diagram $(*)$ is commutative. Therefore (g_{ij}) is a morphism in $\mathscr{V}(\boldsymbol{K}_F)$. Since obviously $\mathfrak{G}(g_{ij}) = (g_i)$ then the functor $\mathfrak{G}$ is full.

In order to prove (2) we put $\boldsymbol{A} = (V_i, t_i)$, $\boldsymbol{B} = (V_i', t_i')$ and we keep the notation above. We define an F_j-subspace $\bar{M}_j$ of M_j by formula

$$\bar{M}_j = \sum_{i<j} \operatorname{Im}\, {}_j\varphi_i, \qquad j = 1, \ldots, n+1.$$

From our assumptions and Lemma 1.4 we easily conclude that

(i) $g_j(\bar{M}_j') \subset \bar{M}_j$,

(ii) $\bar{M}_j = \sum_{i<j} \tilde{b}_{ji}(\operatorname{Im} t_i \otimes_i N_j)$,

(iii) $M_j = \operatorname{Im} t_j \oplus \bar{M}_j$ and $M_j' = \operatorname{Im} t_j' \oplus \bar{M}_j'$.

Assume that (g_i) is an isomorphism. Then every map g_j is an isomorphism and it follows from (i)–(iii) that the map $g_j t_j'$: $V_j' \to \operatorname{Im} t_j \oplus \bar{M}_j$ has the form (g_j', g_j''), where g_j': $V_j' \to \operatorname{Im} t_j$ is an isomorphism. Hence the map g_{jj} in the diagram above is an isomorphism. Therefore the map (g_{ij}) is an isomorphism and (2) follows.

Since the statement (3) follows immediately from the definition of $\mathfrak{G}$ the proof of the theorem is complete.

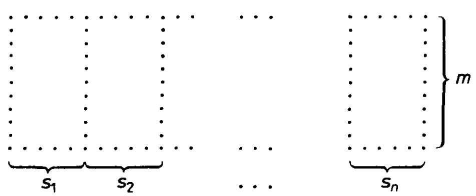
Remark. The method we use in the proof of Theorem 1.1 is similar to that used by Drozd [5].

As an immediate consequence of Theorem 1.1 we have the following

COROLLARY 1.6. *If $\boldsymbol{K}_F$ is a special schurian vector space category, then the category $\mathscr{U}(\boldsymbol{K}_F)$ is of finite representation type if and only if* $l\,\mathrm{her}(\boldsymbol{R}_K)$ *is of finite representation type.*

Remark 1. Theorem 1.1 can be used for computations of indecomposable modules over artinian rings in a way similar to that one the representations of partially ordered sets are used in [11], [14]. In [18] we use Theorem 1.1 for the description of indecomposable modules over l-hereditary artinian PI-rings of finite representation type. A detailed discussion of the use of Theorem 1.1 can be found in [18], Remark 4.

Remark 2. The factorization of the functor Φ through the category $\mathscr{V}(\boldsymbol{K}_F)$ allows us to interprete any object $\mathfrak{F}(A) = (V_j, t_j)$ with $A \in \mathscr{U}(\boldsymbol{K}_F)$ as a block matrix of the form



with coefficients in the field $F = F_{n+1}$, where $m = \dim(V_{n+1})_F$ and $s_j = \dim(V_j)_{F_j}$ for $j = 1, 2, \ldots, n$. In the particular case when $F = F_1 = F_2$

$= \ldots = F_n$ and every non-zero bimodule ${}_iN_j$ is equal to ${}_FF_F$ we are in the position of Nazarova and Rojter [10] with the partially ordered set $N = \boldsymbol{I}_{\boldsymbol{R}_K} \setminus \{n+1\}$. In this case the category $l\,\mathrm{her}(\boldsymbol{R}_K)$ is equivalent with the category N-sp of all finite dimensional N-spaces over F in the sense of Gabriel [6] and therefore the functor $\mathfrak{G}$ establishes a well-known connection between the category of the matrix representations of the partially ordered set N (in the sense of Nazarova and Rojter [10]) and the category N-sp.

2. A connection of subspace categories and Kleisli categories. Now we are going to show that the results in Section 1 allow us to relate the study of subspace category of any special schurian category with Kleisli categories and with the theory of BOCS' [12].

