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On two-modular spaces

1. Let X be a real or complex vector space and let two modulars ϱ and ϱ' in the sense of [5] be given in X. We are going to develop a theory of two-modular spaces, in analogy to the theories of Saks spaces due to W. Orlicz [7], [8] and of two-norm spaces due to A. Alexiewicz [1] and further investigations by A. Alexiewicz and Z. Semadeni [2], and by A. Wiweger [9].

Let X_ϱ and $X_{\varrho'}$ be the modular spaces defined by modulars ϱ and ϱ', respectively. By $|\cdot|_\varrho$ and $|\cdot|_{\varrho'}$ we denote the respective F-norms generated by these modulars. If ϱ or ϱ' is convex, $\|\cdot\|_\varrho$ or $\|\cdot\|_{\varrho'}$ will denote the respective homogeneous norm. Modular convergence $x_n \xrightarrow{\varrho'} x$ in $X_{\varrho'}$ (or $x_n \xrightarrow{\varrho} x$ in X_ϱ) means that $\varrho'\big(k(x_n - x)\big) \to 0$ (or $\varrho\big(k(x_n - x)\big) \to 0$) as $n \to \infty$ for a $k > 0$ depending on the sequence (x_n) (see [5] and [6]).

A sequence (x_n) of elements of X will be called *ϱ-bounded* if for any sequence of numbers $\varepsilon_n \to 0$ there holds $\varepsilon_n x_n \xrightarrow{\varrho} 0$.

1.1. *Let ϱ be convex. A sequence (x_n) of elements of X_ϱ is ϱ-bounded if and only if there exist positive constants k and M such that $\varrho(kx_n) \leqslant M$ for $n = 1, 2, \ldots$*

Proof. Supposing the above condition to be satisfied and $0 < \varepsilon_n < 1$, $\varepsilon_n \to 0$, we get $\varrho(k\varepsilon_n x_n) \leqslant \varepsilon_n \varrho(kx_n) \leqslant \varepsilon_n M \to 0$ as $n \to \infty$, and (x_n) is ϱ-bounded. Conversely, let us suppose that (x_n) is ϱ-bounded with a convex ϱ and for any $k > 0$, $M > 0$ there exists an index n for which $\varrho(kx_n) > M$. Taking $k = m^{-2}$, $M = 1$, we may choose n_m such that $\varrho(m^{-2} \cdot x_{n_m}) > 1$ for $m = 1, 2, \ldots$ Since $x_n \in X_\varrho$, no natural number may appear in the sequence (n_m) infinitely many times. Thus we may extract an increasing subsequence of indices (n_{m_i}), obtaining $\varrho(m_i^{-2} \cdot y_i) > 1$ for $i = 1, 2, \ldots$, where $y_i = x_{n_{m_i}}$. Since (y_i) is also ϱ-bounded, so $\varrho(k \cdot m_i^{-1} \cdot y_i) \to 0$ as $i \to \infty$ for a $k > 0$. Taking i so large that $km_i > 1$ and applying the convexity of ϱ, we obtain

$$1 < \varrho\left(\frac{y_i}{m_i^2}\right) \leqslant \frac{1}{km_i}\, \varrho\left(\frac{ky_i}{m_i}\right) \to 0 \quad \text{as} \quad i \to \infty,$$

a contradiction.

1.2. Now we shall introduce the notion of γ-convergence in the triple $\mathscr{X} = \langle X, \varrho, \varrho'\rangle$. A sequence (x_n) of elements of X is called *γ-convergent* to $x \in X$ if $x_n \xrightarrow{\varrho'} x$ as $n \to \infty$ and (x_n) is ϱ-bounded. We denote this by $x_n \xrightarrow{\gamma} x$.

It is evident that if $x_n', x_n'', x', x'' \in X$, $x_n' \xrightarrow{\gamma} x'$, $x_n'' \xrightarrow{\gamma} x''$, a and b are scalars, then $ax_n' + bx_n'' \xrightarrow{\gamma} ax' + bx''$.

If $\mathscr{X}_1 = \langle X_1, \varrho_1, \varrho_1'\rangle$ and $\mathscr{X}_2 = \langle X_2, \varrho_2, \varrho_2'\rangle$, then an operator $T\colon X_1 \to X_2$ is called *(γ_1, γ_2)-continuous* if $x, x_n \in X_1$ and $x_n \xrightarrow{\gamma_1} x$ imply $T(x_n) \xrightarrow{\gamma_2} T(x)$.

