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On the finest of all linear topologies on Orlicz spaces for which φ-modular convergence implies convergence in these topologies. II

Abstract. Let φ be a φ-function and let (E, Σ, μ) be a measure space with a positive measure. In [3] we have considered the finest of all linear topologies $\mathscr{T}$ on Orlicz spaces $L_E^{*\varphi}$ which satisfy the condition: $x_n \xrightarrow{\varphi} 0$ implies $x_n \xrightarrow{\mathscr{T}} 0$ for every sequence (x_n) in $L_E^{*\varphi}$. We denote this topology by $\mathscr{T}_\varphi^{\wedge}$. In the present paper we prove that a set $K \subset L_E^{*\varphi}$ is bounded in the topology $\mathscr{T}_\varphi^{\wedge}$ if and only if K is bounded in the topology generated by the usual F-norm $\|\cdot\|_\varphi$.

PRELIMINARIES

0.1. Throughout this paper we assume that (E, Σ, μ) is a measure space with a positive measure and φ is a φ-function. By $L^{*\varphi}$ we will denote the Orlicz spaces. For a real valued, μ-measurable function x defined on E we write

$$\varrho_\varphi(x) = \int_E \varphi(|x(t)|)\, d\mu.$$

In $L^{*\varphi}$ and F-norm can be defined as follows:

$$\|x\|_\varphi = \inf\{\lambda > 0\colon\ \varrho_\varphi(x/\lambda) \leqslant \lambda\} \qquad ([1]).$$

We shall denote by $\mathscr{T}_\varphi$ the topology on $L^{*\varphi}$ generated by the F-norm $\|\cdot\|_\varphi$.

0.2. Let ψ and φ be φ-functions. We shall write $\psi \lhd \varphi$ if for an arbitrary $c > 1$ there exists a constant $d > 1$ such that $\psi(cu) \leqslant d\varphi(u)$ ([2], p. 72). For a given φ-function φ by $\Psi^{\lhd\varphi}$ we will denote the set of all φ-functions ψ such that $\psi \lhd \varphi$.

0.3. By $\mathscr{T}_\varphi^{\wedge}$ we will denote the finest of all linear topologies $\mathscr{T}$ on $L^{*\varphi}$ which satisfy the condition: $x_n \xrightarrow{\varphi} 0$ implies $x_n \xrightarrow{\mathscr{T}} 0$ for every sequence (x_n) in $L^{*\varphi}$ (see [3], Theorem 1.2). It has a base of neighbourhoods of 0 consisting of all sets of the form:

$$\bigcup_{N=1}^{\infty} \Big(\sum_{n=1}^{N} U_\varphi(\varepsilon_n)\Big),$$

where (ε_n) is a sequence of positive numbers and $U_\varphi(\varepsilon_n) = \{x \in L^{*\varphi}: \varrho_\varphi(x) < \varepsilon_n\}$ ([3], Definition 1.1). From Theorem 1.2 of [2] and Theorem 2.1 of [3] it follows that all sets of the form: $K_\psi(r) \cap L^{*\varphi}$, where $r > 0$ and $\psi \in \Psi^{\lhd\varphi}$, constitute a base of neighbourhoods of 0 for the topology $\mathscr{T}_\varphi^\wedge$.

RESULTS

1.1. Lemma. *Let φ be a φ-function and let (k_n) be a strictly increasing sequence of natural numbers with $k_1 > 1$. Then there exists a φ-function ψ such that $\psi \lhd \varphi$ and there hold the inequalities:*

(a) $$\psi(2^{-m+2}) \geqslant \varphi(2^{-m-n}) \quad \text{for } nk_n \leqslant m \leqslant (n+1)k_{n+1}-1,$$

(b) $$\psi(2^{m+2}) \geqslant \varphi(2^{m-n}) \quad \text{for } nk_n \leqslant m \leqslant (n+1)k_{n+1}-1.$$

Proof. Let us set

$$p(t) \begin{cases} 0 & \text{for } t = 0, \\ 2^{-n} & \text{for } 2^{-(n+1)(k_{n+1}-1)} < t \leqslant 2^{-n(k_n-1)}, \quad n = 1, 2, \ldots, \\ 1 & \text{for } 2^{-k_1+1} < t \leqslant 2^{k_1-1}, \\ 2^n & \text{for } 2^{n(k_n-1)} < t \leqslant 2^{(n+1)(k_{n+1}-1)}, \quad n = 1, 2, \ldots \end{cases}$$

