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Locally nonconical unit balls in Orlicz spaces

Abstract. The aim of this paper[1] is to investigate the local nonconicality of unit ball in Orlicz spaces, endowed with the Luxemburg norm. A closed convex set Q in a locally convex topological Hausdorff space X is called locally nonconical (LNC), if for every x, $y \in Q$ there exists an open neighbourhood U of x such that $(U \cap Q) + (y-x)/2 \subset Q$. The following theorem is established: An Orlicz space $L^{\varphi}(\mu)$ has an LNC unit ball if and only if either $L^{\varphi}(\mu)$ is finite dimensional or the measure μ is atomic with a positive greatest lower bound and φ satisfies the condition $\Delta_r^0(\mu)$ and is strictly convex on the interval $[0, b]$, or $c(\varphi) = +\infty$ and φ satisfies the condition $\Delta_2(\mu)$ and is strictly convex on $\mathbb{R}$. A similar result is obtained for the space $E^{\varphi}(\mu)$.

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1. Introduction. A convex set Q of a real Hausdorff topological vector space X is called **locally nonconical** (LNC), if for every x, $y \in Q$ there exists an open neighbourhood U of x, such that $(U \cap Q) + (y-x)/2 \subset Q$, cf. [1], [3], [19], [20]. LNC sets are new class of convex sets first considered by N. Weaver, who showed that the range of finite, nonatomic vector measure taking values in $\mathbb{R}^n$ has the LNC property (cf. Theorem 2.7, [19]). It is proved, that intersection and product LNC sets are LNC sets in [1]. Convex sets on the plane, stricly convex or open convex sets, polytypes, cylinders, zonoides (images of vector measures),the unit ball in c_0, (but not the unit ball in l^{∞}) belong to the class of LNC sets. The classical cone is an example of a convex closed set which is not LNC. It is proved in [20], that any LNC set is **stable**, (i. e. the midpoint map $\Phi\colon Q \times Q, \Phi(x, y) = (x + y)/2$ is open with respect to the inherited topology in Q).

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The aim of this paper is to investigate the local nonconicality of the unit ball $B(L^\varphi(\mu))$ (and $B(E^\varphi(\mu))$) of Orlicz spaces $L^\varphi(\mu)$ (and $E^\varphi(\mu)$) of functions defined on an arbitrary σ–finite measure space, endowed with the Luxemburg norm.

2. Basic definitions and auxiliary results. Let (Ω, Σ, μ) be a measure space with a nonnegative, σ–finite and complete measure μ ($\mu(\Omega) > 0$) and $\varphi\colon \mathbb{R} \to [0, +\infty]$ be a convex, even function which is non-identically equal to 0 and left-continuous for $t > 0$, such that $\varphi(0) = 0$, $c(\varphi) := \sup\{\, t > 0 : \varphi(t) < \infty \,\} > 0$. Such functions are called **Young functions**. This definition is somewhat stronger than the one used in [17]. We use the notation $a(\varphi) := \sup\{\, t : \varphi(t) = 0 \,\}$. By an **Orlicz space** $L^\varphi(\mu)$ ([14], [15], [17]), we mean the set of all measurable functions $x\colon \Omega \to \mathbb{R}$, such that $I_\varphi(\lambda x) < \infty$ for some $\lambda > 0$, where **the modular** I_φ is defined by

$$I_\varphi(x) := \int\limits_\Omega \varphi\left(x(\omega)\right)\, d\mu.$$

$L^\varphi(\mu)$ is equipped with **the Luxemburg norm** [13]

$$\|x\|_\varphi := \inf\left\{\, \lambda > 0 : I_\varphi\left(x/\lambda\right) \le 1 \,\right\}.$$

(Note that $\|x\|_\varphi \le 1$ iff $I_\varphi(x) \le 1$; $I_\varphi(x) = 1$ implies $\|x\|_\varphi = 1$; $I_\varphi(x) < 1 \Rightarrow (\|x\|_\varphi = 1$ iff $I_\varphi(\lambda x) = +\infty$ for every $\lambda > 1)$; $\|x_n - x\|_\varphi \to 0$ iff $I_\varphi(\lambda(x_n - x)) \to 0$ for every $\lambda > 0$). If atomless part of μ is positive, then the subspace

$$E^\varphi(\mu) := \left\{\, x \in \mathcal{M} : \forall \lambda > 0 \quad I_\varphi(\lambda x) < +\infty \,\right\}.$$

is called **the space of finite elements**, where $\mathcal{M}$ is the set of all measurable functions $x\colon \Omega \to \mathbb{R}$. If μ is purely atomic then definition of the finite elements is

$$E^\varphi(\mu) := \left\{ (x_n) : \forall \lambda > 0 \exists n_\lambda \in \mathbb{N} \quad \sum_{n=n_\lambda}^{\infty} \varphi(\lambda x_n) < +\infty \right\}.$$

Note, that for $c(\varphi) = \infty$ both definitions are equivalents, but for $L^\varphi(\mu) = l^\infty$ we have $E^\varphi(\mu) = c_0$ (cf.[22], p. 489).

Let $r > 1$. The function φ is said to satisfy **the condition** $\boldsymbol{\Delta_r(\mu)}$, cf. [21], [23] (denoted $\varphi \in \Delta_r(\mu)$) if one of the following three conditions is satisfied:

(a) μ is atomless and there exist constants $c > 1$ and $a_0 \ge 0$ such that $\varphi(a_0) < +\infty$, (or in the case $\mu(\Omega) = +\infty$ then $a_0 = 0$), such that for every $t \ge a_0$, we have $\varphi(rt) \le c\varphi(t)$;

b) when μ is purely atomic measure with $\{\, e_n : n \in N \,\}$, $N \subset \mathbb{N}$, being the set of all atoms of Ω and there exist $b > 0$, $c > 1$ and a nonnegative sequence (d_n) such that $\sum_n d_n < +\infty$, and $\varphi(rt)\mu(e_n) \le c\varphi(t)\mu(e_n) + d_n$ for every t with $\varphi(t)\mu(e_n) \le b$ and every $n \in \mathbb{N}$;

c) a combination of a) and b) when Ω has both an atomless part Ω_1 and purely atomic part Ω_2 of positive measure, such that $\varphi \in \Delta_r(\mu|\Omega_1)$ and $\varphi \in \Delta_r(\mu|\Omega_2)$.

If $c(\varphi) = \infty$, then $\varphi \in \Delta_r(\mu)$ for some $r > 1 \iff \varphi \in \Delta_r(\mu)$ for every $r > 1 \iff \varphi \in \Delta_2(\mu)$.

These equivalences remain true if μ is atomless (in this case $\varphi \in \Delta_r(\mu)$ for some $r > 1$ implies that $c(\varphi) = \infty$). If μ is purely atomic measure with $\sum_n \mu(e_n) = \infty$ and $\varphi \in \Delta_r(\mu)$ for some $r > 1$, then $\varphi(x) = 0 \Leftrightarrow x = 0$ (indeed, $d_n \geq \varphi(ra(\varphi))\mu(e_n)$ for every $n \in \mathbb{N}$). Thus these equivalences hold in the case of a purely atomic measure μ with an infinite number of atoms provided $0 < \inf_n \mu(e_n) \leq \sup \mu(e_n) < \infty$ — no matter whether φ takes only finite values or not (if $\varphi \in \Delta_{r_0}(\mu)$, then evidently $\varphi \in \Delta_r(\mu)$ for every $1 < r \leq r_0$; for $r > r_0$, consider $b_r = \varphi(a'r_0/r) \cdot \inf_n \mu(e_n) > 0$, where $a' = \sup\{\, a > 0 : \varphi(a) \leq b_{r_0}/\sup_n \mu(e_n) \,\} > 0$). If $\dim L^\varphi(\mu) < \infty$ (i. e. , Ω consists of a finite number of atoms), then $\varphi \in \Delta_r(\mu)$ for some $r > 1$ if and only if $L^\varphi(\mu)$ is not isometric to $L^\infty(\mu)$ (take any $a_0 \in (a(\varphi), c(\varphi))$, $1 < r < c(\varphi)/a_0$ and set $b = \varphi(a_0)\cdot\inf_n \mu(e_n) > 0$, $d_n = \varphi(ra_0)\cdot\sup_n \mu(e_n) < \infty$). However, if $0 < a(\varphi) \leq c(\varphi) < \infty$, then φ does not satisfy the condition $\Delta_r(\mu)$ for any $r > c(\varphi)/a(\varphi)$.

Note that if $c(\varphi) = \infty$ and $L^\varphi(\mu)$ is finite dimensional, then $L^\varphi(\mu) = E^\varphi(\mu)$. If $c(\varphi) = \infty$ and $\dim L^\varphi(\mu) = \infty$, the equality $L^\varphi(\mu) = E^\varphi(\mu)$ holds if and only if $\varphi \in \Delta_2(\mu)$ (cf. [14, Theorem 8. 13, p. 52]). Thus, applying the Lebesgue dominated convergence theorem, we obtain

$$(I_\varphi(x) = 1 \iff \|x\|_\varphi = 1) \quad \text{if and only if} \quad \varphi \in \Delta_2(\mu).$$

In fact, we can replace the condition $\Delta_2(\mu)$ by $\Delta_r(\mu)$ for some $r > 1$ in the last equivalence. The assumption $c(\varphi) = \infty$ is used only in the "if" part of the proof. Hence, in any case, it follows that if $\varphi \notin \Delta_r(\mu)$ for any $r > 1$, then there exists $x \in L^\varphi(\mu)$ such that $\|x\| = 1$, but $I_\varphi < 1$ and this is what we need in the sequel.

Let $\{\, e_n : n \in N \,\}$, $N \subset \mathbb{N}$, be the set of all atoms of Ω and let $r > 1$. We say that a function φ satisfies the **condition** $\Delta_r^0(\mu)$(on Ω) — denoted $\varphi \in \Delta_r^0(\mu)$ — if one of the following conditions is satisfied

(a) when the atomless part of Ω is of positive measure there exist $a_0 > 0$ and $c > 1$ such that $0 < \varphi(a_0) < \infty$ and $\varphi(rt) \leq c\varphi(t)$ for every $|t| \leq a_0$:

(b) when μ is purely atomic there exist $a_0 > 0$, $b > 0$, $c > 1$ and a nonnegative sequence (d_n) such that $\sum_n d_n < +\infty$, $0 < \varphi(a_0) < \infty$ and $\varphi(rt)\mu(e_n) \leq c\varphi(t)\mu(e_n) + d_n$ for every $|t| \leq a_0$ with $\varphi(t)\mu(e_n) \leq b$ and every $n \in N$.