We recall from [8] that given a monad $T\colon \mathscr{B} \to \mathscr{B}$ in a category $\mathscr{B}$ the *Kleisli category* $\mathscr{B}_T$ of $\mathscr{B}$ with respect to T is the category having the same objects as $\mathscr{B}$ has whereas the set of maps $(X, Y)_T$ from X into Y in $\mathscr{B}_T$ is the set $\mathscr{B}(X, TY)$ of all maps from X into TY in $\mathscr{B}$. The composition of $f \in (X, Y)_T$ and $g \in (Y, Z)_T$ in $\mathscr{B}_T$ is the composed map

$$X \xrightarrow{f} TY \xrightarrow{Tg} T^2 Z \xrightarrow{m(Z)} TZ,$$

where $m\colon T^2 \to T$ is the multiplication of the monad T. It is easy to see that $\mathscr{B}_T$ is an additive category if so is $\mathscr{B}$.

Now suppose that $\boldsymbol{K}_F$ is a special schurian vector space category and we keep other notation introduced in Section 1. Let $\boldsymbol{R}_K$ be the l-hereditary right QF-2 ring associated with $\boldsymbol{K}_F$ and let us consider the hereditary ring

$$\boldsymbol{S}_K = \begin{bmatrix} F_1 & 0 & 0 & \cdots & 0 & {}_1N_{n+1} \\ & F_2 & 0 & \cdots & 0 & {}_2N_{n+1} \\ & & & \ddots & \vdots & \vdots \\ & & & & F_n & {}_nN_{n+1} \\ & 0 & & & & F_{n+1} \end{bmatrix}$$

The ring embedding $\boldsymbol{S}_K \subset \boldsymbol{R}_K$ induces a monad $T\colon \mathrm{mod}(\boldsymbol{S}_K) \to \mathrm{mod}(\boldsymbol{S}_K)$ given by the formula $T(\cdot) = - \otimes_{S_K} \boldsymbol{R}_K$ and we have the following useful result:

PROPOSITION 2.1. *There is an equivalence of categories* $\mathscr{V}(\boldsymbol{K}_F) \cong \mathrm{mod}(\boldsymbol{S}_K)_T$.

Proof. Given an object $A = (V_i, t_i)$ in $\mathscr{V}(\boldsymbol{K}_F)$ we denote by $H(A)$ the $\boldsymbol{S}_K$-module $(V_i, \tilde{t}_i)$, where $\tilde{t}_i\colon V_i \otimes_i N_{n+1} \to V_{n+1}$ corresponds to t_i via the composed isomorphism

$$\begin{aligned} \mathrm{Hom}_{F_{n+1}}(V_i \otimes_i N_{n+1}, V_{n+1}) &\cong \mathrm{Hom}_{F_i}\big(V_i, \mathrm{Hom}_{F_{n+1}}({}_iN_{n+1}, V_{n+1})\big) \\ &\cong \mathrm{Hom}_{F_i}\big(V_i, (V_{n+1})_{F_i}\big). \end{aligned}$$

Moreover, if (g_{ij}): $A' \to A$ is a map in $\mathscr{V}(K_F)$ and we denote by (H_i, w_i) the right S_K-module $H(A)\otimes_{S_K} R_K$, then obviously $H_j = \bigoplus_{i \leqslant j} V_i \otimes_i N_j$ and in view of diagram $(*)$ we have defined an S_K-homomorphism $H(g_{ij})$: $H(A') \to H(A)\otimes_{S_K} R_K$ which defines a map $H(g_{ij}) \in (H(A'), H(A))_T$. It is easy to check that we have defined an additive functor

$$H\colon \mathscr{V}(K_F) \to \operatorname{mod}(S_K)_T$$

which is an equivalence of categories. The details are left to the reader.

We note that Proposition 2.1 together with results in [1], [7] allows us to apply the methods developed in [12] for the representations of BOCS' to the study of subspace categories $\mathscr{U}(K_F)$, where K_F is special schurian. In particular, the categorical interpretation of the Rojter's classification algorithm given in [1], [7] allows us to define a similar algorithm for our Kleisli category $\operatorname{mod}(S_K)_T$ and therefore we can use it in the study of $\mathscr{U}(K_F)$.