Next, define a function χ by the equality:

$$\chi(u) = \begin{cases} \chi'(u) & \text{for } u \in [0, 1], \\ \chi''(u) & \text{for } u \in [1, \infty), \end{cases}$$

where $\chi'\colon [0, 1] \to [0, u_1]$, $\chi'(u) = \int_0^u p(t)\,dt$, $\frac{1}{2} < u_1 = \chi'(1) < 1$, and $\chi''\colon [1, \infty) \to [u_1, \infty)$, $\chi''(u) = \alpha^{-1}(u)$, $\alpha(u) = \int_{u_1}^u p(t)\,dt + 1$.

Finally, define $\psi(u) = \varphi(\chi(u))$. It can be seen that ψ is a φ-function.

Now, we shall show that $\psi \lhd \varphi$, i.e., for an arbitrary $c > 0$ there exists a constant $d > 1$ such that $\psi(cu) \leqslant d\varphi(u)$ for $u \geqslant 0$. Indeed, let $c > 0$ be given. Choose a natural number n such that $c \leqslant 2^{n-1}$. First, let $u_c' = 2^{-n(k_n-1)} \cdot 2^{-n+1}$. Then for $u \leqslant u_c'$ we have

(1) $$\chi'(cu) \leqslant p(cu)cu \leqslant p(2^{-n(k_n-1)})cu \leqslant 2^{-n}cu < u.$$

Next, let $u_c'' = 2^{(n+1)(k_{n+1}-1)+1}$. Since $u_1 < 1$ for $u \geqslant u_c''$, we have

$$\alpha(u) = \int_{u_1}^u p(t)\,dt + 1 \geqslant \int_1^u p(t)\,dt + \int_0^1 p(t)\,dt = \int_0^u p(t)\,dt \geqslant p(u/2)(u/2)$$
$$= 2^{n-1}u \geqslant cu.$$

Hence

(2) $$\chi''(cu) \leqslant u \quad \text{for } u \geqslant u_c''.$$

From (1) and (2) for $u \leqslant u_c'$ and $u \geqslant u_c''$ we have

$$\psi(cu) = \varphi(\chi(cu)) \leqslant \varphi(u).$$

Suppose $u_c' < u_c''$. Then finally, we obtain

$$\psi(cu) \leqslant d_c \varphi(u) \quad \text{for } u \geqslant 0, \text{ where } d_c = \varphi(u_c'')/\varphi(u_c').$$

Now, we shall show that $\psi(2^{-m+2}) \geqslant \varphi(2^{-m-n})$ for $nk_n \leqslant m \leqslant (n+1)k_{n+1}-1$. In fact, since $2^{-m+1} \geqslant 2^{-(n+1)k_{n+1}+2} > 2^{-(n+2)(k_{n+2}-1)}$ and $2^{-m+2} \leqslant 1$, we have

$$\chi'(2^{-m+2}) \geqslant p(2^{-m+1}) \cdot 2^{-m+1} = 2^{-n-1} \cdot 2^{-m+1} = 2^{-m-n},$$

and hence

$$\psi(2^{-m+2}) = \varphi(\chi'(2^{-m+2})) \geqslant \varphi(2^{-m-n}).$$

Finally, we shall show that $\psi(2^{m+2}) \geqslant \varphi(2^{m-n})$ for $nk_n \leqslant m \leqslant (n+1)k_{n+1}-1$. In fact, since $1 < 2^{n(k_n-1)} \leqslant 2^{m-n} \leqslant 2^{(n+1)(k_{n+1}-1)}$, we have

$$\alpha(2^{m-n}) = \int_{u_1}^{2^{m-n}} p(t)\,dt + 1 \leqslant p(2^{m-n}) \cdot 2^{m-n} + 1 \leqslant 2^n \cdot 2^{m-n} + 1 < 2^{m+2}.$$

Then $\chi''(2^{m+2}) > 2^{m-n}$, and hence

$$\psi(2^{m+2}) = \varphi(\chi''(2^{m+2})) > \varphi(2^{m-n}).$$

1.2. THEOREM. *A set $K \subset L^{*\varphi}$ is bounded in the topology $\mathscr{T}_\varphi^\wedge$ if and only if K is bounded in the topology $\mathscr{T}_\varphi$ generated by the usual F-norm $\|\cdot\|_\varphi$.*

Proof. Since $\mathscr{T}_\varphi^\wedge \subset \mathscr{T}_\varphi$ ([3], Theorem 1.3), it suffices to show that if a set K is not bounded in the topology $\mathscr{T}_\varphi$, then K is not bounded in the topology $\mathscr{T}_\varphi^\wedge$.