If $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ on the atomless part of Ω which is of positive measure, then evidently, $\varphi \in \Delta_r^0(\mu)$ on the whole set Ω. Furthermore, if the measure of the atomless part of Ω is either infinite or equal to zero and $\varphi \in \Delta_r(\mu)$ for some $r > 1$, then $\varphi \in \Delta_r^0(\mu)$. Thus $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ when $\dim L^\varphi(\mu) < \infty$ and $L^\varphi(\mu)$ is not isometric to $L^\infty(\mu)$.

If $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ and $\|x\|_\infty < c(\varphi)$, then

$$I_\varphi(x) = 1 \iff \|x\|_\varphi = 1.$$

(see [23, p. 509]). Note that when φ takes only finite values, $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ iff $\varphi \in \Delta_2^0(\mu)$. Also, analogously to $\Delta_r(\mu)$, if μ is purely atomic measure with $\sum_n \mu(e_n) = \infty$ and $\varphi \in \Delta_r(\mu)$ for some $r > 1$, then $a(\varphi) = 0$.

We will use the following characterization of stability in Orlicz spaces with the Luxemburg norm.

THEOREM 2.1 *[23, Theorem 5, p. 511] $B(L^\varphi(\mu))$ is stable if and only if at least one of the following conditions is satisfied:*

(i) $\dim L^\varphi(\mu) < +\infty$;

(ii) $L^\varphi(\mu)$ *is isometric to* $L^\infty(\mu)$;

(iii) φ *satisfies the condition* $\Delta_r(\mu)$ *for some* $r > 1$;

(iv) φ *satisfies the condition* $\Delta_r^0(\mu)$ *for some* $r > 1$ *provided* $c(\varphi) < +\infty$ *and* $\varphi(c(\varphi)) < +\infty$;

(v) φ *satisfies* $\Delta_r^0(\mu)$ *for some* $r > 1$ *on the purely atomic part of* Ω *whenever* $c(\varphi) < +\infty$, $\varphi(c(\varphi)) < +\infty$ *and the measure of the atomless part of* Ω *is finite;*

(vi) $c(\varphi) < +\infty$, $\varphi(c(\varphi)) < +\infty$ *and* $\mu(\Omega) < +\infty$.

We need the following six technical Lemmas.

LEMMA 2.2 *Let* $\varphi\colon \mathbb{R} \to [0, +\infty]$ *be any Young function. Set* $\mathcal{N} := \mathbb{N}$ *or* $\mathcal{N} := \{1, 2, \ldots, N\}$. *Let* $(x_i)_{i\in\mathcal{N}}$, $(y_i)_{i\in\mathcal{N}}$, $(\varepsilon_i)_{i\in\mathcal{N}}$, $(c_i)_{i\in\mathcal{N}}$ *be sequences of real numbers satisfying the following conditions:*

1. $c_i > 0$ *for* $i \in \mathcal{N}$,
2. $\sum\limits_{i\in\mathcal{N}} \varphi(\lambda x_i + (1-\lambda) y_i) c_i = 1$ *for every* $0 \le \lambda \le 1$,
3. *the set* $J := \{ i \in \mathbb{N} : x_i \neq y_i \wedge (|x_i| > a(\varphi)/2 \vee |y_i| > a(\varphi)/2) \}$ *is finite and for each* $i \in J$ *the following conditions are satisfied:*
   (a) *if* $x_i \neq 0$, *then* $|\varepsilon_i| \le |x_i|$ *and if* $y_i \neq 0$, *then* $|\varepsilon_i| \le |y_i|$
   (b) $|\varepsilon_i| < |y_i - x_i|/2$
   (c) $|\varepsilon_i| < \min\{ |(x_i + y_i)/2|, a(\varphi) - |(x_i + y_i)/2| \}$, *as long as the right-hand side is positive*
   (d) $|\varepsilon_i| < a(\varphi)/2$, *as long as* $a(\varphi) > 0$.

Let

$$F(t) := \sum_{i\in\mathcal{N}} \varphi\left(x_i + t(y_i - x_i)/2 + \varepsilon_i\right) c_i.$$

Then the function F *is nonincreasing on the interval* $[0, 1]$.

PROOF Note that for $\lambda = 0$ and $\lambda = 1$ from Condition 2 we obtain: $\sum\limits_{i\in\mathcal{N}} \varphi(x_i) c_i = \sum\limits_{i\in\mathcal{N}} \varphi(y_i) c_i = 1$. Furthermore, for $0 < \lambda < 1$,

$$1 = \sum_{i\in\mathcal{N}} \varphi\left(\lambda x_i + (1-\lambda) y_i\right) c_i \le \lambda \sum_{i\in\mathcal{N}} \varphi(x_i) c_i + (1-\lambda) \sum_{i\in\mathcal{N}} \varphi(y_i) c_i = 1,$$

which means that for $x_i \neq y_i$ the function φ is affine on $[x_i, y_i]$. The numbers x_i, y_i may be of opposite signs only if $\varphi(x_i) = \varphi(y_i) = 0$, i.e. when $|x_i|, |y_i| \leq a(\varphi)$. Hence, we can assume the following

$$\text{either} \quad x_i, y_i \geq a(\varphi) \geq 0 \quad \text{or} \quad x_i, y_i \in [-a(\varphi), a(\varphi)], \text{or} \quad x_i, y_i \leq -a(\varphi) \leq 0$$

for all $i \in \mathcal{N}$. Let k_i be the gradient of φ on the segment $[x_i, y_i]$ for $i \in J$. Then

$$\varphi\left((x_i + y_i)/2 + \varepsilon_i\right) = \varphi\left((x_i + y_i)/2\right) + k_i \varepsilon_i \quad \text{for} \quad i \in J.$$

Let $k_i(t) \equiv 0$ if $\varepsilon_i = 0$, $k_i(t) \equiv +\infty$ if $\varphi\left(x_i + t(y_i - x_i)/2 + \varepsilon_i\right) = +\infty$ and

$$k_i(t) = \frac{\varphi\left(x_i + t(y_i - x_i)/2 + \varepsilon_i\right) - \varphi\left(x_i + t(y_i - x_i)/2\right)}{\varepsilon_i}.$$

in all other cases for $i \in J$, $0 \leq t \leq 1$. So $k_i(t)$ satisfy

$$\varphi\left(x_i + t(y_i - x_i)/2 + \varepsilon_i\right) = \varphi\left(x_i + t(y_i - x_i)/2\right) + k_i(t)\varepsilon_i.$$

Note that if $|x_i|$, $|y_i| \leq a(\varphi)/2$, then $k_i(t) = 0$ and both sides of the above equality are equal to zero. Consider two cases A) $\quad x_i < y_i$, B) $\quad y_i < x_i$:

A) If $\varepsilon_i > 0$, then $x_i \leq x_i + t(y_i - x_i)/2 + \varepsilon_i \leq (x_i + y_i)/2 + \varepsilon_i \leq (x_i + y_i)/2 + (y_i - x_i)/2 = y_i$, so $k_i(t) \equiv k_i$.

If $\varepsilon_i < 0$, then

$$k_i(t) = \frac{\varphi\left(x_i + t(y_i - x_i)/2\right) - \varphi\left(x_i + t(y_i - x_i)/2 - (-\varepsilon_i)\right)}{-\varepsilon_i}.$$

It is easy to see, that for any convex function ψ and positive constant c the function $\psi(t + c) - \psi(t)$ is nondecreasing. Hence, the function $k_i(t)$ is nondecreasing

B) If $\varepsilon_i < 0$, then $y_i = (x_i + y_i)/2 - (x_i - y_i)/2 \leq (x_i + y_i)/2 + \varepsilon_i \leq x_i + t(y_i - x_i)/2 + \varepsilon_i < x_i + t(y_i - x_i)/2 \leq x_i$, so $k_i(t) \equiv k_i$.

If $\varepsilon_i > 0$, then

$$k_i(t) = \frac{\varphi\left(x_i - t(x_i - y_i)/2 + \varepsilon_i\right) - \varphi\left(x_i - t(x_i - y_i)/2\right)}{\varepsilon_i}.$$

As above, the function $k_i(t)$ is nonincreasing.

In all cases the function:

$$g_i(t) := k_i(t)\varepsilon_i c_i$$

is nonincreasing. Let:

$$I := \{\, i : x_i = y_i \,\}, \quad K := \{\, i : x_i \neq y_i \wedge |x_i|, |y_i| \leq a(\varphi)/2 \,\}.$$

Obviously, $\mathcal{N} = I \dot\cup J \dot\cup K$. Let $\lambda := \lambda(t) = 1 - t/2$. Note that

$$\sum_{i \in K} \varphi\left(x_i + t(y_i - x_i)/2 + \varepsilon_i\right) c_i = \sum_{i \in K} \varphi\left(x_i + t(y_i - x_i)/2\right) c_i = 0,$$

since the arguments of the function φ are contained in $[-a(\varphi), a(\varphi)]$. Hence,

$$\begin{aligned}
F(t) &= \sum_{i\in I}\varphi(x_i+\varepsilon_i)c_i+\sum_{i\in J}\varphi\left(x_i+t(y_i-x_i)/2+\varepsilon_i\right)c_i= \\
&= \sum_{i\in I}\varphi(x_i+\varepsilon_i)c_i+\sum_{i\in J}\varphi\left(x_i+t(y_i-x_i)/2\right)c_i+\sum_{i\in J}k_i(t)\varepsilon_i c_i= \\
&= \sum_{i\in I}\varphi(x_i+\varepsilon_i)c_i+\sum_{i\in\mathcal{N}}\varphi\left(\lambda x_i+(1-\lambda)y_i\right)c_i+ \\
&\quad -\sum_{i\in I}\varphi\left(\lambda x_i+(1-\lambda)y_i\right)c_i+\sum_{i\in K}\varphi\left(x_i+t(y_i-x_i)/2\right)c_i+ \\
&\quad +\sum_{i\in J}g_i(t) \\
&= \sum_{i\in I}\left(\varphi(x_i+\varepsilon_i)-\varphi(x_i)\right)c_i+1+\sum_{i\in J}g_i(t).
\end{aligned}$$

Thus $F(t)$ is nonincreasing on $[0,1]$. ■

Lemma 2.3 *Let $a(\varphi)=0$. If the Young function φ is not strictly convex in any neighbourhood of* 0*, and $(c_n)_{n\in\mathbb{N}}$ is a sequence of real numbers satisfying* $\inf\{c_n : n\in\mathbb{N}\}>0$*, then there exist two sequences of real numbers (x_n), (y_n) such that $y_k<x_k$ for infinitely many $k\in\mathbb{N}$, $\lim_n x_n=\lim_n y_n=0$, $\varphi\restriction[x_n\wedge y_n, x_n\vee y_n]$ is affine for every $n\in\mathbb{N}$ and $\sum_{n\in\mathbb{N}}\varphi(x_n)c_n=\sum_{n\in\mathbb{N}}\varphi(y_n)c_n=1$.*