Note also that Proposition 2.1 allows us to define a sequence of partial Coxeter functors for the category $\operatorname{mod}(S_K)_T$ provided R_K is an artinian PI-ring. In this case there is a sequence of partial Coxeter functors (see [4], [16])

$$\ldots \rightleftarrows \operatorname{mod}(A_{-1}) \underset{s_0^-}{\overset{s_0^+}{\rightleftarrows}} \operatorname{mod}(S_K) \underset{s_1^-}{\overset{s_1^+}{\rightleftarrows}} \operatorname{mod}(A_1) \underset{s_2^-}{\overset{s_2^+}{\rightleftarrows}} \ldots,$$

where A_j are hereditary PI-rings. Then in order to define a sequence of partial Coxeter functors for the Kleisli category $\operatorname{mod}(S_K)_T$ it is enough to define monads T_i: $\operatorname{mod}(A_i) \to \operatorname{mod}(A_i)$ and natural transformations $S_i^+ T_{i-1} \to T_i S_i^+$, $S_i^- T_i \to T_{i-1} S_i^-$ having appropriate "good" properties. Then the sequence above will induce a sequence

$$\ldots \rightleftarrows \operatorname{mod}(A_{-1})_{T_{-1}} \underset{\tilde{s}_0^-}{\overset{\tilde{s}_0^+}{\rightleftarrows}} \operatorname{mod}(S_K)_T \underset{\tilde{s}_1^-}{\overset{\tilde{s}_1^+}{\rightleftarrows}} \operatorname{mod}(A_1)_{T_1} \underset{\tilde{s}_2^-}{\overset{\tilde{s}_2^+}{\rightleftarrows}} \ldots$$

which can be used for the study of the category $\operatorname{mod}(S_K)_T$ in a similar way to that one the partial Coxeter functors are used in the study of hereditary artinian rings [16]. We will discuss the problem in a subsequent paper.

References

[1] R. Bautista, L. Colavita and L. Salmeron, *On adjoint functors in representation theory*, Lecture Notes in Math., No. 903 (1981), 9–25.

[2] R. Bautista and R. Martinez, *Representations of partially ordered sets and 1-Gorenstein Artin algebras*, Proc. Conf. Ring Theory (Antwerp 1978), Marcel Dekker, Inc. New York and Basel 1979.

[3] R. Bautista and D. Simson, *Torsionless modules over 1-Gorenstein l-hereditary artinian rings*, Comm. Algebra 12 (1984).

[4] P. Dowbor and D. Simson, *Quasi-Artin species and rings of finite representation type*, J. Algebra 63 (1980), 435–443.

[5] Ju. A. Drozd, *Coxeter transformations and representations of partially ordered sets*, Funkc. Anal. i Priložen. 8 (1974), 34–42 (in Russian).

[6] P. Gabriel, *Indecomposable representations. II*, in *Symposia Mathemaica*, Vol. XI, 81–104, Academic Press, London 1973.

[7] M. M. Kleiner, *Induced modules and comodules and representations of* BOCS's *and* DGC's, Lecture Notes in Math. No. 903 (1981), 168–175.

[8] Z. Leszczyński and D. Simson, *On triangular matrix rings of finite representation type*, J. London Math. Soc. 20 (1979), 396 402.

[9] S. Mac Lane, *Categories for the Working Mathematicians*, Springer-Verlag, New York–Heidelberg–Berlin 1971.

[10] L. A. Nazarova and A. V. Rojter, *Representations of partially ordered sets*, Zap. Naučn. Sem. LOMI 28 (1972), 5–31.

[11] –, –, *Kategorielle Matrizen-Probleme und die Brauer–Thrall-Vermutung*, Mittelungen Math. Seminar Giessen 115 (1975), 1–153.

[12] A. V. Rojter, *Matrix problems and representations of BOCS's*, Lecture Notes in Math. 831 (1980), 288–324.

[13] C. M. Ringel, *Report on the Brauer–Thrall conjectures*, ibidem 831 (1980), 104–136.

[14] –, *Tame algebras*, ibidem 831 (1980), 137–287.

[15] D. Simson, *Categories of representations of species*, J. Pure Appl. Algebra 14 (1979), 101–114.

[16] –, *Partial Coxeter functors and right pure semisimple hereditary rings*, J. Algebra 71 (1981), 195–218.

[17] –, *Right pure semisimple structures of division rings*, Proceedings of the Eleventh Annual Iranian Math. Conf., March 1980, 186–201.

[18] –, *Right pure semisimple l-hereditary PI-rings*, Rend. Sem. Mat. Univ. Padova, 71 (1984), 1–35.

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