Obviously, there exists in $\mathscr{X}$ also the notion of γ-convergence with respect to norms $|\cdot|_\varrho$ and $|\cdot|_{\varrho'}$, as defined in [1], p. 49.

1.3. *If $x_n, x \in X_\varrho \cap X_{\varrho'}$ and (x_n) is γ-convergent to x in the sense of the norms $|\cdot|_\varrho$, $|\cdot|_{\varrho'}$, then $x_n \xrightarrow{\gamma} x$.*

Proof. We may take $x = 0$. Since $|x_n|_{\varrho'} \to 0$, so $x_n \xrightarrow{\varrho'} 0$. Moreover, taking $\varepsilon_n \to 0$, we have $|\varepsilon_n x_n|_\varrho \to 0$, by assumption. Thus, if $0 < \varepsilon < 1$ is arbitrary, then one can choose an N such that $\varrho(\varepsilon^{-1}\varepsilon_n x_n) < \varepsilon$ for $n > N$. Hence, $\varrho(\varepsilon_n x_n) < \varepsilon$ for $n > N$ and we obtain $\varepsilon_n x_n \xrightarrow{\varrho} 0$.

2. In the following we shall consider a special case of the space $\mathscr{X}$. Let μ be a measure in σ-algebra Σ of subsets of a non-empty set Ω and let X be a vector subspace of the space S of all Σ-measurable, real or complex functions on Ω with equality μ-almost everywhere. Let φ and ψ be φ-functions, i.e. $\varphi(u) > 0$ for $u > 0$, $\varphi(0) = 0$, φ non-decreasing and continuous for $u \geqslant 0$, $\varphi(u) \to \infty$ as $u \to \infty$, and the same about ψ (see [4]). Then ϱ and ϱ' may be defined by the formulae

$$(*) \qquad \varrho(x) = \int_\Omega \varphi\big(|x(t)|\big)\,d\mu, \quad \varrho'(x) = \int_\Omega \psi\big(|x(t)|\big)\,d\mu.$$

Let us remark that taking in this example $\Omega = \langle 0, 1\rangle$, μ = Lebesgue measure, $\varphi(u) = |u|^p$ $(p \geqslant 1)$, $\psi(u) = e^u - u - 1$, we easily observe that the converse statement to 1.3 does not hold. It is sufficient to choose

$$x_n(t) = \begin{cases} 0 & \text{if} \quad 0 \leqslant t < 2^{-n} \text{ or } 2^{-1} \leqslant t \leqslant 1, \\ i/2 & \text{if} \quad 2^{-i-1} \leqslant t < 2^{-i},\ i = 1, 2, \ldots, n-1, \end{cases}$$

$x(t) = \lim\limits_{n\to\infty} x_n(t)$. Then $\varrho'\big(2(x - x_n)\big) = \infty$ but $\varrho'(x - x_n) \to 0$ and $\varrho(x_n) \leqslant \dfrac{1}{2^{p+1}} \sum\limits_{i=1}^{\infty} \dfrac{i^p}{2^i} < \infty$, and so $x_n \xrightarrow{\gamma} x$, but this relation does not hold in the sense of the norms.

Let us remark that the following lemma is true:

2.1. LEMMA. *If ψ is a φ-function, ϱ' is of the form $(*)$ and $x_n \xrightarrow{\varrho'} 0$, then*

(a) *$x_n(t) \to 0$ in measure μ in Ω,*

(b) *if $\mu(\Omega) < \infty$ and ψ is convex, then $\int_\Omega |x_n(t)|\,d\mu \to 0$.*

In order to prove (a) let us choose arbitrary $\varepsilon, \eta > 0$ and let us write

$$E_n^\eta = \left\{t \in \Omega \colon |x_n(t)| > \frac{1}{k}\psi^{-1}(\eta)\right\},$$

the number $k > 0$ being given by the condition

$$(+) \qquad \int_\Omega \psi\big(k|x_n(t)|\big)\,d\mu \to 0 \quad \text{as} \quad n \to \infty.$$