Indeed, let us assume that a set K is not bounded in $\mathscr{T}_\varphi$. This means that there exists a number $r_0 > 0$ such that for every $\lambda > 0$ there exists $x \in K$ such that $\|\lambda x\|_\varphi > r_0$, i.e., $\varrho_\varphi(\lambda x/r_0) > r_0$. From this it follows that there exists a sequence (x_n) in K such that

(1) $$\|4^{-n-1} r_0 x_n\|_\varphi > r_0, \quad n = 1, 2, \ldots,$$

i.e.,

(2) $$\varrho_\varphi(4^{-n-1} x_n) > r_0, \quad n = 1, 2, \ldots$$

In order to prove that K is not bounded in $\mathscr{T}_\varphi^\wedge$, by 0.3 it suffices to show that there exists a φ-function ψ such that

$$\psi \lhd \varphi \quad \text{and} \quad \varrho_\psi(2^{-n+4} x_n) > r_0/9,$$

where $n \geqslant N$ for some natural number N.

For this purpose, for a measurable function $x\colon E \to \boldsymbol{R}$ and $-\infty \leqslant a < b \leqslant +\infty$, let us denote

$$E_a^b(x) = \{t \in E\colon\ 2^a \leqslant |x(t)| < 2^b\}$$

and let $F_a^b(x)\colon E \to \boldsymbol{R}$ be the function defined as

$$F_a^b(x)(t) = \begin{cases} x(t) & \text{for } t \in E_a^b(x), \\ 0 & \text{for } t \in E \setminus E_a^b(x). \end{cases}$$

Now, we shall show that from (1) it follows that there exists a strictly increasing sequence of natural numbers (k_n), $k_1 > 1$, such that for any n holds:

$$\begin{aligned} &\|4^{-n-1} r_0 F_{-((n+1)k_{n+1}+n-2)}^{-(nk_n+n-1)}(x_n)\|_\varphi > r_0/3, \quad \text{or} \\ (3) \qquad &\|4^{-n-1} r_0 F_{-(nk_n+n-1)}^{nk_n+n-1}(x_n)\|_\varphi > r_0/3, \quad \text{or} \\ &\|4^{-n-1} r_0 F_{nk_n+n-1}^{(n+1)k_{n+1}+n-2}(x_n)\|_\varphi > r_0/3. \end{aligned}$$

We shall construct this sequence (k_n) as follows. Let k_1 be an arbitrary natural number such that $k_1 > 1$. Then by (1) we have

$$\|4^{-2} r_0 F_{-\infty}^{-k_1}(x_1)\|_\varphi > r_0/3 \quad \text{or} \quad \|4^{-2} r_0 F_{-k_1}^{k_1}(x_1)\|_\varphi > r_0/3$$
$$\text{or} \quad \|4^{-2} r_0 F_{k_1}^{\infty}(x_1)\|_\varphi > r_0/3.$$

Now, assume that there holds $\|4^{-2} r_0 F_{-\infty}^{-k_1}(x_1)\|_\varphi > r_0/3$ or $\|4^{-2} r_0 F_{k_1}^{\infty}(x_1)\|_\varphi > r_0/3$. Since $\|4^{-2} r_0 F_{-a}^{-k_1}(x_1)\|_\varphi$ and $\|4^{-2} r_0 F_{k_1}^{b}(x_1)\|_\varphi$ are non-decreasing functions of variable $a > k_1$ and $b > k_1$ and

$$\lim_{a\to\infty} \|4^{-2} r_0 F_{-a}^{-k_1}(x_1)\|_\varphi = \|4^{-2} r_0 F_{-\infty}^{-k_1}(x_1)\|_\varphi > r_0/3$$

or

$$\lim_{b\to\infty} \|4^{-2} r_0 F_{k_1}^{b}(x_1)\|_\varphi = \|4^{-2} r_0 F_{k_1}^{\infty}(x_1)\|_\varphi > r_0/3,$$

there exists a natural number $k_2' > k_1$ such that

$$\|4^{-2} r_0 F_{-(2k_2'-1)}^{-k_1}(x_1)\|_\varphi > \tfrac{1}{3} r_0 \quad \text{or} \quad \|4^{-2} r_0 F_{k_1}^{2k_2'-1}(x_1)\|_\varphi > \tfrac{1}{3} r_0.$$