Additionally, if we assume $c(\varphi)=+\infty$ or $\varphi(c(\varphi))\cdot\sum_{n=1}^{\infty}c_n>1$ instead of $\inf\{c_n : n\in\mathbb{N}\}>0$ *we obtain*

$$(\forall\lambda>0)(\exists n_\lambda\in\mathbb{N})\left(\sum_{n=n_\lambda}^{\infty}\varphi(\lambda x_n)c_n<+\infty\quad\text{and}\quad\sum_{n=n_\lambda}^{\infty}\varphi(\lambda y_n)c_n<+\infty\right).$$

Proof First we prove the Lemma with an additional assumption. Fix $N\in\mathbb{N}$ such that $\varphi(c(\varphi))\cdot\sum_{n=1}^{2N-1}c_n>1$ if $c(\varphi)<+\infty$ and $N=1$ if $c(\varphi)=+\infty$. Let $U=\dot{\bigcup}U_i$ be the sum of disjoint open intervals U_i of $(0,c(\varphi))$ such that the restricted function $\varphi\restriction U_i$ is affine. $U\neq\emptyset$ and $\inf U=0$ by assumption. Now we define two sequences (x_n),(y_n), such that for $n\geq 2N$, x_n, $y_n\in U$, $\varphi\restriction[x_n,y_n]$ (respectively $\varphi\restriction[y_n,x_n]$) is affine, $\varphi(nx_n)<1/(c_n 2^n)$, $x_n<1/n$, $\varphi(ny_n)<1/(c_n 2^n)$, $y_n<1/n$ and

$$\sum_{n=2N}^{\infty}\varphi(x_n)c_n=\sum_{n=2N}^{\infty}\varphi(y_n)c_n.$$

The construction will be inductive. Suppose that we have constructed sequences $(x_i)_{2N\leq i\leq 2k-1}$, $(y_i)_{2N\leq i\leq 2k-1}$ for some $k\in\mathbb{N}$ satisfying the assumptions, $y_{2i}<x_{2i}$ for $i<k$ and:

$$\sum_{n=2N}^{2k-1}\varphi(x_n)c_n=\sum_{n=2N}^{2k-1}\varphi(y_n)c_n.$$

For $i \in \{2k, 2k+1\}$ we choose $x_i \in U$ satisfying $\varphi(ix_i) < 1/(c_i 2^i)$, $x_i < 1/i$ and take $y_{2k} < x_{2k}$ sufficiently close to x_{2k} that the appropriate assumptions hold. Moreover, x_{2k} and x_{2k+1} may be such small, that

$$((\varphi(x_{2k}) - \varphi(y_{2k}))\,\frac{c_{2k}}{c_{2k+1}} + \varphi(x_{2k+1}) < \varphi(1/2k+1)$$

and

$$(2k+1)\varphi^{-1}\left((\varphi(x_{2k}) - \varphi(y_{2k}))\,\frac{c_{2k}}{c_{2k+1}} + \varphi(x_{2k+1})\right) < \varphi^{-1}1/(c_{2k+1}2^{2k+1}).$$

Then y_{2k+1} defined by the formula:

$$y_{2k+1} := \varphi^{-1}\left((\varphi(x_{2k}) - \varphi(y_{2k}))\,\frac{c_{2k}}{c_{2k+1}} + \varphi(x_{2k+1})\right)$$

satisfies the appropriate assumptions, too. We may assume that φ is a continuous and increasing function on $[0, c(\varphi))$ and U is an open set. The following formula

$$\varphi(x_{2k})c_{2k} + \varphi(x_{2k+1})c_{2k+1} = \varphi(y_{2k})c_{2k} + \varphi(y_{2k+1})c_{2k+1},$$

holds by construction. Hence, $\sum\limits_{n=2N}^{2k+1} \varphi(x_n)c_n = \sum\limits_{n=2N}^{2k+1} \varphi(y_n)c_n$ and the inductive step is complete. The following inequality

$$\sum_{n=2N}^{\infty} \varphi(x_n)c_n = \sum_{n=2N}^{\infty} \varphi(y_n)c_n \le \sum_{n=2N}^{\infty} \varphi(ny_n)c_n \le \sum_{n=2N}^{\infty} \frac{1}{2^n} \le \frac{1}{2} < 1$$

holds by construction. By the condition put on N at the begining of the proof, there are x_n for $1 \le n \le 2N-1$ such that:

$$\sum_{n=1}^{2N-1} \varphi(x_n)c_n = 1 - \sum_{n=2N}^{\infty} \varphi(x_n)c_n.$$

Let $y_n = x_n$ for $n \le 2N-1$ Hence,

$$\sum_{n\in\mathbb{N}} \varphi(x_n)c_n = \sum_{n\in\mathbb{N}} \varphi(y_n)c_n = 1$$

Now consider $\lambda > 0$. Fix $n_\lambda \in \mathbb{N}$, $n \ge 2N$ such that $\lambda < n_\lambda$. Thus

$$\sum_{i=n_\lambda}^{\infty} \varphi(\lambda x_i)c_i \le \sum_{i=n_\lambda}^{\infty} \varphi(ix_i)c_i \le \sum_{i=n_\lambda}^{\infty} \frac{1}{2^i} < +\infty,$$

what ends the proof.

It remains to prove the Lemma without this additional assumption. We construct sequences $(x_{2k-1})_{k\in\mathbb{N}}$, $(y_{2k-1})_{k\in\mathbb{N}}$ contained in U satisfying

$$0 < 1 - c := \sum_{k\in\mathbb{N}} \varphi(x_{2k-1})c_{2k-1} = \sum_{k\in\mathbb{N}} \varphi(y_{2k-1})c_{2k-1} < 1$$

and $y_{2k-1} < x_{2k-1}$ for infinitely many $k \in \mathbb{N}$ using the methods from the previous case. To complete the proof it suffices to construct a sequence (x_{2k}) converging to zero such that $\sum\limits_{k\in\mathbb{N}} \varphi(x_{2k})c_{2k} = c$ and set $y_{2k} := x_{2k}$. This sequence is constructed by induction in the following way:

Let $x_2 \in U$ be any real number satisfying $\varphi(x_2)c_2 < c$ and $x_{2(k+1)} = x_{2k}$, as long as $\sum\limits_{i=1}^{k+1} \varphi(x_{2i})c_{2i} < c$. If $\sum\limits_{i=1}^{k+1} \varphi(x_{2i})c_{2i} \geq c$ we set $x_{2(k+1)}$ to be any $x \in U$, such that $\sum\limits_{i=1}^{k} \varphi(x_{2i})c_{2i} + \varphi(x)c_{2(k+1)} < c$. Observe

$$0 < \sum_{i\in\mathbb{N}} \varphi(x_{2i}) \leq \frac{1}{\inf\{\, c_i : i \in \mathbb{N}\,\}} \sum_{i\in\mathbb{N}} \varphi(x_{2i})c_{2i} < +\infty.$$

Thus $\lim\limits_{n} x_{2n} = 0$. Hence, there exists infinitely many $k \in \mathbb{N}$ such that $x_{2(k+1)} < x_{2k}$ it follows that:

$$c \geq \sum_{i=1}^{k} \varphi(x_{2i})c_{2i} > c - \varphi(x_{2k})c_{2k}$$

Letting k tend to infinity, it follows that $\sum\limits_{i\in\mathbb{N}} \varphi(x_{2i})c_{2i} = c$, what completes the proof. ■

Lemma 2.4 *If* $0 < c(\varphi) < +\infty$, $\varphi(c(\varphi)) < +\infty$ *and if there exists an infinite sequence of disjoint sets of finite positive measure* (A_n) *such that*

$$\varphi(c(\varphi)) \cdot \sum_{n=1}^{\infty} \mu(A_n) \leq 1,$$

then $B(L^{\varphi}(\mu))$ *is not LNC. In particular,* $B(L^{\infty}(\mu))$ *is not LNC when* $\dim L^{\infty}(\mu) = \infty$.

Proof Let $n_0 \in \mathbb{N}$, $n_0 > 1/c(\varphi)$. Set

$$x := \sum_{n=n_0}^{\infty} (c(\varphi) - 1/n)\, \chi_{A_n} \quad \text{ and } \quad y := \sum_{n=n_0}^{\infty} c(\varphi)\chi_{A_n}$$

By assumption $I_{\varphi}(y) = \int\limits_{\Omega} \varphi(y)\, d\mu = \sum\limits_{n=n_0}^{\infty} \varphi(c(\varphi))\mu(A_n) \leq 1$. It follows that $I_{\varphi}(x) \leq 1$. Moreover, for $\lambda > 1$ and $m > \max\{\, n_0, \lambda/((\lambda-1)c(\varphi))\,\}$ the following holds:

$$I_{\varphi}(\lambda x) = \sum_{n=n_0}^{\infty} \varphi\left(\lambda\left(c(\varphi) - 1/n\right)\right)\mu(A_n) \geq \varphi\left(\left(\lambda - \lambda/(mc(\varphi))\right)c(\varphi)\right)\mu(A_m) = +\infty$$

and $I_{\varphi}(\lambda y) = +\infty$. Thus $\|x\|_{\varphi} = \|y\|_{\varphi} = 1$. Set $x_n := x + (1/n)\chi_{A_n}$ for $n \geq n_0$. Then $I_{\varphi}(x_n) \leq I_{\varphi}(y) \leq 1$, so $\|x_n\|_{\varphi} \leq 1$. Moreover, the following holds for $\lambda > 0$:

$\lim_{n\to\infty} I_\varphi(\lambda(x_n - x)) = \lim_{n\to\infty} \varphi(\lambda/n)\mu(A_n) = 0$. Hence, $\lim_{n\to\infty} \|x_n - x\|_\varphi = 0$. However,

$$\begin{aligned} I_\varphi\left(x_n + (y-x)/2\right) &= I_\varphi\left(x_n - x + (x+y)/2\right) = \\ &= \int_\Omega \varphi\left((1/n)\chi_{A_n} + \sum_{k=n_0}^{\infty} \left(c(\varphi) - 1/2k\right)\chi_{A_k}\right) d\mu \geq \\ &\geq \int_\Omega \varphi\left((1/n)\chi_{A_n} + \left(c(\varphi) - 1/2n\right)\chi_{A_n}\right) d\mu = \\ &= \int_{A_n} \varphi\left(c(\varphi) + 1/2n\right) d\mu = \varphi\left(c(\varphi) + 1/2n\right)\mu(A_n) = +\infty. \end{aligned}$$

Thus $x_n + \frac{1}{2}(y-x) \notin B(L^\varphi(\mu))$, which means $B(L^\varphi(\mu))$ is not LNC. ■

Note that the assumptions of the above Lemma are satisfied provided $c(\varphi) < +\infty$, $\varphi(c(\varphi)) < +\infty$ and either the atomless part of Ω has positive measure or $\inf\{\,\mu(e_n) : n \in \mathbb{N}\,\} = 0$.