Taking N so large that $\int_\Omega \psi\big(k|x_n(t)|\big)\,d\mu < \varepsilon\eta$ for $n > N$ we obtain

$$\varepsilon\eta > \int_{E_n^\eta} \psi\big(k|x_n(t)|\big)\,d\mu \geqslant \eta\mu(E_n^\eta) \quad \text{for} \quad n > N,$$

whence $\mu(E_n^\eta) \to 0$ as $n \to \infty$. In order to show (b) we take an arbitrary $\eta > 0$ and we choose $M_\eta > 0$ such that $\psi(u)/u \geqslant M_\eta$ for $u \geqslant \eta$. Taking $k > 0$ from condition $(+)$ and writing $A_n = \{t \in \Omega \colon |x_n(t)| \geqslant \eta/k\}$, we get

$$k \int_\Omega |x_n(t)|\,d\mu \leqslant \eta\mu(\Omega) + \frac{1}{M_\eta}\int_\Omega \psi\big(k|x_n(t)|\big)\,d\mu,$$

and this shows the required property.

3. Supposing the measure μ to be atomless, we are going to prove some embedding theorems for two-modular spaces $\mathscr{X} = \langle L^\varphi, \varrho, \varrho'\rangle$ and $\mathscr{X}' = \langle L^\psi, \varrho, \varrho'\rangle$, with various functions φ and ψ.

3.1. *Let $\varphi_1, \varphi_2, \psi_1$ and ψ_2 be φ-functions and let*

$$\varrho_i(x) = \int_\Omega \varphi_i\big(|x(t)|\big)\,d\mu, \qquad \varrho_i'(x) = \int_\Omega \psi_i\big(|x(t)|\big)\,d\mu,$$

$$\mathscr{X}_i = \langle L^{\varphi_i}, \varrho_i, \varrho_i'\rangle, \qquad \mathscr{X}_i' = \langle L^{\psi_i}, \varrho_i, \varrho_i'\rangle \quad \textit{for } i = 1, 2.$$

Let us still recall the relations $\psi_2 \prec^l \psi_1$ and $\psi_2 \prec^a \psi_1$ for pairs of φ-functions ψ_1 and ψ_2 (see [4], p. 123). The relation $\psi_2 \prec^l \psi_1$ means that there exist positive numbers A, B, u_0 such that $\psi_2(Au) \leqslant B\psi_1(u)$ for $u \geqslant u_0$. Moreover, $\psi_2 \prec^a \psi_1$ means that there are positive numbers A, B for which $\psi_2(Au) \leqslant B\psi_1(u)$ for all $u \geqslant 0$.

γ_i-convergence of (x_n) to zero will mean that $x_n \xrightarrow{\varrho_i'} 0$ and (x_n) is ϱ_i-bounded, where $i = 1, 2$.

First, we shall give necessary and sufficient conditions in order that $\mathscr{X}_1' \subset \mathscr{X}_2'$, this being understood in the sense of a (γ_1, γ_2)-continuous embedding. The same problem will be solved also for the case $\mathscr{X}_1 \subset \mathscr{X}_2$. First, we show that

3.2. *If* $L^{\psi_1}(\Omega) \subset L^{\psi_2}(\Omega)$ *(as sets), then* $x_n \in L^{\psi_1}(\Omega)$, $x_n \xrightarrow{\varrho_1'} 0$ *implies* $x_n \xrightarrow{\varrho_2'} 0$.

Indeed, the supposed inclusion implies $\psi_2 \prec^l \psi_1$ in the case of $\mu(\Omega) < \infty$ and $\psi_2 \prec^a \psi_1$ in the case of $\mu(\Omega) = \infty$ (see [4], p. 131). Let us limit ourselves to the case $\mu(\Omega) < \infty$. From $\psi_2 \prec^l \psi_1$ we conclude in the well-known manner that for every $u_1 > 0$ there exists a $B_1 > 0$ such that $\psi_2(Au) \leqslant B_1\psi_1(u)$ for $u \geqslant u_1$. By assumption, there is a constant $k > 0$ such that $\varrho_1'(kx_n) \to 0$. Let us put $\Omega_n = \{t \in \Omega: k|x_n(t)| > u_1\}$. Then

$$\varrho_2'(Akx_n) \leqslant B_1 \int_{\Omega_n} \psi_1(k|x_n(t)|)\,d\mu + \int_{\Omega\setminus\Omega_n} \psi_2(Ak|x_n(t)|)\,d\mu$$
$$\leqslant B_1\varrho_1'(kx_n) + \psi_2(Au_1)\mu(\Omega),$$

and the condition $x_n \xrightarrow{\varrho_2'} 0$ follows by continuity of ψ_2 at 0.