Now, suppose there are already chosen natural numbers $k_1 < k_2 < \ldots < k_{n-1}$. Let $k_n = k_n' > k_{n-1}$ in the case where k_n' was already chosen in the previous step or, otherwise, let k_n be an arbitrary natural number such that $k_n > k_{n-1}$. Then by (1) we have

$$\|4^{-n-1} r_0 F_{-\infty}^{-(nk_n+n-1)}(x_n)\|_\varphi > \tfrac{1}{3} r_0 \quad \text{or} \quad \|4^{-n-1} r_0 F_{-(nk_n+n-1)}^{nk_n+n-1}(x_n)\|_\varphi > \tfrac{1}{3} r_0$$
$$\text{or} \quad \|4^{-n-1} r_0 F_{nk_n+n-1}^{\infty}(x_n)\|_\varphi > \tfrac{1}{3} r_0.$$

Now, assume that there holds $\|4^{-n-1} r_0 F_{-\infty}^{-(nk_n+n-1)}(x_n)\|_\varphi > \tfrac{1}{3} r_0$ or

$\|4^{-n-1} r_0 F^{\infty}_{nk_n+n-1}(x_n)\|_\varphi > \frac{1}{3} r_0$. Then as above (for $n=1$) from these inequalities it follows that there exists a natural number $k'_{n+1} > k_n$ such that

$$\|4^{-n-1} r_0 F^{-(nk_n+n-1)}_{-((n+1)k'_{n+1}+n-2)}(x_n)\|_\varphi > \tfrac{1}{3} r_0$$

or

$$\|4^{-n-1} r_0 F^{(n+1)k'_{n+1}+n-2}_{nk_n+n-1}(x_n)\|_\varphi > \tfrac{1}{3} r_0.$$

Thus, we constructed a strictly increasing sequence (k_n) which satisfies (3).

From (3) we have

$$\text{(I)}\qquad \begin{aligned} \tfrac{1}{3} r_0 &< \varrho_\varphi\big(4^{-n} F^{-(nk_n+n-1)}_{-((n+1)k_{n+1}+n-2)}(x_n)\big) \\ &\leqslant \sum_{m=nk_n+n-1}^{(n+1)k_{n+1}+n-3} \varphi(4^{-n}\cdot 2^{-m})\,\mu\big(E^{-m}_{-m-1}(x_n)\big) \\ &= \sum_{m=nk_n}^{(n+1)k_{n+1}-2} \varphi(2^{-m-n-1})\,\mu\big(E^{-m+n-1}_{-m+n-2}(x_n)\big) \end{aligned}$$

or

$$\text{(II)}\qquad \begin{aligned} \tfrac{1}{3} r_0 &< \varrho_\varphi\big(4^{-n} F^{nk_n+n-1}_{-(nk_n+n-1)}(x_n)\big) \\ &\leqslant \sum_{m=-k_1+1}^{k_1} \varphi(4^{-n}\cdot 2^{+m})\,\mu\big(E^{m}_{m-1}(x_n)\big) + \sum_{m=k_1+1}^{nk_n+n-1} \varphi(4^{-n}\cdot 2^{m})\,\mu\big(E^{m}_{m-1}(x_n)\big) + \\ &\qquad + \sum_{m=k_1}^{nk_n+n-2} \varphi(4^{-n}\cdot 2^{-m})\,\mu\big(E^{-m}_{-m-1}(x_n)\big) \end{aligned}$$

or

$$\text{(III)}\qquad \begin{aligned} \tfrac{1}{3} r_0 &< \varrho_\varphi\big(4^{-n} F^{(n+1)k_{n+1}+n-2}_{nk_n+n-1}(x_n)\big) \\ &\leqslant \sum_{m=nk_n+n}^{(n+1)k_{n+1}+n-2} \varphi(4^{-n}\cdot 2^{m})\,\mu\big(E^{m}_{m-1}(x_n)\big) \\ &= \sum_{m=nk_n}^{(n+1)k_{n+1}-2} \varphi(2^{m-n})\,\mu\big(E^{m+n}_{m+n-1}(x_n)\big). \end{aligned}$$