In the next Lemma we assume that μ has infinitely many disjoint atoms $\{\,e_n : n \in \mathbb{N}\,\}$ and set

$$c_n := \mu(e_n)$$

Lemma 2.5 *Let (x_n), (y_n) be sequences of nonnegative real numbers, such that either $x_k = y_k$ or φ is affine and increasing on $[x_k \wedge y_k, x_k \vee y_k]$ for $k \in \mathbb{N}$. Assume $\sum_{i\in\mathbb{N}} \varphi(x_i)c_i = \sum_{i\in\mathbb{N}} \varphi(y_i)c_i = 1$. Then $I_\varphi(u_k) = 1$ and $I_\varphi(u_n + \frac{1}{2}(y-x)) > 1$ for any $n \in \mathbb{N}$ satisfying $y_n < x_n$, where $u_k := x - 2x_k\chi_{e_k}$, $x := \sum_{i\in\mathbb{N}} x_i\chi_{e_i}$, and $y := \sum_{i\in\mathbb{N}} y_i\chi_{e_i}$.*

Proof Note $I_\varphi(u_n) = \sum_{i\in\mathbb{N}\setminus\{n\}} \varphi(x_i)c_i + \varphi(-x_n)c_n = \sum_{i\in\mathbb{N}} \varphi(x_i)c_i = 1$ what completes the proof of the first part. Set $\delta_i := (y_i - x_i)/2$. Let $n \in \mathbb{N}$ be such that $y_n < x_n$. Thus $\varphi(x_n + \delta_n) < \varphi(x_n - \delta_n)$. Hence,

$$\begin{aligned} I_\varphi\left(u_n + \frac{1}{2}(y-x)\right) &= I_\varphi\left(x - 2x_n\chi_{e_n} + \frac{1}{2}(y-x)\right) = \\ &= \sum_{i\in\mathbb{N}\setminus\{n\}} \varphi(x_i+\delta_i)c_i + \varphi(-x_n+\delta_n)c_n = \\ &= \sum_{i\in\mathbb{N}} \varphi(x_i+\delta_i)c_i - \varphi(x_n+\delta_n)c_n + \varphi(x_n-\delta_n)c_n > \\ &> \sum_{i\in\mathbb{N}} \varphi(x_i+\delta_i)c_i = \sum_{i\in\mathbb{N}} \varphi\left(\frac{x_i+y_i}{2}\right)c_i = \\ &= \sum_{i\in\mathbb{N}} \left(\frac{1}{2}\varphi(x_i)c_i + \frac{1}{2}\varphi(y_i)c_i\right) = 1, \end{aligned}$$

what ends the proof of the Lemma. ■

LEMMA 2.6 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of disjoint sets with positive measure such that $\lim_{n\to\infty}\mu(A_n)=0$ and let a,b be real numbers such that $0<a<b$ and the function φ is affine and increasing on $[a,b]$. Assume that $\mu(A)<(\varphi((a+b)/2))^{-1}$ for $A:=\bigcup_{n\in\mathbb{N}}A_n$. Let $B\subset\Omega\setminus A$ be a set of finite positive measure. Fix $d>0$ satisfing $\varphi(d)\mu(B)=1-\mu(A)\varphi((a+b)/2)$. Let*

$$0<\delta<\min\left\{\frac{b-a}{2},\quad \frac{\mu(A\setminus A_1)}{\mu(A_1)}\cdot\frac{b-a}{2}\right\}\quad \textit{and let}\quad \eta:=\frac{\delta\mu(A_1)}{\mu(A\setminus A_1)}$$

$$\textit{Put}\quad x:=\frac{a+b}{2}\chi_A+d\chi_B,\quad y:=\left(\frac{a+b}{2}-\eta\right)\chi_{A\setminus A_1}+\left(\frac{a+b}{2}+\delta\right)\chi_{A_1}+d\chi_B,$$

$$x_n:=x\left(\mathbf{1}-2\chi_{A_n}\right)\quad \textit{for}\quad n\in\mathbb{N}.$$

$$\textit{Then}\quad I_\varphi(x)=I_\varphi(y)=I_\varphi(x_n)=1\quad \textit{and}\quad I_\varphi\left(x_n+\frac{y-x}{2}\right)>1\quad \textit{for}\ \ n\geq 2.$$

PROOF Observe that $0<\delta,\,\eta<(b-a)/2$. Thus $x(A)\cup y(A)\subset[a,b]$. Let $k>0$ be the slope of the function φ on $[a,b]$. We have $I_\varphi(x)=\varphi\left((a+b)/2\right)\mu(A)+\varphi(d)\mu(B)=1$ and

$$\begin{aligned}
I_\varphi(y) &= \varphi\left((a+b)/2-\eta\right)\mu(A\setminus A_1)+\varphi\left((a+b)/2+\delta\right)\mu(A_1)+\varphi(d)\mu(B)=\\
&= \left(\varphi\left((a+b)/2\right)-k\eta\right)\mu(A\setminus A_1)+\left(\varphi\left((a+b)/2\right)+k\delta\right)\mu(A_1)+\varphi(d)\mu(B)=\\
&= \varphi\left((a+b)/2\right)\left(\mu(A\setminus A_1)+\mu(A_1)\right)+k\left(\delta\mu(A_1)-\eta\mu(A\setminus A_1)\right)+\varphi(d)\mu(B)=\\
&= \varphi\left((a+b)/2\right)\mu(A)+0+\varphi(d)\mu(B)=1.
\end{aligned}$$

Hence,

$$I_\varphi(x_n)=\int\limits_{\Omega\setminus A_n}\varphi(x)\,d\mu+\int\limits_{A_n}\varphi(-x)\,d\mu=\int\limits_{\Omega}\varphi(x)\,d\mu=1.$$

Moreover, for all $n \geq 2$, the following equalities hold:

$$\begin{aligned}
I_\varphi\left(x_n + \frac{y-x}{2}\right) &= \int_\Omega \varphi\left(x_n - x + \frac{x+y}{2}\right) d\mu = \int_\Omega \varphi\left(-(a+b)\chi_{A_n} + \frac{x+y}{2}\right) d\mu = \\
&= \int_{A_n} \varphi\left(-(a+b) + \frac{x+y}{2}\right) d\mu + \int_{A\setminus A_n} \varphi\left(\frac{x+y}{2}\right) d\mu + \int_{\Omega\setminus A} \varphi(x)\, d\mu = \\
&= \int_{A_n} \varphi\left(a+b - \frac{x+y}{2}\right) d\mu + \int_A \varphi\left(\frac{x+y}{2}\right) d\mu - \int_{A_n} \varphi\left(\frac{x+y}{2}\right) d\mu + \\
&\quad + \frac{1}{2}\int_{\Omega\setminus A} \varphi(x)\, d\mu + \frac{1}{2}\int_{\Omega\setminus A} \varphi(y)\, d\mu = \\
&= \int_{A_n} \varphi\left(a+b - \frac{x+y}{2}\right) d\mu - \int_{A_n} \varphi\left(\frac{x+y}{2}\right) d\mu + \int_A \frac{\varphi(x)+\varphi(y)}{2}\, d\mu + \\
&\quad + \int_{\Omega\setminus A} \frac{\varphi(x)+\varphi(y)}{2}\, d\mu = \int_{A_n} \left(\varphi\left(a+b-\frac{x+y}{2}\right) - \varphi\left(\frac{x+y}{2}\right)\right) d\mu + \\
&\quad + \frac{1}{2}\int_\Omega \varphi(x)\, d\mu + \frac{1}{2}\int_\Omega \varphi(y)\, d\mu = \\
&= \int_{A_n} \left(\varphi\left(a+b-\frac{a+b-\eta}{2}\right) - \varphi\left(\frac{a+b-\eta}{2}\right)\right) d\mu + 1 = \\
&= 1 + \left(\varphi\left(\frac{a+b}{2} + \frac{\eta}{2}\right) - \varphi\left(\frac{a+b}{2} - \frac{\eta}{2}\right)\right)\mu(A_n) > 1,
\end{aligned}$$

what ends the proof of the Lemma. ∎

Lemma 2.7 *If $X \subset L^\varphi(\mu)$ is a linear subspace of $L^\varphi(\mu)$ with the norm inherited from $L^\varphi(\mu)$, such that the equivalence*

$$\|x\|_\varphi = 1 \quad \Leftrightarrow \quad I_\varphi(x) = 1,$$

holds for any $x \in X$ and φ is strictly convex, then $B(X)$ is strictly convex.

Proof Suppose there exist $x, y \in X$, $x \neq y$, $\|x\|_\varphi$, $\|y\|_\varphi \leq 1$, $0 < \alpha < 1$ such that $\|\alpha x + (1-\alpha)y\| = 1$. Since φ is strictly convex, we have

$$\varphi(\alpha x(\omega) + (1-\alpha)y(\omega)) < \alpha\varphi(x(\omega)) + (1-\alpha)\varphi(y(\omega))$$

on the set $\{\omega : x(\omega) \neq y(\omega)\}$ wich is of positive measure. Thus

$$\begin{aligned}
1 &= \|\alpha x + (1-\alpha)y\|_\varphi = I_\varphi(\alpha x + (1-\alpha)y) = \int_\Omega \varphi(\alpha x + (1-\alpha)y)\, d\mu < \\
&< \int_\Omega (\alpha\varphi(x) + (1-\alpha)\varphi(y))\, d\mu = \alpha I_\varphi(x) + (1-\alpha)I_\varphi(y) \leq \alpha + (1-\alpha) = 1
\end{aligned}$$

and we get a contradiction. ∎

3. Main results. Now we characterize the *LNC* properties of unit balls in Orlicz spaces $L^\varphi(\mu)$ and the space of finite elements $E^\varphi(\mu)$ respectively equipped with the Luxemburg norm assuming that φ is a Young function and μ is a σ–finite measure. In the cases of a purely atomic measure μ, we fix a partition of Ω on disjoint atoms $\{ e_n : n \in \mathcal{N} \}$, where $\mathcal{N}$ denotes $\mathbb{N}$ if the set of atoms is infinite (i. e. if $\dim L^\varphi(\mu) = \infty$) or is equal to the number of atoms if the cardinality of this set is finite (i. e. if $\dim L^\varphi(\mu) < \infty$).