3.3. *The following conditions are equivalent for* $\mu(\Omega) < \infty$ *and for* μ *atomless:*

(a) $\mathscr{X}_1' \subset \mathscr{X}_2'$ *and* $x_n \xrightarrow{\gamma_1} 0$ *implies* $x_n \xrightarrow{\gamma_2} 0$,

(b) $\psi_2 \prec^l \psi_1$ *and there are constants* $A, B, u_0 > 0$ *such that*

$$\text{(i)} \qquad \varphi_2(Au) \leqslant B\max(\varphi_1(u), \psi_1(u)) \quad \textit{for} \quad u \geqslant u_0.$$

Proof. Let us suppose that (b) holds. It is known that $L^{\psi_1}(\Omega) \subset L^{\psi_2}(\Omega)$ is equivalent to $\psi_2 \prec^l \psi_1$ (see [4]). Let $x_n \in L^{\psi_1}(\Omega)$, $x_n \xrightarrow{\varrho_1'} 0$, (x_n) is ϱ_1-bounded. By 3.2, $x_n \xrightarrow{\varrho_2'} 0$. It remains to show that (x_n) is ϱ_2-bounded. Let $\varepsilon_n \to 0$; then writing $y_n = \varepsilon_n x_n$, we obtain $y_n \in L^{\varphi_1}(\Omega) \cap L^{\psi_1}(\Omega)$ for sufficiently large n, $\varrho_1'(ky_n) \to 0$ and $\varrho_1(ky_n) \to 0$ for some $k > 0$.

Let us remark that from (b) it follows that for any $u_1 > 0$ there exists a $B_1 > 0$ such that $\varphi_2(Au) \leqslant B_1\max(\varphi_1(u), \psi_1(u))$ for $u \geqslant u_1$. Hence, writing $E_n = \{t \in \Omega: k|y_n(t)| > u_1\}$ and arguing as in the proof of 3.2, we obtain

$$\varrho_2(Aky_n) \leqslant B_1\varrho_1'(ky_n) + B_1\varrho_1(ky_n) + \varphi_2(Au_1)\mu(\Omega),$$

which shows that $y_n \xrightarrow{\varrho_2} 0$. Thus, (x_n) is ϱ_2-bounded.

Now, let us suppose (a). Then $L^{\psi_1}(\Omega) \subset L^{\psi_2}(\Omega)$ and so $\psi_2 \prec^l \psi_1$. It remains to prove the second condition in (b). We show now $L^{\varphi_1}(\Omega) \cap L^{\psi_1}(\Omega) \subset L^{\varphi_2}(\Omega)$. Let $x \in L^{\varphi_1}(\Omega) \cap L^{\psi_1}(\Omega)$; then $x_n = x/n$ tends to zero with respect to any of the modulars ϱ_1 and ϱ_1'. Consequently, $x_n \xrightarrow{\gamma_1} 0$. By (a), $x_n \xrightarrow{\gamma_2} 0$. Hence (x_n) is ϱ_2-bounded, and so $\varrho_2(k \cdot n^{-2} \cdot x) \to 0$ as $n \to \infty$ for some $k > 0$. Thus, $x \in L^{\varphi_2}(\Omega)$. Applying the above proved inclusion and modified Theorem 2.21 from [4], p. 131, we immediately conclude the second condition from (b).

3.4. Let us remark that the Theorem 3.3 remains valid also for an atomless infinite measure μ if we replace (b) by the following condition

(c) $\psi_2 \prec^a \psi_1$ *and there are constants* $A, B > 0$ *such that* (i) *holds for all* $u \geqslant 0$.

Since it is evident that $\varphi_2 \prec^l \varphi_1$ implies (i) from 3.3 (b), so it is easily observed that in the case of $\mathscr{X}_1$ and $\mathscr{X}_2$ the following theorem is obtained analogously:

3.5. *If* $\mu(\Omega) < \infty$ *and* μ *is atomless, then the following conditions are equivalent*:

(a′) $\mathscr{X}_1 \subset \mathscr{X}_2$ *and* $x_n \xrightarrow{\gamma_1} 0$ *implies* $x_n \xrightarrow{\gamma_2} 0$,

(b′) $\varphi_2 \prec^l \varphi_1$.