Now, we define the φ-function ψ as in the proof of Lemma 1.1. First, let us assume that (I) holds. Then from (a) of Lemma 1.1 we obtain

$$\begin{aligned} \varrho_\psi(2^{-n+4} x_n) &\geqslant \varrho_\psi\big(2^{-n+4} F^{-(nk_n+n-3)}_{-((n+1)k_{n+1}+n-4)}(x_n)\big) \\ &\geqslant \sum_{m=nk_n+n-2}^{(n+1)k_{n+1}+n-4} \psi(2^{-n-m+4})\,\mu\big(E^{-m+1}_{-m}(x_n)\big) \\ &= \sum_{m=nk_n}^{(n+1)k_{n+1}-2} \psi(2^{-m+2})\,\mu\big(E^{-m+n-1}_{-m+n-2}(x_n)\big) \\ &\geqslant \sum_{m=nk_n}^{(n+1)k_{n+1}-2} \varphi(2^{-m-n})\,\mu\big(E^{-m+n-1}_{-m+n-2}(x_n)\big) > \tfrac{1}{3} r_0. \end{aligned}$$

Now, let us assume that (III) holds. Then from (b) of Lemma 1.1 we get

$$\begin{aligned}\varrho_\psi(2^{-n+4}x_n) &\geqslant \varrho_\psi\big(2^{-n+3}F_{nk_n+n-1}^{(n+1)k_{n+1}+n-2}(x_n)\big)\\ &\geqslant \sum_{m=nk_n+n-1}^{(n+1)k_{n+1}+n-3} \psi(2^{-n+m+3})\,\mu\big(E_m^{m+1}(x_n)\big)\\ &= \sum_{m=nk_n}^{(n+1)k_{n+1}-2} \psi(2^{m+2})\,\mu\big(E_{m+n-1}^{m+n}(x_n)\big)\\ &\geqslant \sum_{m=nk_n}^{(n+1)k_{n+1}-2} \varphi(2^{m-n})\,\mu\big(E_{m+n-1}^{m+n}(x_n)\big) > \tfrac{1}{3}r_0.\end{aligned}$$

Finally, let us assume that (II) holds. Then there holds:

$$\sum_{m=-k_1+1}^{k_1} \varphi(4^{-n}\cdot 2^m)\,\mu\big(E_{m-1}^m(x_n)\big) > \tfrac{1}{9}r_0 \tag{II$_1$}$$

or

$$\sum_{m=k_1+1}^{nk_n+n-1} \varphi(4^{-n}\cdot 2^m)\,\mu\big(E_{m-1}^m(x_n)\big) > \tfrac{1}{9}r_0 \tag{II$_2$}$$

or

$$\sum_{m=k_1}^{nk_n+n-2} \varphi(4^{-n}\cdot 2^{-m})\,\mu\big(E_{-m-1}^{-m}(x_n)\big) > \tfrac{1}{9}r_0. \tag{II$_3$}$$

First, let us assume that (II$_1$) holds and let $n \geqslant 2k_1+2$. Then

$$\tfrac{1}{9}r_0 < \sum_{m=-k_1+1}^{k_1} \varphi(4^{-n}\cdot 2^m)\cdot\mu\big(E_{m-1}^m(x_n)\big) \leqslant \varphi(2^{k_1-2n})\,\mu\big(E_{-k_1}^{k_1}(x_n)\big).$$

We have

$$\begin{aligned}\varrho_\psi(2^{-n+4}x_n) &\geqslant \varrho_\psi\big(2^{-n+4}F_{-k_1}^{k_1}(x_n)\big)\\ &\geqslant \sum_{m=-k_1}^{k_1-1} \psi(2^{-n+m+4})\,\mu\big(E_m^{m+1}(x_n)\big) \geqslant \psi(2^{-n+4-k_1})\,\mu\big(E_{-k_1}^{k_1}(x_n)\big)\\ &= \psi(2^{-(n+k_1)+4})\,\mu\big(E_{-k_1}^{k_1}(x_n)\big).\end{aligned}$$