THEOREM 3.1 *The unit ball $B(L^\varphi(\mu))$ is LNC if and only if at least one of the following conditions is satisfied:*

(i) $\dim L^\varphi(\mu) < \infty$

(ii) *μ is purely atomic measure with* $\inf\{ \mu(e_n) : n \in \mathbb{N} \} > 0$*, $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ and φ is strictly convex on the interval* $[0, b]$ *for some $b > 0$.*

(iii) *$c(\varphi) = +\infty$, $\varphi \in \Delta_2(\mu)$ and φ is strictly convex on $\mathbb{R}$.*

PROOF ($\Rightarrow$) Assume that $B(L^\varphi(\mu))$ is *LNC* and $L^\varphi(\mu)$ is infinite dimensional. It is necessary to prove that *(ii)* or *(iii)* are satisfied. We consider two cases:

A) Suppose μ is purely atomic and $\inf\{ \mu(e_n) : n \in \mathbb{N} \} > 0$. Because $B(L^\varphi(\mu))$ is *LNC*, it follows from [20, Theorem 3.1, p. 196] that it is stable. From Wisła's Theorem one of the six conditions from Theorem 2.1 hold, ([23], Theorem 5). Obviously, *(i)* is excluded by asumption and *(ii)* is excluded by Lemma 2.4, *(vi)* is excluded by assumption A). Thus one of conditions *(iii)—(v)* is satisfied. Either $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ when $c(\varphi) < +\infty$ and $\varphi(c(\varphi)) < +\infty$, or $\varphi \in \Delta_r(\mu)$. Hence, $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ in both cases. Thus $a(\varphi) = 0$.

Suppose that *(ii)* is not satisfied. Thus the assumptions of Lemma 2.3 are satisfied. Let sequences (x_n) and (y_n) satisfy the conditions given in Lemma 2. Let x, y and u_n ($n \in \mathbb{N}$) satisfy the conditions given in Lemma 2.5. In order to show that $B(L^\varphi(\mu))$ is not *LNC*, it suffices to prove that $\lim_{n\to\infty} \|u_n - x\| = 0$.

Because $\varphi(x_n)c_n \to 0$ we have $\lim_n I_\varphi(\frac{1}{2}(x - u_n)) = lim_n I_\varphi(x_n \chi_{e_n}) = 0$. By condition $\Delta_r^0(\mu)$ we have $\|x - u_n\| \to 0$. This completes the proof for case A).

B) It remains to consider the case when either the measure μ is not purely atomic or $\inf\{ \mu(e_n) : n \in \mathbb{N} \} = 0$. In other words, we may assume that there exists a sequence of mutually disjoint sets of positive finite measures (A_n), such that $\lim_n \mu(A_n) = 0$. Fix such a sequence. We will show that condition *(iii)* of Theorem 1 is satisfied.

Because $B(L^\varphi(\mu))$ is *LNC* and thus stable, at least one of the conditions *(i)—(vi)* of Theorem 2.1 is satisfied. Conditions *(i)* and *(ii)* are excluded, as in case A). We claim that the case where both $c(\varphi) < +\infty$ and $\varphi(c(\varphi)) < +\infty$ is excluded, i.e. conditions *(iv)—(vi).* are excluded, too. Suppose $c(\varphi) < +\infty$ and $\varphi(c(\varphi)) < +\infty$. Then there exists an increasing sequence of positive integers (n_k) such that

$$\varphi(c(\varphi)) \cdot \sum_{k\in\mathbb{N}} \mu\left(A_{n_k}\right) \leq 1.$$

Hence, by Lemma 2.4, the unit ball $B(L^{\varphi}(\mu))$ is not LNC — a contradiction. Thus condition *(iii)* of Theorem 2.1 is satisfied, so $\varphi \in \Delta_r(\mu)$ for some $r > 1$.

We claim $c(\varphi) = +\infty$. If μ has positive atomless part Ω_1, then $\varphi \in \Delta_r(\mu \restriction \Omega_1)$. Hence, $c(\varphi) = +\infty$.

If μ is purely atomic and $\inf\{\, \mu(e_n) : n \in \mathbb{N}\,\} = 0$, then there exists $n \in \mathbb{N}$ such that

$$\mu(e_n) \le b \Big/ \varphi\left(\frac{c(\varphi)}{r'}\right)$$

for all $r' \in (1, r)$ (the constant b is choosen according to the definition of $\Delta_r(\mu)$). Thus $\varphi\big(c(\varphi)/r'\big)\mu(e_n) \le b$. Hence,

$$+\infty = \varphi\left(r\frac{c(\varphi)}{r'}\right)\mu(e_n) \le c\varphi\left(\frac{c(\varphi)}{r'}\right)\mu(e_n) + d_n < +\infty$$

and we obtain a contradiction, which proves our claim. Hence, $\varphi \in \Delta_r(\mu)$ for any $r > 1$, in particular $\varphi \in \Delta_2(\mu)$.

Now we show $a(\varphi) = 0$. Assume $a(\varphi) > 0$. Let x_1 be a positive real number such that $\varphi(x_1)\mu(A_1) = 1$. Set

$$x := x_1\chi_{A_1}, \qquad y := x_1\chi_{A_1} + a(\varphi)\cdot\sum_{i=2}^{+\infty}\chi_{A_i}.$$

and

$$u_n := x + a(\varphi)\chi_{A_n} \qquad \text{for} \qquad n = 2, 3, \dots\ .$$

Then

$$I_{\varphi}(x) = I_{\varphi}(y) = \varphi(x_1)\mu(A_1) = 1, \qquad I_{\varphi}(u_n) = 1.$$

Thus x, y, $u_n \in B(L^{\varphi}(\mu))$. Let $\lambda > 0$. Then:

$$I_{\varphi}\left(\lambda(u_n - x)\right) = I_{\varphi}\left(\lambda a(\varphi)\chi_{A_n}\right) = \varphi(\lambda a(\varphi))\mu(A_n).$$

Hence, $\lim_n I_{\varphi}(\lambda(u_n - x)) = 0$ and we have $\lim\limits_n \|u_n - x\| = 0$. Moreover,

$$\begin{aligned}
I_{\varphi}\left(u_n + \frac{1}{2}(y - x)\right) &= I_{\varphi}\left(u_n - x + \frac{1}{2}(x + y)\right)\\
&= I_{\varphi}\left(a(\varphi)\chi_{A_n} + x_1\chi_{A_1} + \sum_{i=2}^{\infty}\frac{a(\varphi)}{2}\chi_{A_i}\right) =\\
&= I_{\varphi}\left(x_1\chi_{A_1} + \frac{3}{2}a(\varphi)\chi_{A_n} + \sum_{i\in\mathbb{N}\setminus\{\,1,n\,\}}\frac{a(\varphi)}{2}\chi_{A_i}\right) =\\
&= \varphi(x_1)\mu(A_1) + \varphi\left(\frac{3}{2}a(\varphi)\right)\mu(A_n) = 1 + \varphi\left(\frac{3}{2}a(\varphi)\right)\mu(A_n)\\
&> 1.
\end{aligned}$$

Thus $u_n + \frac{1}{2}(y - x) \notin B(L^{\varphi}(\mu))$, which contradicts the assumption that $B(L^{\varphi}(\mu))$ is LNC. It follows that $a(\varphi) = 0$.

To prove *(iii)* suppose that φ is not strictly convex. Let x, y, x_n, $n \in \mathbb{N}$ be defined as in Lemma 2.6. We have x, y, $x_n \in B(L^\varphi(\mu))$ and $x_n + (y-x)/2 \notin B(L^\varphi(\mu))$ for $n \geq 2$. Moreover,

$$I_\varphi\left(\lambda(x-x_n)\right) = \int\limits_{A_n} \varphi(2\lambda x)\, d\mu = \int\limits_{A_n} \varphi(\lambda(a+b))\, d\mu = \varphi(\lambda(a+b))\mu(A_n)$$

for $\lambda > 0$. It follows that $\varphi(\lambda(a+b)) < +\infty$, since $c(\varphi) = +\infty$. Also, $\lim_{n\to\infty} I_\varphi(\lambda(x - x_n)) = 0$, since $\lim_{n\to\infty}\mu(A_n) = 0$. Therefore, $\lim_{n\to\infty}\|x - x_n\| = 0$. Hence, we obtain a contradiction.

($\Leftarrow$) We divide the proof into three parts.

a) Suppose that condition *(i)* holds, so μ is purely atomic and has a finte number of atoms $\{\, e_i : 1 \leq i \leq N\,\}$. Set $c_i = \mu(e_i)$ for $i = 1, 2, \ldots, N$. Let x, $y \in B(L^\varphi(\mu))$, $u_n \in B(L^\varphi(\mu))$ for $n \in \mathbb{N}$, $\lim_n \|u_n - x\|_\varphi = 0$. It is necessary to prove that for some $n_0 \in \mathbb{N}$, $u_n + \frac{1}{2}(y-x) \in B(L^\varphi(\mu))$ for $n \geq n_0$. Without loss of generality we can assume that $\|\lambda x + (1-\lambda)y\|_\varphi = 1$ for any $0 \leq \lambda \leq 1$. Set

$$x = \sum_{i=1}^N x_i \chi_{e_i}, \qquad y = \sum_{i=1}^N y_i \chi_{e_i}, \qquad u_n = \sum_{i=1}^N u_i^n \chi_{e_n}$$

for $i = 1, 2, \ldots, N$, $n \in \mathbb{N}$, where x_i, y_i, u_i^n are real numbers. We consider two cases:

1. There exists $\lambda_0 \in [0,1]$ such that $I_\varphi(\lambda_0 x + (1-\lambda_0)y) < 1$. It is easy to see that the function $g(\lambda) := I_\varphi(y + \lambda(x-y))$ is convex. From $g(0) \leq 1$, $g(1) \leq 1$ and $g(\lambda_0) < 1$ we have $\sum\limits_{i=1}^N \varphi(\lambda x_i + (1-\lambda)y_i)c_i < 1$ for any $\lambda \in (0,1)$. In particular, $\sum\limits_{i=1}^N \varphi((x_i + y_i)/2)c_i < 1$. We have $I_\varphi(\alpha(x+y)/2) = +\infty$ for $\alpha > 1$, since $\|(x+y)/2\|_\varphi = 1$. Hence $\sum_{i=1}^N \varphi(\alpha(x_i + y_i)/2)c_i = +\infty$. This is possible only if there exists k, $1 \leq k \leq N$, such that $|(x_k + y_k)/2| = c(\varphi) < +\infty$. If $x_k \neq y_k$ then the expression $|\lambda x_k + (1-\lambda)y_k|$ for $\lambda \in [0,1]$ attains its supremum at one of the ends of the interval. Hence, either $|x_k| > c(\varphi)$ or $|y_k| > c(\varphi)$. But then either $I_\varphi(x) = +\infty$ or $I_\varphi(y) = +\infty$, what contradicts to x, $y \in B(L^\varphi(\mu))$. Hence, $x_k = y_k = \pm c(\varphi)$ for some fixed k, $1 \leq k \leq N$. Thus $\lim_{n\to\infty} I_\varphi(\alpha(x - u_n)) = 0$ holds for $\alpha > 0$. So $\lim_{n\to\infty}\sum_{i=1}^N \varphi(\alpha(x_i - u_i^n))c_i = 0$. Hence, $\lim\limits_n u_i^n = x_i$ for every $1 \leq i \leq N$. Otherwise, there exists a natural number $1 \leq i \leq N$, increasing sequence n_k of natural numbers and $\varepsilon > 0$, such that $|u_i^{n_k} - x_i| > \varepsilon$. Setting $a := \max\{\, a(\varphi), 1\,\}$ and $\alpha := \frac{2a}{\varepsilon}$ we obtain $\sum_{j=1}^N \varphi(\alpha(x_j - u_j^{n_k}))c_j \geq \varphi(2a)c_i > 0$ for $k \in \mathbb{N}$, which is contradiction. Set

$$I := \{\, i : x_i = y_i = \pm c(\varphi)\,\}, \qquad J := \{1, 2, \ldots, N\,\} \setminus I.$$

For any $i \in J$ there exists $n(i) \in \mathbb{N}$, such

$$\left|\varphi\left(u_i^n - x_i + \frac{x_i + y_i}{2}\right) - \varphi\left(\frac{x_i + y_i}{2}\right)\right| c_i < \frac{1}{N}\left(1 - I_\varphi\left(\frac{x+y}{2}\right)\right),$$

for all $n \geq n(i)$, because $|(x_i + y_i)/2| < c(\varphi)$ and φ is continuous on $(-c(\varphi), c(\varphi))$. Hence,

$$\begin{aligned}
& I_\varphi\left(u_n + \frac{1}{2}(y-x)\right) = \sum_{i\in I}\varphi(u_i^n)c_i + \sum_{i\in J}\varphi\left(u_i^n - x_i + \frac{x_i+y_i}{2}\right)c_i \leq \\
\leq\ & \sum_{i\in I}\varphi\left(\frac{x_i+y_i}{2}\right)c_i + \sum_{i\in J}\left(\varphi\left(u_i^n - x_i + \frac{x_i+y_i}{2}\right) - \varphi\left(\frac{x_i+y_i}{2}\right)\right) + \\
& + \sum_{i\in J}\varphi\left(\frac{x_i+y_i}{2}\right)c_i \leq \\
\leq\ & I_\varphi\left(\frac{x+y}{2}\right) + \sum_{i\in J}\left|\varphi\left(u_i^n - x_i + \frac{x_i+y_i}{2}\right) - \varphi\left(\frac{x_i+y_i}{2}\right)\right|c_i \leq \\
\leq\ & I_\varphi\left(\frac{x+y}{2}\right) + N\cdot\frac{1}{N}\left(1 - I_\varphi\left(\frac{x+y}{2}\right)\right) = 1.
\end{aligned}$$

Thus, there exists an n_0 such that $u_n + \frac{1}{2}(y-x) \in B(L^\varphi(\mu))$ for $n \geq n_0$. Thus, $B(L^\varphi(\mu))$ is LNC.

2. We have $I_\varphi(\lambda x + (1-\lambda)y) = 1$ for every $0 \leq \lambda \leq 1$.

Let n_0 be large enough that for $n \geq n_0$ and for $\varepsilon_i := u_i^n - x_i$ the conditions *(a)—(d)* in item *3* of Lemma 2.2 are satisfied. It follows from this lemma that

$$\begin{aligned}
I_\varphi\left(u_n + \frac{1}{2}(y-x)\right) &= \sum_{i=1}^N \varphi\left(\varepsilon_i + \frac{x_i+y_i}{2}\right)c_i = F(1) \leq F(0) = \\
&= \sum_{i=1}^N \varphi(x_i+\varepsilon_i)c_i = \sum_{i=1}^N \varphi(u_i^n)c_i = I_\varphi(u_n) \leq 1,
\end{aligned}$$

for fixed $n \geq n_0$, such that $u_n + (y-x)/2 \in B(L^\varphi(\mu))$ for $n \geq n_0$. Thus $B(L^\varphi(\mu))$ is LNC, which finishes the proof of part *a)*.

b) Now we assume that condition *(ii)* is satisfied.

Fix $x, y \in B(L^\varphi(\mu))$, $x \neq y$, $u_n \in B(L^\varphi(\mu))$ for $n \in \mathbb{N}$, $\lim_n \|u_n - x\|_\varphi = 0$. We can assume that $\|\lambda x + (1-\lambda)y\|_\varphi = 1$ for every $\lambda \in [0,1]$. It is necessary to prove that there exists n_0, such that for $n \geq n_0$, $u_n + \frac{1}{2}(y-x) \in B(L^\varphi(\mu))$. Analogously to part *a)*, set $x = \sum_{i\in\mathbb{N}} x_i\chi_{e_i}$, $y = \sum_{i\in\mathbb{N}} y_i\chi_{e_i}$, $u_n = \sum_{i\in\mathbb{N}} u_i^n\chi_{e_i}$ $n \in \mathbb{N}$. By assumption $\inf\{\, c_i : i \in \mathbb{N}\,\} > 0$. In particular, $\sum_{i\in\mathbb{N}}\mu(e_i) = +\infty$ and hence, $a(\varphi) = 0$. As in part *a)* we consider two cases:

1. There exists $\lambda_0 \in [0,1]$ such that $I_\varphi(\lambda_0 x + (1-\lambda_0)y) < 1$.

Analogously to part *a,)*, $I_\varphi(\lambda x + (1-\lambda)y) < 1$ for every $\lambda \in (0,1)$. In particular, $\sum_{i=1}^\infty \varphi((x_i+y_i)/2)c_i < 1$. We obtain $\lim_{n\to\infty} u_i^n = x_i$ for $i \in \mathbb{N}$, because $a(\varphi) = 0$.

From the following obvious inequality $0 \leq \varphi(x_i) \leq (1/\inf\{\, c_j : j \in \mathbb{N}\,\})\cdot\varphi(x_i)c_i$, together with $\sum_{i=1}^\infty \varphi(x_i)c_i \leq 1$ we obtain $\lim_i \varphi(x_i) = 0$. Hence, $\lim_i x_i = 0$.

Analogously, $\lim_i y_i = 0$ and $\lim_i u_i^n = 0$ for $n \in \mathbb{N}$. Set

$$J := \{\, k \in \mathbb{N} : |x_k| = c(\varphi) \quad \text{or} \quad |y_k| = c(\varphi)\,\}.$$

Obviously J is finite. Define the following two complementary subspaces of $L^\varphi(\mu)$:

$$X_1 := \operatorname{span}\{\, \chi_{e_k} : k \notin J\,\} \qquad X_2 := \operatorname{span}\{\, \chi_{e_k} : k \in J\,\}$$

Set

$$x' := \operatorname{Pr}_{X_1} x, \qquad x'' := \operatorname{Pr}_{X_2} x, \qquad y' := \operatorname{Pr}_{X_1} y, \qquad y'' := \operatorname{Pr}_{X_2} y$$

and

$$u_n' := \operatorname{Pr}_{X_1} u_n, \qquad u_n'' := \operatorname{Pr}_{X_2} u_n \qquad \text{for } \; n \in \mathbb{N},$$

where Pr denotes the natural projection. Obviously, $x = x' + x''$, $y = y' + y''$, $u_n = u_n' + u_n''$, and $\|x'\|_\infty < c(\varphi)$, $\|y'\|_\infty < c(\varphi)$. Thus $\|(x'+y')/2\|_\infty < c(\varphi)$. From this we obtain $\|(x'+y')/2\|_\varphi < 1$, since $I_\varphi((x'+y')/2) < 1$ (see the remarks after the definition of $\Delta_r^0(\mu)$). Moreover,

$$0 \le I_\varphi\left(\lambda(x' - u_n')\right) = \sum_{k \notin J} \varphi\left(\lambda(x_k - u_k^n)\right) c_k \le I_\varphi(\lambda(x - u_n)) \longrightarrow 0$$

for $\lambda > 0$. Therefore, $\lim_{n\to\infty} \|x' - u_n'\|_\varphi = 0$. Since

$$\left\| u_n' + \frac{1}{2}(y' - x') \right\|_\varphi = \left\| u_n' - x' + \frac{x' + y'}{2} \right\|_\varphi \le \|u_n' - x'\|_\varphi + \left\| \frac{x'+y'}{2} \right\|_\varphi,$$

we obtain

$$\exists \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \ge n_0 \qquad \left\| u_n' + \frac{1}{2}(y' - x') \right\|_\varphi < 1 - \varepsilon$$

and $I_\varphi(u_n' + (y' - x')/2) < 1 - \varepsilon$.

We can find $n_1 \ge n_0$ such that

$$I_\varphi\left(u_n'' + \frac{1}{2}(y'' - x'')\right) = \sum_{k \in J} \varphi\left(u_k^n + \frac{1}{2}(y_k - x_k)\right) c_k < \varepsilon$$

for any $n \ge n_1$, because $\lim_k u_n^k = x_n$ for $k \in J$ and J is finite. Hence,

$$\begin{aligned} I_\varphi\left(u_n + \frac{1}{2}(y - x)\right) &= \sum_{k \notin J} I_\varphi\left(u_k^n + \frac{1}{2}(y_k - x_k)\right) + \sum_{k \in J} I_\varphi\left(u_k^n + \frac{1}{2}(y_k - x_k)\right) = \\ &= I_\varphi\left(u_n' + \frac{1}{2}(y' - x')\right) + I_\varphi\left(u_n'' + \frac{1}{2}(y'' - x'')\right) \\ &< 1 - \varepsilon + \varepsilon = 1 \end{aligned}$$

for $n \ge n_1$, so $u_n + (y - x)/2 \in B(L^\varphi(\mu))$.