3.6. As in the previous case we may remark that Theorem 3.5 remains valid also in the case where μ is atomless and infinite if we replace (b′) by

(c′) $\varphi_2 \prec^a \varphi_1$.

4. We start with the definition of γ-completeness in the general case of $\langle X, \varrho, \varrho' \rangle$, where ϱ is convex. A sequence (x_n) of elements of X satisfies the ϱ'-*Cauchy condition* if there exists a constant $k > 0$ with the property that for every $\varepsilon > 0$ there is an N such that $\varrho'\big(k(x_n - x_m)\big) < \varepsilon$ for $m, n > N$. A set $\{x \in X_\varrho : \varrho(k_0 x) \leqslant M_0\}$ with fixed k_0, $M_0 > 0$ will be termed a ϱ-*ball* in X_ϱ. $\langle X, \varrho, \varrho' \rangle$ will be called γ-*complete* if for every fixed ϱ-ball K in X_ϱ, and sequence (x_n) of elements of K satisfying the ϱ'-Cauchy condition, is γ-convergent to an element of K.

4.1. $\mathscr{X}' = \langle L^\varphi, \varrho, \varrho' \rangle$ *with* ϱ, ϱ' *given by* (*), φ *convex, is* γ-*complete.*

Let (x_n) be ϱ-bounded with constants k_0, $M_0 > 0$ and let it satisfy the ϱ'-Cauchy condition with a constant $k > 0$. Since $L^\psi(\Omega)$ is complete with respect to ϱ'-convergence, there is an $x_0 \in L^\psi(\Omega)$ such that $\varrho'\big(k(x_n - x_0)\big) \to 0$, with the same constant k. Hence $x_n \to x_0$ in measure (see [2], Lemma (a)), and so $\varphi\big(k_0|x_{n_i}(t)|\big) \to \varphi\big(k_0|x_0(t)|\big)$ for a subsequence (x_{n_i}) of (x_n). Applying Fatou's lemma, we observe that $\varrho(k_0 x_0) \leqslant M_0$. Thus, $\mathscr{X}'$ is γ-complete.

Arguing as in the above proof, we easily get that

4.2. $\langle L^\varphi \cap L^\psi, \varrho, \varrho' \rangle$ *with* ϱ, ϱ' *given by* (*), φ *convex, is* γ-*complete.*

From 4.2 it immediately follows:

4.3. *Let* φ *be convex and let* $\psi \prec^l \varphi$ *in the case of* $\mu(\Omega) < \infty$, $\psi \prec^a \varphi$ *in the case of* $\mu(\Omega) = \infty$. *Then* $\mathscr{X} = \langle L^\varphi, \varrho, \varrho' \rangle$ *with* ϱ, ϱ' *given by* (*) *is* γ-*complete.*

Remark. The results above remain valid if we replace the space under consideration by the Saks spaces $\langle L^\varphi, \|\cdot\|_\varrho, \|\cdot\|_{\varrho'} \rangle$, $\langle L^\varphi, \|\cdot\|_\varrho, \|\cdot\|_{\varrho'} \rangle$,

$\langle L^\varphi \cap L^\psi, \|\cdot\|_\varrho, \|\cdot\|_{\varrho'}\rangle$, because, for convex ϱ, ϱ-boundedness is equivalent to boundedness in norm $\|\cdot\|_\varrho$, and completeness in norm $\|\cdot\|_{\varrho'}$ means ϱ'-completeness with every $k > 0$.

5. We shall say that a set $X_0 \subset X$ is *γ-dense* in $\langle X, \varrho, \varrho'\rangle$ if for every $x \in X$ there exists a sequence (x_n) of elements of X_0 such that $x_n \xrightarrow{\gamma} x$. $\langle X, \varrho, \varrho'\rangle$ will be called *γ-separable* if there exists a countable, γ-dense subset X_0 of X. First, we prove the following lemma.

5.1. *If φ is a φ-function, then the set of simple functions integrable in Ω is ϱ-dense in the Orlicz space $L^\varphi(\Omega)$.*

Proof. Let $x \in L^\varphi(\Omega)$, $x(t) \geqslant 0$ in Ω. Taking a sequence (x_n) of simple functions such that $0 \leqslant x_n(t) \nearrow x(t)$ in Ω, we observe that x_n are integrable. Taking $k > 0$ such that $\varrho(kx) = \int\limits_\Omega \varphi(kx(t))\,d\mu < \infty$ and applying the Lebesgue dominated convergence theorem, we easily get $\varrho(k(x-x_n)) \to 0$, i.e. $x_n \xrightarrow{\varrho} x$. In the general case the proof is obtained writing $x(t)$ as the difference of its positive and negative part.