But $lk_l \leqslant n+k_1-2 \leqslant (l+1)k_{l+1}-1$ for some natural number l such that $1 \leqslant l \leqslant n-2k_1+1$ ($n \geqslant 2k_1+1$). Hence by (a) of Lemma 1.1 we get

$$\psi(2^{-(n+k_1-2)+2}) \geqslant \varphi(2^{-n-k_1+2-l}) \geqslant \varphi(2^{-n-k_1-n+2k_1+1}) \geqslant \varphi(2^{-2n+k_1})$$

and therefore we obtain

$$\varrho_\psi(2^{-n+4}x_n) \geqslant \tfrac{1}{9}r_0.$$

Now, let us assume that (II_2) holds and let $n \geqslant 2k_1+1$. Then

$$\begin{aligned}\tfrac{1}{9}r_0 &< \sum_{m=k_1+1}^{nk_n+n-1} \varphi(4^{-n}\cdot 2^m)\,\mu\big(E^m_{m-1}(x_n)\big)\\ &= \sum_{i=1}^{n-1}\Big(\sum_{m=ik_i+i}^{(i+1)k_{i+1}+i} \varphi(2^{-2n+m})\,\mu\big(E^m_{m-1}(x_n)\big)\Big)\\ &= \sum_{i=1}^{n-1}\Big(\sum_{m=ik_i}^{(i+1)k_{i+1}} \varphi(2^{-2n+m+i})\,\mu\big(E^{m+i}_{m+i-1}(x_n)\big)\Big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\varrho_\psi(2^{-n+4}x_n) &\geqslant \varrho_\psi\big(2^{-n+3}F^{nk_n+n-1}_{k_1}(x_n)\big)\\ &\geqslant \sum_{m=k_1}^{nk_n+n-2} \psi(2^{-n+m+3})\,\mu\big(E^{m+1}_m(x_n)\big)\\ &= \sum_{i=1}^{n-1}\Big(\sum_{m=ik_i+i-1}^{(i+1)k_{i+1}+i-1} \psi(2^{-n+m+3})\,\mu\big(E^{m+1}_m(x_n)\big)\Big)\\ &= \sum_{i=1}^{n-1}\Big(\sum_{m=ik_i}^{(i+1)k_i+1} \psi(2^{(m+i-n)+2})\,\mu\big(E^{m+i}_{m+i-1}(x_n)\big)\Big).\end{aligned}$$

We shall show that $\psi(2^{(m+i-n)+2}) \geqslant \varphi(2^{-2n+m+i})$ for $ik_i \leqslant m \leqslant (i+1)k_{i+1}$, where $1 \leqslant i \leqslant n-1$. For this purpose, we shall consider three cases:

(α) Let $m+i-n \leqslant -k_1$. Then $k_1 \leqslant n-i-m \leqslant (n-1)k_{n-1}-1$. Hence $lk_l \leqslant n-i-m \leqslant (l+1)k_{l+1}-1$ for some natural number l such that $1 \leqslant l \leqslant n-2$. Therefore by (a) of Lemma 1.1 we obtain

$$\psi(2^{(m+i-n)+2}) = \psi(2^{-(n-i-m)+2}) \geqslant \varphi(2^{-(n-i-m)-l}) \geqslant \varphi(2^{-2n+m+i}).$$

(β) Let $m+i-n \geqslant k_1$. Hence $lk_l \leqslant m+i-n \leqslant (l+1)k_{l+1}-1$ for some natural number $1 \leqslant l \leqslant n-1$. Then by (b) of Lemma 1.1 we get

$$\psi(2^{(m+i-n)+2}) \geqslant \varphi(2^{m+i-n-l}) \geqslant \varphi(2^{-2n+m+i}).$$

(γ) At last, let $-k_1 < m+i-n < k_1$. Then $k_1 < -(m+i-n-2k_1) < 3k_1 < 2k_2-1$. Then by (a) of Lemma 1.1 for $n \geqslant 2k_1+1$ we obtain

$$\begin{aligned}\psi(2^{(m+n-i)+2}) &= \psi(2^{-(-(m+i-n-2k_1))+2k_1+2}) \geqslant \psi(2^{-(-(m+i-n-2k_1))+2})\\ &\geqslant \varphi(2^{-(-(m+i-n-2k_1))-1}) = \varphi(2^{m+i-n-2k_1-1}) \geqslant \varphi(2^{-2n+m+i}).\end{aligned}$$