2. We have $I_\varphi(\lambda x + (1 - \lambda)y) = 1$ for every $0 \le \lambda \le 1$. Obviously, there exists $n_0 \in \mathbb{N}$ such that $x_i,\ y_i \in [-b, b]$ for $i > n_0$. Set

$$I := \{\, i \in \mathbb{N} : x_i = y_i\,\}, \qquad J := \{\, i \in \mathbb{N} : x_i \ne y_i\,\}.$$

Since

$$1 = \sum_{i\in\mathbb{N}} \varphi\left(\lambda x_i + (1-\lambda)y_i\right) c_i \le \lambda \sum_{i\in\mathbb{N}} \varphi(x_i)c_i + (1-\lambda)\sum_{i\in\mathbb{N}} \varphi(y_i)c_i = 1,$$

it follows that φ is affine on every interval $[x_i, y_i]$ (or $[y_i, x_i]$) for $i \in J$. Because φ is strictly convex on $[-b, b]$, we obtain $i \notin J$ for $i > n_0$. Hence, $J \subseteq \{1, 2, \ldots, n_0\}$. In particular, $x_i = y_i$ for $i > n_0$. Therefore, setting $\varepsilon_i = u_i^n - x_i$ for sufficently large n, the conditions given in Lemma 2.2 for the case $N = +\infty$ are satisfied. Therefore, from Lemma 2.2

$$I_\varphi\left(u_n + \frac{1}{2}(y-x)\right) = F(1) \le F(0) = I_\varphi(u_n) \le 1.$$

c) Suppose that *(iii)* is satisfied. From $\Delta_2(\mu)$ and $c(\varphi) = \infty$ it is known that $L^\varphi(\mu)$ satisfies the following condition:

$$\|x\|_\varphi = 1 \Leftrightarrow I_\varphi(x) = 1 \qquad x \in L^\varphi(\mu).$$

From Lemma 2.7 $B(L^\varphi(\mu))$ is strictly convex and so LNC. The proof of the Theorem is complete. ■

Corollary 3.2 *If μ is an atomless measure, then the following conditions are equivalent:*

(i) *$B(L^\varphi(\mu))$ is LNC.*

(ii) *φ is strictly convex, $c(\varphi) = +\infty$ and φ satisfies either Δ_2 globally when $\mu(\Omega) = +\infty$ or for sufficently large t when $\mu(\Omega) < +\infty$.*

(iii) *$B(L^\varphi(\mu))$ is strictly convex.*

Proof *(i) ⇒ (ii)* Since μ is atomless, conditions *(i)* and *(ii)* from Theorem 3.1 are not satisfied. Therefore, condition *(iii)* of this theorem must hold. The result follows from the fact that for atomless measures the condition $\Delta_2(\mu)$ is equivalent to the classic condition for Δ_2 based on $\mu(\Omega)$.

(ii) ⇒ (iii) Follows from the last part of *c)* in the proof of Theorem 3.1.

(iii) ⇒ (i) Obvious. ■

Corollary 3.3 *If μ is purely atomic measure with an infinite number of atoms and*
$0 < \inf\{\mu(e_n) : n \in \mathbb{N}\} \le \sup\{\mu(e_n) : n \in \mathbb{N}\} < +\infty$
then the following conditions are equivalent:

(i) *$B(L^\varphi(\mu))$ is LNC.*

(ii) *φ satisfies the condition δ_2 and is strictly convex on $[0, b]$ for some $b > 0$.*

Proof *(i)* $\Rightarrow$ *(ii)* Condition *(ii)* holds because conditions *(ii)* and *(iii)* of Theorem 3.1 are satisfied. From the remark given after the definition of $\Delta_r^0(\mu)$ we see that φ satisfies the condition δ_2.

(ii) $\Rightarrow$ *(i)* follows from Theorem 3.1 and from the remark mentioned above. ■

From the last corollary it follows that $B(l^\varphi)$ is LNC iff the condition *(ii)* of Corollary 3.3 is satisfied. Obviously, $B(l^p)$ is LNC for $1 < p < +\infty$. Note that $B(l^1)$ and $B(l^\infty)$ are stable, but not LNC (see also [1], [20]).

Example 3.4 A ball $B(l^\varphi)$ satisfying LNC need not be strictly convex.

Indeed, for

$$\varphi(t) = \begin{cases} t^2, & \text{for} |t| \le \frac{1}{2} \\ |t| - \frac{1}{4}, & \text{for} |t| \ge \frac{1}{2} \end{cases}$$

the unit ball $B(l^\varphi)$ is LNC from Corollary 3.3, but

$$I_\varphi(\lambda x + (1-\lambda)y) = I_\varphi\left(\left(\frac{3}{4} + \frac{1}{4}\lambda, \frac{3}{4} - \frac{1}{4}\lambda, 0, 0, \ldots\right)\right) = \frac{1}{2} + \frac{1}{4}\lambda + \frac{1}{2} - \frac{1}{4}\lambda = 1,$$

for $x = (1, 1/2, 0, 0, \ldots)$, $y = (3/4, 3/4, 0, 0, \ldots)$ and $\lambda \in [0,1]$. Hence, $\|z\|_\varphi = 1$ for any $z \in [x,y]$, i.e. $B(l^\varphi)$ is not strictly convex.

Now we present a characterization of the LNC property for the unit ball $B(E^\varphi(\mu))$.

Theorem 3.5 *$B(E^\varphi(\mu))$ is LNC iff at least one of the following conditions is satisfied:*

(i) *μ is purely atomic measure with a finite number of atoms (equivalently $\dim E^\varphi(\mu) < \infty$).*

(ii) *$c(\varphi) = +\infty$, μ is purely atomic measure with an infinite number of atoms, $\inf\{\mu(e_n) : n \in \mathbb{N}\} > 0$ and either $a(\varphi) > 0$ or φ is strictly convex on $[0,b]$ for some $b > 0$.*

(iii) *$c(\varphi) = +\infty$, φ is strictly convex on $\mathbb{R}$ and μ either has a positive atomless part or $\inf\{\mu(e_n) : n \in \mathbb{N}\} = 0$.*

(iv) *$c(\varphi) < +\infty$ and $a(\varphi) > 0$.*

(v) *$c(\varphi) < +\infty$ and φ is strictly convex on $[0,b]$ for some $b > 0$.*

(vi) *$c(\varphi) < +\infty$, $\mu(\Omega)\varphi(c(\varphi)) \le 1$.*

(vii) *$c(\varphi) < +\infty$ and μ has a positive atomless part.*

Proof ($\Rightarrow$) Assume that $B(E^\varphi(\mu))$ is LNC. Suppose $\dim E^\varphi(\mu) = \infty$. We consider three cases:

1. $c(\varphi) = +\infty$ and μ is purely atomic measure with an infinite number of atoms, such that $\inf\{\mu(e_n) : n \in \mathbb{N}\} > 0$. We prove that in this case *(ii)* is satisfied. Consider the case $a(\varphi) = 0$. Suppose that φ is not strictly convex on any interval of the form $[0, b]$ where $b > 0$. Let x, y be defined as in Lemma 2.3 and u_n as in Lemma 2.5. Then x, y, $u_n \in B(E^\varphi(\mu))$. Moreover, for each $\lambda > 0$ and $n > 2\lambda$ we have

$$I_\varphi(\lambda(x - u_n)) = \varphi(2\lambda x_n)\mu(e_n) = \varphi(nx_n)\mu(e_n) < \frac{1}{2^n} \to 0$$

holds. Hence, $\lim_{n\to\infty} I_\varphi(\lambda(x - u_n)) = 0$ and $\lim_{n\to\infty} u_n = x$ in the space $E^\varphi(\mu)$. This contradicts the fact that $B(E^\varphi(\mu))$ is LNC.

2. $c(\varphi) = +\infty$, μ either has a positive atomless part or $\inf\{\mu(e_n) : n \in \mathbb{N}\} = 0$.

To start, we prove that $a(\varphi) = 0$. By assumption there exists an infinite sequence of mutually disjoint sets with positive measures (A_n), such that $\sum_{n\in\mathbb{N}} \mu(A_n) < +\infty$. Suppose that $a := a(\varphi) > 0$. Set

$$x := d\chi_{A_1} + a\chi_{\bigcup\limits_{i=2}^{\infty} A_n}, \qquad y := d\chi_{A_1} - a\chi_{\bigcup\limits_{i=2}^{\infty} A_n},$$

where d is a positive number, such that $\varphi(d)\mu(A_1) = 1$.

Set $u_n := x - 2a\chi_{A_n}$. Then

$$I_\varphi(\lambda x) = \varphi(\lambda d)\mu(A_1) + \varphi(\lambda a)\mu\left(\bigcup_{i=2}^{\infty} A_n\right) < +\infty$$

for $\lambda > 0$. Hence, $x \in E^\varphi(\mu)$ and similarly y, $u_n \in E^\varphi(\mu)$. Moreover,

$$\lim_n I_\varphi(x - u_n) = \lim_n \varphi(2a)\mu(A_n) = 0, \qquad \text{Thus} \quad \lim_n \|u_n - x\| = 0$$

Also,

$$I_\varphi(x) = \varphi(d)\mu(A_1) + \varphi(a)\mu\left(\bigcup_{i=2}^{\infty} A_n\right) = 1$$

and analogously $I_\varphi(y) = 1$ and $I_\varphi(u_n) = 1$ for $n \in \mathbb{N}$. Hence, x, y, $u_n \in B(E^\varphi(\mu))$. But

$$u_n + \frac{1}{2}(y - x) = u_n - x + \frac{x + y}{2} = -2a\chi_{A_n} + d\chi_{A_1}$$

and

$$I_\varphi\left(u_n + \frac{1}{2}(y - x)\right) = \varphi(2a)\mu(A_n) + \varphi(d)\mu(A_1) = 1 + \varphi(2a)\mu(A_n) > 1.$$

Therefore, $u_n + \frac{1}{2}(y - x) \notin B(E^\varphi(\mu))$, which means that $B(E^\varphi(\mu))$ is not LNC. Thus $a(\varphi) = 0$.

Now we prove that the function φ is strictly convex.

To obtain a contradiction, suppose that φ is not strictly convex. Note that x, y, $x_n \in E^\varphi(\mu)$ for $n \in \mathbb{N}$ (defined in Lemma 2.6), since they are linear combinations of characteristic functions of sets of finite positive measures which, by $c(\varphi) = +\infty$, belong to $E^{\varphi}(\mu)$. From Lemma 2.6 it follows that $B(E^{\varphi}(\mu))$ is not LNC. Thus we obtain a contradiction.

3. Remaining cases. Then $c(\varphi) < +\infty$. If μ has positive atomless part then *(vii)* is satisfied. Let μ be purely atomic measure with an infinite numbers of atoms. If $a(\varphi) > 0$ or φ is strictly convex on $[0, b]$ for some $b > 0$ then *(iv)* or *(v)* is satisfied. Let $a(\varphi) = 0$ and let φ be not strictly convex in any neighbourhood of 0. We need to prove that $\mu(\Omega)\varphi(c(\varphi)) \leq 1$. Suppose $\varphi(c(\varphi)) \sum_{n=1}^{\infty} \mu(e_n) > 1$. Let x, y be defined as in Lemma 2.3 and u_n as in Lemma 2.5. In an analogous way as in the step 1 we prove that $B(E^{\varphi}(\mu))$ is LNC.