5.2. *Let φ and ψ be φ-functions. Then the set of simple functions integrable in Ω is γ-dense in $\langle L^\varphi \cap L^\psi, \varrho, \varrho'\rangle$, where ϱ and ϱ' are given by* (*).

Proof. Let $x \in L^\varphi(\Omega) \cap L^\psi(\Omega)$, $x(t) \geqslant 0$ in Ω. Let (x_n) be the sequence of simple functions from the proof of 5.1. Then, by 5.1, $x_n \xrightarrow{\varrho'} x$. Now, let $0 < \varepsilon_n \leqslant 1$, $\varepsilon_n \to 0$. Then $\varphi(k\varepsilon_n x_n(t)) \to 0$ and $\varphi(k\varepsilon_n x_n(t)) \leqslant \varphi(kx(t))$ almost everywhere in Ω. By the Lebesgue dominated convergence theorem, $\varrho(k\varepsilon_n x_n) \to 0$ as $n \to \infty$ for sufficiently small $k > 0$. Hence (x_n) is ϱ-bounded. If we drop the assumption $x(t) \geqslant 0$, we obtain the result splitting $x(t)$ into positive and negative part.

From the above result it follows at once that:

5.3. *Let $\varphi \prec^l \psi$ in case of $\mu(\Omega) < \infty$, $\varphi \prec^a \psi$ in case of $\mu(\Omega) = \infty$. Then the set of simple function integrable in Ω is γ-dense in $\langle L^\psi, \varrho, \varrho'\rangle$.*

5.4. *Let $\psi \prec^l \varphi$ in case of $\mu(\Omega) < \infty$, $\psi \prec^a \varphi$ in case of $\mu(\Omega) = \infty$. Then the set of simple functions integrable in Ω is γ-dense in $\langle L^\varphi, \varrho, \varrho'\rangle$.*

5.5. *Let φ and ψ be φ-functions and let ϱ and ϱ' are given by* (*). *If the measure μ is separable, then $\langle L^\varphi \cap L^\psi, \varrho, \varrho'\rangle$ is γ-separable.*

Proof. First, let us suppose that $\mu(\Omega) < \infty$. Let Σ_0 be a countable family of sets from Σ such that for any $A \in \Sigma$ there is a sequence of sets $A^n \in \Sigma_0$ for which $\mu(A^n \dot{-} A) \to 0$ as $n \to \infty$. Let $x \in L^\varphi(\Omega) \cap L^\psi(\Omega)$, $x(t) \geqslant 0$ in Ω and let (x_n) be a sequence of simple functions such that $0 \leqslant x_n(t) \nearrow x(t)$ in Ω. By 5.2, $x_n \xrightarrow{\gamma} x$. Let $x_n = \sum_{i=1}^{p_n} c_i \chi_{A_i}$, where $A_i \in \Sigma$ are pairwise disjoint and χ_{A_i} is the characteristic function of the set A_i. We choose sets $A_i^n \in \Sigma_0$, $i = 1, 2, \ldots, p_n$, pairwise disjoint, for which

$$\mu(A_i^n \dot{-} A_i) < \left[n \cdot p_n \cdot \max_{1 \leqslant i \leqslant p_n} \max(\varphi(c_i), \psi(c_i))\right]^{-1}.$$

Let us take positive rational numbers c_i' such that $|c_i'-c_i|<1/n$ for $i=1,2,\ldots,p_n$, and let us put $y_n=\sum_{i=1}^{p_n} c_i'\chi_{A_i^n}$. Let $0<\varepsilon_n<1$, $\varepsilon_n\to 0$, and let $0<k<1/2$ be so small that $\varrho(k\varepsilon_n x_n)\to 0$ and $\varrho'\big(k(x_n-x)\big)\to 0$. We shall prove that $y_n\xrightarrow{\gamma} x$. First, we show that (y_n) is ϱ-bounded. We have

$$\varrho(\tfrac{1}{2}k\varepsilon_n y_n)\leqslant \varrho\big(k\varepsilon_n(y_n-x_n)\big)+\varrho(k\varepsilon_n x_n).$$