Finally, let us assume that (II_3) holds and let $n \geqslant 2k_1$. We have

$$\tfrac{1}{9} r_0 < \sum_{m=k_1}^{nk_n+n-2} \varphi(4^{-n}\cdot 2^{-m})\,\mu\big(E_{-m-1}^{-m}(x_n)\big)$$
$$= \sum_{i=1}^{n-1}\Big(\sum_{m=ik_i+i-1}^{(i+1)k_{i+1}+i-1} \varphi(2^{-2n-m})\,\mu\big(E_{-m-1}^{-m}(x_n)\big)\Big)$$
$$= \sum_{i=1}^{n-1}\Big(\sum_{m=ik_i}^{(i+1)k_{i+1}} \varphi(2^{-2n-m+i-1})\,\mu\big(E_{-m+i-2}^{-m+i-1}(x_n)\big)\Big).$$

On the other hand, we get

$$\varrho_\psi(2^{-n+4}x_n) \geqslant \varrho_\psi\big(2^{-n+4}F_{-(nk_n+n-3)}^{-(k_1-2)}(x_n)\big)$$
$$\geqslant \sum_{m=k_1-1}^{nk_n+n-3} \psi(2^{-n-m+4})\,\mu\big(E_{-m}^{-m+1}(x_n)\big)$$
$$= \sum_{i=1}^{n-1}\Big(\sum_{m=ik_i+i-2}^{(i+1)k_{i+1}+i-2} \psi(2^{-n-m+4})\,\mu\big(E_{-m}^{-m+1}(x_n)\big)\Big)$$
$$= \sum_{i=1}^{n-1}\Big(\sum_{m=ik_i}^{(i+1)k_{i+1}} \psi(2^{(-n-m+i)+2})\,\mu\big(E_{-m+i-2}^{-m+i-1}(x_n)\big)\Big).$$

We shall show that $\psi(2^{(-n-m+i)+2}) \geqslant \varphi(2^{-2n-m+i-1})$ for $ik_n \leqslant m \leqslant (i+1)k_{i+1}$, where $1 \leqslant i \leqslant n-1$. We have $lk_l \leqslant m+n-i \leqslant (l+1)k_{l+1}$ for some natural number l such that $1 \leqslant l \leqslant n-1$. Therefore by (a) of Lemma 1.1 we get

$$\psi(2^{(-n-m+i)+2}) = \psi(2^{-(n+m-i)+2}) \geqslant \varphi(2^{-n-m+i-l}) \geqslant \varphi(2^{-2n-m+i-1}).$$

Thus we showed that

$$\varrho_\psi(2^{-n+4}x_n) \geqslant \tfrac{1}{9} r_0 \quad \text{for } n \geqslant 2k_1+1$$

and this proves that a set K is not bounded in the topology $\mathcal{T}_\varphi^\wedge$.

Remark. In the case where $(L^{*\varphi}, \mathcal{T}_\varphi)$ is a locally bounded space, then we have immediately that subsets of $L^{*\varphi}$ bounded in $\mathcal{T}_\varphi$ are bounded in $\mathcal{T}_\varphi$.

Indeed, from [2] (Theorem 2.1 and Theorem 2.2) we have the inclusions of topologies

$$(+) \qquad \mathcal{T}^{\triangleleft\varphi} \subset \mathcal{T}^{1\varphi} \subset \mathcal{T}_\varphi,$$

where $\mathcal{T}^{\triangleleft\varphi}$ and $\mathcal{T}^{1\varphi}$ denote the linear topologies on $L^{*\varphi}$ defined in [2]. Moreover, there holds the equality $\mathcal{T}^{1\varphi} = \mathcal{T}_\varphi^\wedge$ ([3], Theorem 2.1). In [4] (Theorem 2.7) it is proved that in the case when $(L^{*\varphi}, \mathcal{T}_\varphi)$ is a locally bounded space, then a subset of $L^{*\varphi}$ is bounded in $\mathscr{T}^{\lessdot\varphi}$ if and only if it is bounded in $\mathscr{T}_{\varphi}$. Then, in virtue of $(+)$ a subset of $L^{*\varphi}$ is bounded in $\mathscr{T}^{\wedge}_{\varphi}$ if and only if it is bounded in $\mathscr{T}_{\varphi}$.

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