($\Leftarrow$) We divide the proof into six parts.

a) Assume that condition *(i)* holds. But then $E^{\varphi}(\mu) = L^{\varphi}(\mu)$ is of finite dimension and the thesis follows from Theorem 3.1.

b) Assume that condition *(ii)* holds. Let x, $y \in B(E^{\varphi}(\mu))$, $u_n \in B(E^{\varphi}(\mu))$ for $n \in \mathbb{N}$. Let $\lim_{n\to\infty} \|u_n - x\|_{\varphi} = 1$ for every $0 \leq \lambda\lambda \leq 1$. It is necessary to prove that there exists n_0, such that for $n \geq n_0$, $u_n + (y - x)/2 \in B(E^{\varphi}(\mu))$. Without loss of generality we assume $\|\lambda x + (1-\lambda)y\| = 1$ for $0 \leq \lambda \leq 1$. Hence $I_{\varphi}(\lambda x + (1-\lambda)y) = 1$ for $0 \leq \lambda \leq 1$. If not then $I_{\varphi}(\alpha(\lambda x + (1-\lambda)y)) = +\infty$ every $\alpha > 1$, what contradicts $\lambda x + (1 - \lambda)y \in E^{\varphi}(\mu)$ in case $c(\varphi) = +\infty$. We set

$$x = \sum_{i\in\mathbb{N}} x_i \chi_{e_i}, \qquad y = \sum_{i\in\mathbb{N}} y_i \chi_{e_i}, \qquad u_n = \sum_{i\in\mathbb{N}} u_i^n \chi_{e_i}$$

for $n \in \mathbb{N}$, where x_i, y_i, u_i^n are real numbers. We prove that $\lim_n x_n = \lim_n y_n = 0$. Suppose for instance that $\lim\limits_n x_n \neq 0$. Then there exists $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $|x_{n_k}| > \varepsilon$. Let $\lambda > 0$ satisfy $\varphi(\varepsilon\lambda) = 1$. Then

$$\begin{aligned} I_{\varphi}(\lambda x) &= \sum_{n\in\mathbb{N}} \varphi(\lambda x_n)\mu(e_n) \geq \sum_{k\in\mathbb{N}} \varphi(\lambda x_{n_k})\mu(e_{n_k}) \geq \\ &\geq \sum_{k\in\mathbb{N}} \varphi(\lambda\varepsilon)\mu(e_{n_k}) \geq \sum_{k\in\mathbb{N}} \inf\{\, \mu(e_n) : n \in \mathbb{N}\,\} = +\infty, \end{aligned}$$

which is impossible for $x \in E^{\varphi}(\mu)$. Thus $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. Hence, there exists $n_0 \in \mathbb{N}$ such that for $i > n_0$, either x_i, $y_i \in [-a(\varphi)/2, a(\varphi)/2]$ in the case $a(\varphi) > 0$, or x_i, $y_i \in [-b, b]$ in the case $a(\varphi) = 0$ and φ is strictly convex on $[0, b]$. Denote

$$I := \{\, i \in \mathbb{N} : x_i = y_i \,\}, \quad J := \left\{\, i \in \mathbb{N} : x_i \neq y_i \text{ and } \left(|x_i| > \frac{a(\varphi)}{2} \text{ or } |y_i| > \frac{a(\varphi)}{2} \right) \right\}.$$

Analogously to the proof of Theorem 3.1, we observe that the function φ is affine on every interval of the form $[x_i, y_i]$ or $[y_i, x_i]$ for $i \in I$ such that $x_i \neq y_i$. Moreover, $i \notin J$ for $i > n_0$, because either $i \in I$ in the case $a(\varphi) = 0$ (from the strict convexity of φ on $[-b, b]$), or $|x_i|$, $|y_i| \leq a(\varphi)/2$ in the case $a(\varphi) > 0$. Thus J is finite and the assumptions of Lemma 2.2 are satisfied for $\varepsilon_i := u_i^n - x_i$, where n is a fixed, sufficently large natural number. Hence, there exists n_0, such that for $n \geq n_0$

$$I_\varphi\left(u_n + \frac{1}{2}(y-x)\right) = F(1) \leq F(0) = I_\varphi(u_n) \leq 1.$$

It follows that $B(E^\varphi(\mu))$ is LNC.

c) Assume that condition *(iii)* holds. Note that $E^\varphi(\mu)$ satisfies the assumption of Lemma 2.7, so $B(E^\varphi(\mu))$ is strictly convex. Hence, it is also LNC.

d) Assume that condition *(iv)* or *(v)* holds. Put x, y, u_n as in case *b)*. We prove that $\lim_n x_n = \lim_n y_n = 0$. Suppose that there exists $\varepsilon > 0$ and subsequence x_{n_k} such that $|x_{n_k}| > \varepsilon$. Let $\lambda > 0$ satisfy $\lambda > c(\varphi)$. Then for any $n \in \mathbb{N}$

$$\sum_{i=n}^{\infty} \varphi(\lambda x_n)\mu(e_n) = +\infty$$

holds. This cotradicts $x \in E^\varphi(\mu)$. Without loss of generality we assume $\|\lambda x + (1-\lambda)y\| = 1$ for $0 \leq \lambda \leq 1$. If $I_\varphi(\lambda x + (1-\lambda)y) = 1$ for $0 \leq \lambda \leq 1$, then proof is analogous to prooff of case *b)*. If not then $I_\varphi(\alpha(\lambda x + (1-\lambda)y)) = +\infty$ for every $\alpha > 1$. It is possible if finite set $\{n \in \mathbb{N} \colon x_i = y_i = \pm c(\varphi)\}$ is nonempty. Proof in this case is analogous to proof respectively case of Theorem 3.1 and we omit details.

e) Assume that condition *(vi)* holds. In this case for every $x \in E^\varphi(\mu)$ we have $I_\varphi(x) \leq 1$ or $I_\varphi(x) = +\infty$. Let x, y, $u_n \in B(E^\varphi(\mu))$ and $\lim u_n = x$. Let $N \in \mathbb{N}$ be such that $\sum_{k=N+1}^{\infty} \varphi(x_k + y_k)\mu(e_n) < +\infty$. Then

$$\begin{aligned} & I_\varphi\left(u_n + \frac{1}{2}(y-x)\right) = I_\varphi\left(\frac{1}{2}(2(u_n - x)) + \frac{1}{2}(x+y)\right) \leq \\ \leq\ & \sum_{k=1}^{N} \varphi\left(u_k^n + \frac{1}{2}(y_k - x_k)\right)\mu(e_k) + \frac{1}{2} I_\varphi(2(u_n - x)) + \sum_{k=N+1}^{\infty} \varphi(x_k + y_k)\mu(e_k) \\ <\ & +\infty \end{aligned}$$

for n large enough. So $u_n + \frac{1}{2}(y-x) \in B(E^\varphi(\mu))$ and $B(E^\varphi(\mu))$ is LNC.

f) Assume that condition *(vii)* holds. In this case $E^\varphi(\mu) = \{0\}$ and the thesis is trivial.

This completes the proof. ∎

4. The positive part of the unit ball.

Now we consider the following question: under what assumptions do the positive parts of balls $B^+(L^\varphi(\mu))$ and $B^+(E^\varphi(\mu))$ satisfy LNC.

THEOREM 4.1 *The positive part of the unit ball $B^+(L^\varphi(\mu))$ is LNC iff $\dim L^\varphi(\mu) < \infty$. We can replace $L^\varphi(\mu)$ by $E^\varphi(\mu)$.*

PROOF ($\Rightarrow$) Suppose that $\dim L^{\varphi}(\mu) = \infty$. Let $x := \sum_{n\in\mathbb{N}} x_n \chi_{A_n}$, where the x_n are positive and chosen in such a way that $\varphi(nx_n)\mu(A_n) < 1/2^n$. Hence, for each $\lambda > 0$ and $n > \lambda$, we have

$$\sum_{k=n}^{\infty} \varphi(\lambda x_k)\mu(A_k) \leq \sum_{k=n}^{\infty} \varphi(kx_k)\mu(A_k) \leq \sum_{k=n}^{\infty} 2^{-k} < \infty.$$

Hence $x \in E^{\varphi}(\mu)$. Let $y \equiv 0$. Set $u_n := x - \chi_{A_n} \cdot x_n$. If $x \in E^{\varphi}(\mu)$, then $u_n \in E^{\varphi}(\mu)$ for $n \in \mathbb{N}$. Moreover, $I_{\varphi}(u_n) = \sum_{k\in\mathbb{N}\setminus\{n\}} \varphi(x_k)\mu(A_k) \leq 1$. Thus x, y, $u_n \in B^+(L^{\varphi}(\mu))$. Moreover,

$$I_{\varphi}\left(\lambda(x - u_n)\right) = I_{\varphi}\left(\lambda x_n \chi_{A_n}\right) = \varphi(\lambda x_n)\mu(A_n) \leq \varphi(nx_n)\mu(A_n) \leq \frac{1}{2^n}$$

for $\lambda > 0$ and $n > \lambda$. Therefore, $\lim_{n\to\infty} \|u_n - x\|_{\varphi} = 0$. But we have

$$\Big(u_n + \frac{1}{2}(y - x)\Big)(\omega) = \Big(\frac{1}{2}x - x_n\chi_{B_n}\Big)(\omega) = -\frac{1}{2}x_n < 0,$$

for $\omega \in A_n$. Hence, $u_n + (y-x)/2 \notin B^+(L^{\varphi}(\mu))$ (respectively $B^+(E^{\varphi}(\mu))$). It follows that $B^+(L^{\varphi}(\mu))$ is not LNC.

($\Leftarrow$) Let $\dim L^{\varphi}(\mu) < \infty$. Then $L^{\varphi}(\mu)^+$ is a finite cartesian product of halflines which are LNC, so it is also LNC. It follows from either Theorem 3.1 or 3.5 that the unit ball $B(L^{\varphi}(\mu))$ is LNC. Hence, $B^+(L^{\varphi}(\mu))$ is LNC as it is the intersection of LNC sets. ∎

The following can be derived using a similar argument to the one used in the above proof (compare with [1]):

COROLLARY 4.2 *The positive cone of* $L^{\varphi}(\mu)$ *(or* $E^{\varphi}(\mu)$*) is LNC iff* $\dim L^{\varphi}(\mu) < \infty$ *(respectively* $\dim E^{\varphi}(\mu) < \infty$*).*

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