But

$$\begin{aligned}\varrho\big(k\varepsilon_n(y_n-x_n)\big)&\leqslant \int_\Omega \varphi\Big(k\varepsilon_n\sum_{i=1}^{p_n}[|c_i'-c_i|\chi_{A_i^n}+c_i|\chi_{A_i^n}-\chi_{A_i}|]\Big)d\mu\\&=\int_\Omega \varphi\Big(k\varepsilon_n\sum_{i=1}^{p_n}|c_i'-c_i|\chi_{A_i^n}+k\varepsilon_n\sum_{i=1}^{p_n}c_i\chi_{A_i^n\dot{-}A_i}\Big)d\mu\\&\leqslant \int_\Omega \varphi\Big(2k\varepsilon_n\sum_{i=1}^{p_n}|c_i'-c_i|\chi_{A_i^n}\Big)d\mu+\int_\Omega \varphi\Big(2k\varepsilon_n\sum_{i=1}^{p_n}c_i\chi_{A_i^n\dot{-}A_i}\Big)d\mu\\&<\varphi(2k\varepsilon_n\cdot n^{-1})\mu(\Omega)+\sum_{i=1}^{p_n}\varphi(2k\varepsilon_n c_i)\mu(A_i^n\dot{-}A_i)\\&\leqslant \varphi(2k\varepsilon_n\cdot n^{-1})\mu(\Omega)+\sum_{i=1}^{p_n}\varphi(c_i)\mu(A_i^n\dot{-}A_i)<\varphi(2k\varepsilon_n\cdot n^{-1})\mu(\Omega)+n^{-1},\end{aligned}$$

and so

$$\varrho(\tfrac{1}{2}k\varepsilon_n y_n)<\varphi(2k\varepsilon_n\cdot n^{-1})\mu(\Omega)+n^{-1}+\varrho(k\varepsilon_n x_n)\to 0 \text{ as } n\to\infty.$$

Hence (y_n) is ϱ-bounded. Similar calculation shows that

$$\varrho'\big(k(y_n-x_n)\big)<\psi(2k\cdot n^{-1})\mu(\Omega)+n^{-1}.$$

Consequently,

$$\begin{aligned}\varrho'\big(\tfrac{1}{2}k(y_n-x)\big)&\leqslant \varrho'\big(k(y_n-x_n)\big)+\varrho'\big(k(x_n-x)\big)\\&<\psi(2k\cdot n^{-1})\mu(\Omega)+n^{-1}+\varrho'\big(k(x_n-x)\big)\to 0 \text{ as } n\to\infty,\end{aligned}$$

and thus, $y_n\xrightarrow{\varrho'} x$. Hence we conclude that $y_n\xrightarrow{\gamma} x$.

If we omit the assumption $x(t)\geqslant 0$ in Ω, we obtain the proof by splitting x into positive and negative parts.

This shows that the countable set of simple functions which are rational linear combinations of characteristic functions of sets from Σ_0 is γ-dense in $L^\varphi\cap L^\psi$.

Now, let $\mu(\Omega)=\infty$ and let Σ_0 be defined as before, where $A\in\Sigma$ is of finite measure μ. Let us choose an arbitrary $\varepsilon>0$ and let us take for a given $x\in L^\varphi(\Omega)\cap L^\psi(\Omega)$, $x(t)\geqslant 0$ in Ω, a $k>0$ so small that $\varrho(kx)<\infty$ and $\varrho'(kx)<\infty$. Then there exists a set $\Omega_0\in\Sigma$ of finite measure μ for which $\varrho'(\frac{1}{2}kx\chi_{\Omega\setminus\Omega_0}) < \varepsilon/2$. Now, let us define the sequences (x_n) and (y_n) as in the previous part of the proof, replacing Ω by Ω_0. Of course, (y_n) remains ϱ-bounded. Moreover, we have

$$\varrho'\big(\tfrac{1}{4}k(y_n-x)\big) \leqslant \varrho'\big(\tfrac{1}{2}k(y_n-x\chi_{\Omega_0})\big)+\varrho'(\tfrac{1}{2}kx\chi_{\Omega\setminus\Omega_0})$$
$$< \psi(2k\cdot n^{-1})\mu(\Omega_0)+n^{-1}+\varrho'\big(k(x_n-x)\chi_{\Omega_0}\big)+\varepsilon/2.$$

This shows that $y_n \xrightarrow{\varrho'} x$. Thus we may conclude the proof.

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